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EULERIAN FRACTIONS AND STIRLING, BERNOULLI AND EULER FUNCTIONS WITH COMPLEX ORDER PARAMETERS AND THEIR IMPACT ON THE POLYLOGARITHM FUNCTION

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We first study some generalizations of Eulerian fractions with complex order parameter and investigate their interrelationship with likewise generalized Eulerian functions as well as Stirling functions. We apply the new approach to polylogarithms of non-integral order, for which only a few values are known in closed form. In particular, we present a structural solution of the counterpart of an old conjecture of Mengoli and Euler in the polylogarithm case with the aid of Riemann's zeta function and the Dirichlet eta and beta functions.

1. INTRODUCTION

Among the important special functions in analytic number theory, the polylogarithm function (also known as Jonquière's function or Li-function) plays a prominent role. It is defined for any complex γ and z as the analytic continuation of the Dirichlet series

$$Li_\gamma(z) := \sum_{k\geq 1} \frac{z^k}{k^\gamma} \quad (|z| < 1, z, \gamma \in \mathbb{C}). \tag{1}$$

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For $\gamma = 1$, the Li-function is just the ordinary logarithm function $Li_1(z) = -\ln(1-z)$, while the special cases $\gamma = 2$ and $\gamma = 3$ are known as Euler' s dilogarithm (also referred to as the Spence' s function) and the trilogarithm, respectively. More generally, the polylogarithm is associated with the Lerch transcendent (cf., e.g.,§1.11 of [**17**] and [**18**])

(2)
$$\Psi(z,\gamma,\nu) = \sum_{n=0}^{\infty}(\nu+n)^{-\gamma}z^n, \nu \neq 0,-1,-2,\ldots, |z| \leq 1, \operatorname{Re}\gamma > 0 \text{ or } z = 1, \operatorname{Re}\gamma > 1,$$

in view of the obvious correspondence $Li_\gamma(z) = z\Psi(z,\gamma,1)$.

Many mathematicians, including Euler, Kummer, Abel, Spence and Nielsen, have contributed to the theory of polylogarithms. Perhaps more often than in other fields of mathematics, previously obtained results have been overlooked and properties repeatedly rediscovered. This topic treated a subject which has regained considerable interest in the last few decades on account of its applications in several fields of pure and applied mathematics, for instance in connection with non-Euclidean geometry and group theory and in quantum electrodynamics, where the use of modern computational facilities, in particular formula manipulation, allows certain multidimensional integrals over rational functions to be expressed in closed form by means of polylogarithms and related functions. For more materials, one may see [**25, 34, 35**].

A major purpose of this paper is to investigate the polylogarithm function by exhibiting and employing its close relationship with the generalized Eulerian fractions for complex order parameter. The classical Eulerian fractions

$$\alpha_n(z) = \sum_{k=0}^{\infty}(k+1)^n z^k \quad (n \in \mathbb{N}_0, |z| < 1, z \in \mathbb{C}) \tag{3}$$

are obtained in the study of the classical Eulerian numbers and their associated Eulerian polynomials,

$$E(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+1-j)^n \quad (n,k \in \mathbb{N}_0)$$
$$A_n(z) = \sum_{k=0}^{n}E(n,k)z^k \quad (n \in \mathbb{N}_0, |z| < 1, z \in C)$$

by inverting the Carlitz identity (cf., e.g., [**10, 12, 13, 29**])

$$A_n(z) = (1-z)^{n+1}\alpha_n(z). \tag{4}$$

Comparing the coefficients of z^n on both sides of the identity $\alpha_n(z) = A_n(z)(1-z)^{-n-1}$, we may obtain the Worpitzky's identity (cf. [**29**])

$$(k+1)^n = \sum_{i=0}^{n-1} E(n,i)\binom{n+k-i}{n}.$$

The functions $\alpha_n(z)$ are a powerful tool in many applications as, e.g., in number theory, combinatorics or computational mathematics [**9, 13, 16, 19, 20, 22, 21, 33**]). For instance, the particular series $\alpha_n(1/2)$ has interesting combinatorial applications found by Velleman and Call [**33**] and by Lengyel [**24**]. Even the partial sums of the Eulerian fraction series or the sums of the so-called arithmetic-geometric progression have been studied for a long time due to their important applications, see, e.g., [**9, 16**] or [**28**] in case $n = 1$. In addition, various series summation problems treated in Schwatt's book [**31**] are of this form.

The definition (3) of the Eulerian fraction now clearly suggests that its generalization to a complex order parameter, say $\alpha_\gamma(z), \gamma \in \mathbb{C}$, would lead to new valuable representations of the polylogarithm (1) by observing that

$$Li_\gamma(z) = z\alpha_{-\gamma}(z) \quad (|z| < 1, z, \gamma \in C). \tag{5}$$

This is carried out in Section 2 (cf. Corollary 4). In addition, we will make use of the fact that the classical Eulerian fractions are closely related to the Stirling numbers of the second kind $S(n,k)$ via (cf., e.g., [**7**] as well as Theorem 8)

$$\alpha_n(z) = \sum_{k=1}^{n} k!S(n,k)\frac{z^{k-1}}{(1-z)^{k+1}}, \quad n \in \mathbb{N}. \tag{6}$$

The quantities $S(n,k)$, also denoted by $\left\{{n \atop k}\right\}$, count the number of ways to partition a set of n elements into k nonempty subsets, and are given by (see [**13, 23**])

$$S(n,k) = \frac{1}{k!}\Delta^k x^n\Big|_{x=0} = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n \quad (n \in \mathbb{N}_0, k \in \mathbb{N}),\ S(0,0) = 1.$$

Here, Δ is the forward difference operator, $\Delta f(x) = f(x+1) - f(x)$, and $\Delta^{k+1} = \Delta(\Delta^k), k \in \mathbb{N}$. There is a huge literature connected with these numbers. For instance, in [**32**], the numbers $L_n^{(r)}$, generated by the Taylor expansion

$$\left(\frac{z}{\ln(1+z)}\right)^r = \sum_{n\geq 0} L_n^{(r)}\frac{z^n}{n!} \quad (|z| < 1)$$

satisfy the remarkable formula $\binom{m-1}{n}L_n^{(m)} = S(m, m-n)$ for $n \in \mathbb{N}_0, m \in N$ such that $0 \leq n \leq m-1$. In order to extend the identities (4) and (6) to complex order

$\gamma \in \mathbb{C}$, we need appropriate generalizations of the Eulerian numbers and functions, $E(\gamma, k)$ and $A_\gamma(z)$, as well as of the Stirling numbers of the second kind, $S(\gamma, j)$. These latter quantities have been defined already for $\gamma \in \mathbb{R}$ [**4**] and later for general $\gamma \in \mathbb{C}$ [**8**]. Moreover, we will investigate two more types of Eulerian fractions which, besides $S(\gamma, j)$, depend on an additional parameter $a \in \mathbb{R}$ (see Proposition 2.10).

It has always been of great interest and an enormous challenge to represent the polylogarithm function $Li_\gamma(z)$ for particular entries of either the argument z or the order γ, but still very few of these values are known in closed form. Another aim of this paper, as pursued in Section 3, is to investigate the polylogarithm with imaginary arguments $Li_\gamma(\pm i)$. Here we restrict ourselves to the parameter range $\operatorname{Re}\gamma > 1$ to be sure that the representations described in the Corollaries 2.3 and 2.8 can be continued to the boundary values $z = \pm i$. Moreover, we will make use of the fact that the polylogarithm is linked, as a function of its order γ, to the Riemann zeta-function

$$\zeta(\gamma) := Li_\gamma(1) = \sum_{k=1}^{\infty} \frac{1}{k^\gamma}, \quad \operatorname{Re}\gamma > 1, \tag{7}$$

as well as to the Dirichlet eta function (or alternating zeta function)

$$\eta(\gamma) := -Li_\gamma(-1) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^\gamma}, \quad \operatorname{Re}\gamma > 0. \tag{8}$$

These two functions are also connected with the lambda function

$$\lambda(\gamma) := \sum_{k=0}^{\infty} \frac{1}{(2k+1)^\gamma}, \quad \operatorname{Re}\gamma > 1,$$

in view of the two obvious identities

$$\lambda(\gamma) = (1 - 2^{-\gamma})\zeta(\gamma), \quad \zeta(\gamma) = (1 - 2^{1-\gamma})^{-1}\eta(\gamma), \tag{9}$$

which implies $\lambda(\gamma) = 2^{-\gamma}\zeta(\gamma) + \eta(\gamma)$. Finally, the Dirichlet (or Catalan) beta function is defined by

$$\beta(\gamma) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^\gamma}, \quad \operatorname{Re}\gamma > 0.$$

This function is particularly interesting for us because of the following connection with the zeta and eta functions (7) and (8) (cf. 5.4 of [**34**]).

Lemma 1. *Let $\gamma \in \mathbb{C}$, $Re\,\gamma > 1$. The polylogarithm* (1) *with imaginary arguments satisfies*

$$Li_\gamma(\pm i) = -2^{-\gamma}\eta(\gamma) \pm i\beta(\gamma) = -2^{-\gamma}(1 - 2^{1-\gamma})\zeta(\gamma) \pm i\beta(\gamma). \tag{10}$$

Proof. By definition (1), one has

$$Li_\gamma(\pm i) = \sum_{k=1}^{\infty} \frac{(\pm i)^k}{k^\gamma} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^\gamma} \pm i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^\gamma} = -2^{-\gamma}\eta(\gamma) \pm i\beta(\gamma),$$

which yields the second identity in (10) in view of (9). □

If the parameter γ is an even integer, the zeta function and thus also the eta and lambda functions (see p. 125 of [**1**]) may be stated in closed form in terms of the Bernoulli numbers B_n, while for γ being an odd integer, the beta function is linked to the Euler numbers E_n. This is due to the two famous identities already established by Euler in 1735 and 1747,

$$\zeta(2m) = (-1)^{m+1}(2\pi)^{2m} \frac{B_{2m}}{2(2m)!} \quad (m \in \mathbb{N}), \tag{11}$$

$$\beta(2m+1) = (-1)^m \left(\frac{\pi}{2}\right)^{2m+1} \frac{E_{2m}}{2(2m)!} \quad (m \in \mathbb{N}). \tag{12}$$

Here the Bernoulli and Euler numbers are the particular values $B_n \equiv B_n(0)$ and $E_n \equiv 2^n E_n(1/2)$ of the Bernoulli and Euler polynomials, $B_n(x)$ and $E_n(x)$, which are usually defined via their exponential generating functions

$$\frac{we^{xw}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{w^n}{n!} \quad (|w| < 2\pi) \tag{13}$$

$$\frac{2e^{xw}}{e^w + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{w^n}{n!} \quad (|w| < \pi) \tag{14}$$

for $w \in \mathbb{C}$ and $x \in \mathbb{R}$. By expanding the left-hand sides of (13) and (14) as products of two power series, it is easy to see that

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k, \quad E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k} 2^{k-n} \left(x - \frac{1}{2}\right)^k \quad (n \in \mathbb{N}_0).$$

In particular, when inserting the first non-vanishing values of the Bernoulli and Euler numbers

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{2\cdot 3\cdot 5},\ B_6 = \frac{1}{2\cdot 3\cdot 7},\ B_8 = -\frac{3}{2\cdot 5\cdot 9},\ B_{10} = \frac{5}{2\cdot 3\cdot 11},$$
$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385,$$

into the formulas (11) and (12), it follows that

$$\zeta(2)=\frac{2^0}{3!}\pi^2, \zeta(4)=\frac{1}{3}\cdot\frac{2^2}{5!}\pi^4, \zeta(6)=\frac{1}{3}\cdot\frac{2^4}{7!}\pi^6, \zeta(8)=\frac{3}{5}\cdot\frac{2^6}{9!}\pi^8, \zeta(10)=\frac{5}{3}\cdot\frac{2^8}{11!}\pi^{10},$$
$$\beta(1)=\frac{1}{2^2}\pi, \beta(3)=\frac{1}{2^4 2!}\pi^3, \beta(5)=\frac{5}{2^6 4!}\pi^5, \beta(7)=\frac{61}{2^8 6!}\pi^7, \beta(9)=\frac{1385}{2^{10}8!}\pi^9.$$

Already in 1650, Mangoli [**26**] raised the question of evaluating $\zeta(2m+1)$. Some numerical approximation and computational results of $\zeta(2m+1)$ can be found in [**14, 15**] and their references. The Mangoli's problem was successfully attached by the first-named author in joint work with M. Hauss and M. Leclerc [**6**]. In order to find an appropriate counterpart of identity (11) for odd integer values, they first extended the classical Fourier series of the Bernoulli polynomials to Bernoulli functions with complex index $\gamma \in \mathbb{C}$, $\mathrm{Re}\gamma > 1$,

$$B_\gamma(x) = -2\Gamma(\gamma+1)\sum_{k=1}^{\infty}\frac{\cos(2\pi kx-\gamma\pi/2)}{(2\pi k)^\gamma} \quad (0\le x<1) \tag{15}$$

and then defined the conjugate Bernoulli functions as their Hilbert transforms

$$B_\gamma^\sim(x) = -2\Gamma(\gamma+1)\sum_{k=1}^{\infty}\frac{\sin(2\pi kx-\gamma\pi/2)}{(2\pi k)^\gamma} \quad (0\le x<1) \tag{16}$$

This led to a far-reaching generalization of identity (11) involving the Bernoulli numbers $B_\gamma := B_\gamma(0)$ and their conjugates $B_\gamma^\sim := B_\gamma^\sim(0)$ with complex order γ (see Proposition 3.5). In particular, they arrived at

$$\zeta(2m+1) = (-1)^m(2\pi)^{2m}\frac{B_{2m+1}^\sim}{2(2m+1)!} \quad (m\in N). \tag{17}$$

For a detailed examination of the values $B_{2m+1}^\sim$ see [**2**].

Shortly afterwards, a similar approach was used in [**3**] to generalize the representation (12) of the beta function at odd integers to complex order values. Here, the final result (see Proposition 3.6) is based on defining the Euler functions and their conjugates for $\gamma\in\mathbb{C}$, $\mathrm{Re}\gamma > -1$. On the interval $0 \le x < 1$, their Fourier series read

$$E_\gamma(x) = 4\Gamma(\gamma+1)\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x-\gamma\pi/2)}{((2k+1)\pi)^{\gamma+1}}, \tag{18}$$

$$E_\gamma^\sim(x) = -4\Gamma(\gamma+1)\sum_{k=1}^{\infty}\frac{\cos((2k+1)\pi x-\gamma\pi/2)}{((2k+1)\pi)^{\gamma+1}}. \tag{19}$$

Now define the generalized Euler numbers by $E_\gamma := 2^\gamma E_\gamma(1/2)$ and their conjugates by $E_\gamma^\sim := 2^\gamma E_\gamma^\sim(1/2)$. Then the latter quantities yield

$$\beta(2m) = (-1)^m \left(\frac{\pi}{2}\right)^{2m} \frac{E_{2m-1}^\sim}{2(2m-1)!} \quad (m \in \mathbb{N}). \tag{20}$$

Finally we notice that in view of (15) and (18), and even for $\gamma \in \mathbb{C}\backslash\mathbb{Z}^+, x \in \mathbb{C}\backslash\mathbb{R}_0^-$ (cf. Theorem 3.7 of [**3**]), the Euler functions can be expressed in terms of the Bernoulli functions by

$$E_\gamma(x) = \frac{2^{\gamma+1}}{\gamma+1}\left\{B_{\gamma+1}\left(\frac{x+1}{2}\right) - B_{\gamma+1}\left(\frac{x}{2}\right)\right\} \quad (0 \le x < 1).$$

Analogously, it follows from (16) and (19) that the conjugates of the Euler and Bernoulli functions are related to each other by

$$E_\gamma^\sim(x) = \frac{2^{\gamma+1}}{\gamma+1}\left\{B_{\gamma+1}^\sim\left(\frac{x+1}{2}\right) - B_{\gamma+1}^\sim\left(\frac{x}{2}\right)\right\} \quad (0 \le x < 1).$$

Consequently, the generalized Euler numbers and their conjugates satisfy

$$\begin{aligned} E_{\gamma-1} &= 2^{\gamma-1}E_{\gamma-1}\left(\frac{1}{2}\right) = \frac{2^{2\gamma-1}}{\gamma}\left\{B_\gamma\left(\frac{3}{4}\right) - B_\gamma\left(\frac{1}{4}\right)\right\}, \\ E_{\gamma-1}^\sim &= 2^{\gamma-1}E_{\gamma-1}^\sim\left(\frac{1}{2}\right) = \frac{2^{2\gamma-1}}{\gamma}\left\{B_\gamma^\sim\left(\frac{3}{4}\right) - B_\gamma^\sim\left(\frac{1}{4}\right)\right\}. \end{aligned} \tag{21}$$

In Section 3, we use Lemma 1 together with the relationships between the zeta and beta functions, on the one hand, and the generalized Bernoulli and Euler numbers and their conjugates, on the other hand, to establish further representations of the Li-function at the values $\pm i$.

2. EULERIAN FUNCTIONS AND FRACTIONS, STIRLING FUNCTIONS AND THE POLYLOGARITHM

In order to generalize the classical Eulerian fractions to complex order parameters, we can use mixed techniques shown in [**4, 5, 11, 9, 19, 20, 22, 21**]. In [**5**], the Eulerian numbers (1.1) and the (slightly modified) Eulerian polynomials (1.2) have been extended already to the fractional situation

(22)

$$E(\gamma,k) = \sum_{j=0}^{k}(-1)^j\binom{\gamma+1}{j}(k+1-j)^\gamma, \quad A_\gamma(z) = \sum_{k=0}^{\infty}E(\gamma,k)z^k \quad (\gamma \in \mathbb{R}, |z| < 1).$$

But the functions $A_\gamma(z)$ are also valid for complex values of the order parameter γ. In fact, the generalized binomial coefficient in the definition of $E(\gamma,k)$ can be estimated by (cf. (1.51) of Samko, Kilbas, and Marichev [**30**])

$$\left|\binom{\gamma+1}{j}\right| := \frac{|(\gamma+1)\gamma\ldots(\gamma-j+2)|}{j!} \le \frac{A}{j^{\operatorname{Re}\gamma+2}} \quad (\gamma\in\mathbb{C}, \gamma\neq -2,-3,\ldots)$$

with some constant $A>0$, and

$$|(k+1-j)^\gamma| = |(k+1-j)^{\operatorname{Re}\gamma} e^{i\operatorname{Im}\gamma\log(k+1-j)}| = (k+1-j)^{\operatorname{Re}\gamma}, \quad 0\le j\le k.$$

Substituting now $E(\gamma,k)$ into the power series of $A_\gamma(z)$, interchanging the order of summation and using the generalized binomial theorem, we obtain

$$\begin{aligned} A_\gamma(z) &= \sum_{k=0}^\infty\sum_{j=0}^k (-1)^j\binom{\gamma+1}{j}(k+1-j)^\gamma z^k = \sum_{j=0}^\infty\sum_{k=j}^\infty\cdots \\ &= \sum_{j=0}^\infty(-1)^j\binom{\gamma+1}{j}z^j\sum_{k=0}^\infty(k+1)^\gamma z^k \\ &= (1-z)^{\gamma+1}\sum_{k=0}^\infty(k+1)^\gamma z^k. \end{aligned}$$

Since both sums in the second line are convergent within the unit disk $|z|<1$ for any $\gamma\in\mathbb{C}$, we can even extend the parameter γ of the Eulerian function $A_\gamma(z)$ to the whole complex plane. This enables us to generalize the classical identity (4) as follows.

Definition 2. *For $\gamma\in\mathbb{C}$, the generalized Eulerian function is defined by*

$$(23)\qquad A_\gamma(z) = (1-z)^{\gamma+1}a_\gamma(z), \quad a_\gamma(z) = \sum_{k=0}^\infty (k+1)^\gamma z^k \qquad (|z|<1, z\in\mathbb{C}).$$

Here, the function $a_\gamma(z)$ is called the generalized Eulerian fraction. In terms of the Lerch transcendent (2), *it is given by $\alpha_\gamma(z) = \Psi(z,-\gamma,1)$.*

By definition of $\alpha_\gamma(z)$, we readily obtain the following statements.

Corollary 3. *Let $\gamma\in\mathbb{C}$. The generalized Eulerian fraction satisfies the differentiation formula*

$$\frac{d}{dz}\left[za_\gamma(z)\right] = a_{\gamma+1}(z) \quad (|z|<1, z\in\mathbb{C}).$$

Corollary 4. *Let $\gamma \in \mathbb{C}$. The polylogarithm* (1) *is related to the generalized Eulerian fraction and thus to the generalized Eulerian numbers by*

$$
\begin{aligned}
Li_\gamma(z) :=& za_{-\gamma}(z) = z(1-z)^{\gamma-1}A_{-\gamma}(z) \\
=& (1-z)^{\gamma-1}\sum_{k=0}^{\infty} E(-\gamma,k)z^{k+1} \quad (|z|<1, z\in C).
\end{aligned} \tag{24}
$$

Applying again the generalized binomial theorem, the latter expression is equivalent to

$$
\begin{aligned}
Li_\gamma(z) =& \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{\gamma-1}{j}E(-\gamma,k)z^{j+k+1} \\
=& \sum_{j=0}^{\infty}\left\{\sum_{k=0}^{j}(-1)^{j-k}\binom{\gamma-1}{j-k}E(-\gamma,k)\right\}z^{j+1} \quad (|z|<1, z\in\mathbb{C}).
\end{aligned} \tag{25}
$$

Remark 5. *Alternatively, we obtain identity* (25) *by observing that for all $j \in \mathbb{N}_0$, the inner sum is equal to*

$$
\begin{aligned}
&\sum_{k=0}^{j}(-1)^{j-k}\binom{\gamma-1}{j-k}\sum_{n=0}^{k}(-1)^{k-n}\binom{1-\gamma}{k-n}(n+1)^{-\gamma} \\
&= \sum_{n=0}^{j}(-1)^{j-n}(n+1)^{-\gamma}\sum_{k=n}^{j}\binom{\gamma-1}{j-k}\binom{1-\gamma}{k-n} \\
&= \sum_{n=0}^{j}(-1)^{j-n}(n+1)^{-\gamma}\delta_{j-n,0} = (j+1)^{-\gamma}.
\end{aligned}
$$

If $\gamma = -n, n \in \mathbb{N}$, the sum in the second identity of (24) terminates since $E(n,k) = 0, k \geq n$. As was shown already by Wood [**34**], the identity (24) then reduces to

$$
Li_{-n}(z) = (1-z)^{-n-1}\sum_{k=1}^{n} E(n,k-1)z^k.
$$

In particular, one has

$$
\begin{aligned}
Li_0(z) =& za_0(z) = \frac{z}{1-z}, \\
Li_{-1}(z) =& za_1(z) = z\frac{d}{dz}\left[\frac{1}{1-z}\right] = \frac{z}{(1-z)^2}, \\
Li_{-2}(z) =& za_2(z) = z\frac{d}{dz}\left[za_1(z)\right] = \frac{z(1+z)}{(1-z)^3}.
\end{aligned}
$$

Theorem 6. *For $\gamma \in \mathbb{C}$, $Re\,\gamma > 0$, $\mu \in [0,1)$, $|z| < 1$, and $|t| < 1$, there holds*

$$\sum_{n=0}^{\infty} \alpha_{\gamma+n\mu}(z)\frac{t^n}{n!} = \sum_{k=0}^{\infty}(k+1)^{\gamma} z^k \exp((k+1)^{\mu} t). \tag{26}$$

In particular, if $\gamma = 0$ and $\mu \to 1$, one arrives at the generating function of the classical Eulerian fractions defined in (4), namely

$$\sum_{n=0}^{\infty} \alpha_n(z)\frac{t^n}{n!} = \frac{e^t}{1-e^t z}.$$

Proof. Using the definition of the generalized Eulerian fractions on the left-hand side of identity (26) and interchanging the order of summation in the resulting double sum, we immediately arrive at the right-hand side of (26). □

We now present the connection of the Eulerian fractions with the numbers $S(\gamma, k), \gamma \in \mathbb{C}$, that generalize the classical Stirling numbers defined in (6).

Definition 7. *((7.1) of [**8**]) The generalized Stirling numbers of the second kind with complex parameter $\gamma \in \mathbb{C}$ are defined by*

$$S(\gamma, k) := \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j} j^{\gamma} \quad (\gamma \neq 0, k \in \mathbb{N}), \tag{27}$$

while $S(0,0) = 1$, $S(0,k) = 0$, $k \in \mathbb{N}$, and $S(\gamma, 0) = 0$ $(\gamma \neq 0)$.

It is not surprising, that these numbers naturally arise in this context: As was shown already for positive real γ in Theorem 6 of [**5**], the generalized Stirling numbers are mutually related to the Eulerian numbers $E(\gamma, k)$ by two summation formulas, which clearly hold for any $\gamma \in \mathbb{C}$ and $k \in \mathbb{N}_0$

$$\begin{aligned} E(\gamma, k) &= \sum_{j=0}^{k}(-1)^{k-j}\binom{\gamma - j - 1}{k-j}(j+1)!S(\gamma, j+1) \\ &= \sum_{j=0}^{k}\binom{k-\gamma}{k-j}(j+1)!S(\gamma, j+1) \end{aligned} \tag{28}$$

$$(k+1)!S(\gamma, k+1) = \sum_{j=0}^{k}\binom{\gamma - j - 1}{k-j}E(\gamma, j)$$

Notice that the second line in (28) follows since $(-1)^m\binom{b}{m} = \binom{m-b-1}{m}$, $b \in \mathbb{C}, m \in \mathbb{N}_0$.

Theorem 8. *For $\gamma \in \mathbb{C}$, $\gamma \neq 0$, let $\alpha_\gamma(z)$ be the generalized Eulerian fraction* (23), *and $S(\gamma, k)$, the generalized Stirling numbers of the second kind defined in* (27). *Then*

$$\alpha_\gamma(z) = \sum_{k=1}^{\infty} k! S(\gamma, k) \frac{z^{k-1}}{(1-z)^{k+1}} \qquad (|z| < 1, \operatorname{Re} z < \frac{1}{2}). \tag{29}$$

In particular, if γ is replaced by $n \in \mathbb{N}$, identity (29) *reduces to* (6).

Proof. Substituting the definition (27) of $S(\gamma, k)$ on the right-hand side of (29), and interchanging the order of the double sum, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^\gamma \right\} \frac{z^{k-1}}{(1-z)^{k+1}} = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} \cdots \\
&= \sum_{j=1}^{\infty} \frac{j^\gamma z^{j-1}}{(1-z)^{j+1}} \sum_{k=0}^{\infty} (-1)^k \binom{k+j}{j} \frac{z^k}{(1-z)^k} = \sum_{j=1}^{\infty} j^\gamma z^{j-1} \equiv \alpha_\gamma.
\end{aligned}$$

Here we used the generalized binomial theorem to evaluate the inner sum over k as

$$\sum_{k=0}^{\infty} \binom{k+j}{k} \left(\frac{z}{z-1} \right)^k = \left(1 - \frac{z}{z-1} \right)^{-j-1} = (1-z)^{j+1} \qquad \left(\left| \frac{z}{z-1} \right| < 1 \right).$$

If $\gamma = n \in \mathbb{N}$, the sum in (29) is finite since $S(n, k) = 0$ for each $k > n$. This yields identity (6). $\square$

Corollary 9. *Let $\gamma \in \mathbb{C}.\gamma \neq 0$. In terms of the generalized Stirling numbers of the second kind, the polylogarithm* (1) *is given by*

$$Li_\gamma(z) := z a_{-\gamma}(z) = \sum_{k=1}^{\infty} k! S(-\gamma, k) \frac{z^k}{(1-z)^{k+1}} \qquad (|z| < 1, \operatorname{Re} z < \frac{1}{2}). \tag{30}$$

From (30) *it follows that*

$$\begin{aligned}
Li_\gamma(z) &= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \binom{j+k}{j} k! S(-\gamma, k) z^{j+k} \\
&= \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{j} \binom{j}{j-k} k! S(-\gamma, k) \right\} z^j \qquad (|z| < 1, z \in \mathbb{C}).
\end{aligned} \tag{31}$$

Notice that the identity (31) can be proved directly by evaluating the inner sum. In fact, since

$$\binom{j}{j-k} \binom{k}{n} = \binom{j}{n} \binom{j-n}{k-n}, \quad 0 \le n \le k \le j < \infty,$$

we arrive at

$$\sum_{k=1}^{j}\binom{j}{j-k}\sum_{n=1}^{k}(-1)^{k-n}\binom{k}{n}n^{-\gamma}=\sum_{n=1}^{j}\binom{j}{n}n^{-\gamma}\sum_{k=n}^{j}(-1)^{k-n}\binom{j-n}{k-n}$$
$$=\sum_{n=1}^{j}\binom{j}{n}n^{-\gamma}\delta_{j-n,0}=j^{-\gamma}.$$

We close this section by introducing an even more general type of Eulerian fraction. To this end we replace the right-hand side of (29) by a sum containing the generalized Stirling numbers $S(\gamma,k)$ and, in addition, some formal power series $B(z), |z|<1$. Roughly speaking, we use a symbolic operator approach which, in a wider sense, proceeds from the generating function of the generalized Sheffer polynomials (cf. [**19**]).

Definition 10. *Let $B(z)$ be a formal power series convergent in $|z|<1$ with $B(0)=1$, and let the generalized Stirling numbers $S(\gamma,k)$ be defined as in* (27)*. In terms of $B(z)$, the generalized Eulerian fraction of type-one with order $\gamma\in\mathbb{C},\gamma\neq 0$, is defined by*

$$\alpha_\gamma(z;B(z)):=\sum_{k=1}^{\infty}S(\gamma,k)B^{(k)}(z)z^{k-1}\qquad(|z|<1).$$

If $B(z)=\sum_{j=0}^{\infty}z^j=(1-z)^{-1}$, the function $\alpha_\gamma(z;B)$ coincides with the original $\alpha_\gamma(z)$ in (29), since $B^{(k)}(z)=k!(1-z)^{-k-1}, k\in\mathbb{N}$, We are particularly interested in the two situations

$$B_1(z,a)=(1+z)^a,\quad B_2(z,a)=(1-z)^{-a-1},\quad a\in\mathbb{R},\tag{32}$$

which give rise to the following kinds of generalized Eulerian fractions of type-one.

Proposition 11. *For $a\in\mathbb{R}$ we have*

$$\alpha_\gamma(z;(1+z)^a)=\sum_{k=1}^{\infty}\binom{a}{k}\frac{k!S(\gamma,k)z^{k-1}}{(1+z)^{k-a}}\qquad(|z|<1),\tag{33}$$

$$\alpha_\gamma(z;(1+z)^{-a-1})=\sum_{k=1}^{\infty}\binom{a+k}{k}\frac{k!S(\gamma,k)z^{k-1}}{(1-z)^{a+k+1}}\qquad(|z|<1,\operatorname{Re}z<\frac{1}{2}).\tag{34}$$

Proof. Denoting the falling and rising factorials, respectively, by

$$[a]_k:=a(a-1)\dots(a-k+1)=\binom{a}{k}k!,$$

$$(a+1)_k := (a+1)(a+2)\dots(a+k) = \binom{a+k}{k} k!,$$

the identities (33) and (34) follow since $B_1^{(k)}(z,a) = [a]_k(1+z)^{a-k}, B_2^{(k)}(z,a) = (a+1)_k(1-z)^{-a-k-1}$. □

In the cases $\gamma = n \in \mathbb{N}$, the corresponding type-one Eulerian fractions (33)-(34) have been defined and treated in [**22**].

In the following, we focus on the choice $B(z) \equiv B_2(z,a)$ in (32).

Definition 12. *Let $\gamma \in \mathbb{C}, \gamma \neq 0,$ and let $\alpha_\gamma(z;(1-z)^{-a-1}), a \in \mathbb{R}$, be the generalized Eulerian fractions of type-one given in* (34)*. The corresponding generalized Eulerian functions and numbers of type-one are defined, respectively, by*

$$A_\gamma(z;(1-z)^{-a-1}) = (1-z)^{a+\gamma+1}\alpha_\gamma(z;(1-z)^{-a-1}) \quad (|z|<1), \tag{35}$$

$$E(\gamma,k;(1-z)^{-a-1}) = \left[z^k\right] A_\gamma(z;(1-z)^{-a-1}) \quad (k \in \mathbb{N}_0). \tag{36}$$

Theorem 13. *Following Definition 12, one has*

$$\begin{aligned} &A_\gamma(z;(1-z)^{-a-1}) \\ &= \sum_{k=1}^{\infty}(-1)^k z^{k-1}\left\{\sum_{j=1}^{k}(-1)^j\binom{a+j}{j}\binom{\gamma-j}{k-j} j!S(\gamma,j)\right\} \\ &= \sum_{k=1}^{\infty} z^{k-1}\left\{\sum_{j=1}^{k}\binom{a+j}{j}\binom{k-\gamma-1}{k-j} j!S(\gamma,j)\right\} \qquad (|z|<1), \end{aligned} \tag{37}$$

$$\begin{aligned} &E(\gamma,k;(1-z)^{-a-1}) \\ &= (-1)^{k+1}\sum_{j=1}^{k+1}(-1)^j\binom{a+j}{j}\binom{\gamma-j}{k-j+1} j!S(\gamma,j) \\ &= \sum_{j=1}^{k+1}\binom{a+j}{j}\binom{k-\gamma}{k-j+1} j!S(\gamma,j) \qquad (k \in \mathbb{N}_0). \end{aligned} \tag{38}$$

Proof. In view of (35) and (36)), we find that,

$$\begin{aligned} A_\gamma(z;(1-z)^{-a-1}) &= (1-z)^{a+\gamma+1}\alpha_\gamma(z;(1-z)^{-a-1}) \\ &= \sum_{j=1}^{\infty}\binom{a+j}{j}(1-z)^{\gamma-j} j!S(\gamma,j) z^{\gamma-1} \\ &= \sum_{j=1}^{\infty}\binom{a+j}{j}\sum_{k=0}^{\infty}(-1)^k\binom{\gamma-j}{k} j!S(\gamma,j) z^{k+j-1} \end{aligned}$$

$$= \sum_{j=1}^{\infty} \binom{a+j}{j} \sum_{k=j}^{\infty} (-1)^{k-j} \binom{\gamma - j}{k-j} j! S(\gamma, j) z^{k-1}$$
$$= \sum_{k=1}^{\infty} (-1)^k z^{k-1} \sum_{j=1}^{k} (-1)^j \binom{a+j}{j} \binom{\gamma - j}{k-j} j! S(\gamma, j).$$

The interchange of the order of the summation in the last step is clear. This implies both (37) and (38). The second identities in (37) and (38) just follow as in (28). □

Remark 14. *In particular, if $a = 0$, (37) reduces to*

$$A_\gamma(z; (1-t)^{-1}) = \sum_{k=1}^{\infty} (-1)^k z^{k-1} \left(\sum_{j=1}^{k} (-1)^j \binom{\gamma - j}{k-j} j! S(\gamma, j) \right)$$
$$= \sum_{k=1}^{\infty} z^{k-1} \left\{ \sum_{j=1}^{k} \binom{k-\gamma-1}{k-j} j! S(\gamma, j) \right\},$$

while (38) coincides with the identities stated in (28), namely

$$E(\gamma, k; (1-t)^{-1}) = \sum_{j=1}^{k+1} (-1)^{k+1-j} \binom{\gamma - j}{k+1-j} j! S(\gamma, j) = \sum_{j=1}^{k+1} \binom{k-\gamma}{k+1-j} j! S(\gamma, j).$$

Substituting (27) into (35) and (36) and taking into account that

$$\sum_{j=n}^{k} \binom{a+j}{j} \binom{\gamma - j}{k-j} \binom{j}{n} = \binom{a+n}{n} \sum_{j=n}^{k} \binom{a+j}{j-n} \binom{\gamma - j}{k-j} = \binom{a+n}{n} \binom{\gamma + a + 1}{k-n},$$

we obtain the following results.

Corollary 15. *Let $A_\gamma(z; (1-z)^{-a-1})$ and $E(\gamma, k; (1-z)^{-a-1})$ ($\gamma \in \mathbb{C}, \gamma \neq 0, k \in \mathbb{N}_0$) be defined by (35) and (36), respectively. Then*

$$A_\gamma(z; (1-z)^{-a-1}) = \sum_{k=1}^{\infty} (-1)^k z^{k-1} \sum_{n=1}^{k} n^\gamma (-1)^n \binom{a+n}{n} \binom{\gamma + a + 1}{k-n}$$
$$= \sum_{k=1}^{\infty} z^{k-1} \sum_{n=1}^{k} n^\gamma \binom{a+n}{n} \binom{k-n-\gamma-a-2}{k-n} \qquad (|z| < 1),$$

$$E(\gamma, k; (1-z)^{-a-1}) = (-1)^{k+1} \sum_{n=1}^{k+1} n^\gamma (-1)^n \binom{a+n}{n} \binom{\gamma + a + 1}{k-n+1}$$
$$= \sum_{n=1}^{k+1} n^\gamma \binom{a+n}{n} \binom{k-n-\gamma-a-1}{k-n+1} \qquad (k \in \mathbb{N}_0).$$

3. REPRESENTATIONS OF THE POLYLOGARITHM FUNCTION WITH IMAGINARY ARGUMENT

As far as we know, there are only eight values of z for which $Li_2(z)$ can be given in closed form. Setting $\omega := (1+\sqrt{5})/2$, these are [**27**]

$$
\begin{aligned}
&Li_2(0) = 0, \quad Li_2(1) = \frac{\pi^2}{6}, \quad Li_2(-1) = -\frac{\pi^2}{12}, \quad Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2,\\
&Li_2(-\omega) = -\frac{\pi^2}{10} + \frac{1}{2}\log^2 \omega, \quad Li_2(1-\omega) = \frac{\pi^2}{15} + \frac{1}{2}\log^2 \omega,\\
&Li_2(\omega - 1) = \frac{\pi^2}{10} - \log^2 \omega, \quad Li_2(2-\omega) = \frac{\pi^2}{15} - \log^2 \omega.
\end{aligned}
$$

The only known full values for $z = \frac{1}{2}$ seem to be

$$
Li_1\left(\frac{1}{2}\right) = \log 2,\ Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2,\ Li_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log 2 + \frac{1}{6}\log^3 2, \ldots
$$

In the following, we apply the various results of the preceding sections to represent the values $Li_\gamma(\pm i)$ with complex order parameter γ. Throughout we assume that $\operatorname{Re}\gamma > 1$, which guarantees that the fundamental relationship $Li_\gamma(z) = z\alpha_{-\gamma}(z), |z| < 1$, can be continued to $z = \pm i$, analytically (cf. (5), Definition 2, and the proof of Lemma 1). Moreover, it suffices to treat the case $z = i$. The results for $Li_\gamma(-i)$ then follow analogously.

To begin with, we employ the identity (25) to express $Li_\gamma(i)$ in terms of the generalized Eulerian numbers $E(-\gamma, k)$.

Theorem 16. *For $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > 1$, let $E(-\gamma, k)$ be given as in (22). Then*

$$
\begin{aligned}
Li_\gamma(i) &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{2n+1}(-1)^{n-k}\binom{\gamma-1}{2n+1-k}E(-\gamma,k)\right\}\\
&+ i\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{2n}(-1)^{n-k}\binom{\gamma-1}{2n-k}E(-\gamma,k)\right\}\\
&=: I_1 + iI_2,
\end{aligned}
\tag{39}
$$

where

$$\begin{aligned}I_1 = \sum_{n=0}^{\infty}\Bigg\{&\sum_{k=0}^{2n+1}(-1)^{n-k}\binom{\gamma-1}{2n+1-k}\operatorname{Re}E(-\gamma,k)\\&-\sum_{k=0}^{2n}(-1)^{n-k}\binom{\gamma-1}{2n-k}\operatorname{Im}E(-\gamma,k)\Bigg\},\\I_2 = \sum_{n=0}^{\infty}\Bigg\{&\sum_{k=0}^{2n}(-1)^{n-k}\binom{\gamma-1}{2n-k}\operatorname{Re}E(-\gamma,k)\\&+\sum_{k=0}^{2n+1}(-1)^{n-k}\binom{\gamma-1}{2n+1-k}\operatorname{Im}E(-\gamma,k)\Bigg\}.\end{aligned}$$

Similarly, the identity (29) leads to a counterpart of Theorem 16 in terms of the generalized Stirling numbers of the second kind $S(-\gamma,k)$.

Theorem 17. *For $\gamma\in\mathbb{C}$, $\operatorname{Re}\gamma>1$, let $S(-\gamma,k)$ be given via Definition 7. Then*

$$\begin{aligned}Li_\gamma(i) &= \sum_{n=1}^{\infty}(-1)^n\left\{\sum_{k=1}^{2n}\binom{2n}{k}k!S(-\gamma,k)\right\}\\&\quad + i\sum_{n=0}^{\infty}(-1)^n\left\{\sum_{k=1}^{2n+1}\binom{2n+1}{k}k!S(-\gamma,k)\right\}\\&=: J_1+iJ_2,\end{aligned}\tag{40}$$

where

$$J_1 = \sum_{n=1}^{\infty}(-1)^n\left\{\sum_{k=1}^{2n}\binom{2n}{k}k!\operatorname{Re}S(-\gamma,k)-\sum_{k=1}^{2n+1}\binom{2n+1}{k}k!\operatorname{Im}S(-\gamma,k)\right\},$$
$$J_2 = \sum_{n=1}^{\infty}(-1)^n\left\{\sum_{k=1}^{2n+1}\binom{2n+1}{k}k!\operatorname{Re}S(-\gamma,k)+\sum_{k=1}^{2n}\binom{2n}{k}k!\operatorname{Im}S(-\gamma,k)\right\}.$$

If $\gamma\in\mathbb{R}$, both double sums in the representations (39) and (40) are real, so that the two Theorems 16 and 17, when combined with the relationship (10) in Lemma 1, give rise to the following representations of the zeta and beta functions.

Corollary 18. *Let $\gamma\in\mathbb{R},\gamma>1$, and $c_\gamma := -2^\gamma(1-2^{1-\gamma})^{-1}$. Then*

$$\begin{aligned}\zeta(\gamma) := c_\gamma\operatorname{Re}\,[Li_\gamma(i)] &= c_\gamma\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{2n+1}(-1)^{n+1-k}\binom{\gamma-1}{2n+1-k}E(-\gamma,k)\right\}\\&= c_\gamma\sum_{n=0}^{\infty}(-1)^{n+1}\left\{\sum_{k=1}^{2n}\binom{2n}{k}k!S(-\gamma,k)\right\}\end{aligned}\tag{41}$$

and

$$\beta(\gamma) := \operatorname{Im}\left[Li_\gamma(i)\right] = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{2n}(-1)^{n-k}\binom{\gamma-1}{2n-k}E(-\gamma,k)\right\}$$
$$= \sum_{n=0}^{\infty}(-1)^n\left\{\sum_{k=1}^{2n+1}\binom{2n+1}{k}k!S(-\gamma,k)\right\}. \tag{42}$$

If γ is real, we can also proceed directly from identity (24) *to achieve*

$$Li_\gamma(i) = (1-i)^{\gamma-1}\sum_{k=0}^{\infty}E(-\gamma,k)i^{k+1}$$
$$= 2^{(\gamma-1)/2}\left\{\cos\frac{(\gamma-1)\pi}{4} - i\sin\frac{(\gamma-1)\pi}{4}\right\}$$
$$\left\{\sum_{n=0}^{\infty}(-1)^{n+1}E(-\gamma,2n+1) + i\sum_{n=0}^{\infty}(-1)^nE(-\gamma,2n)\right\}.$$

Alternatively to the first lines of (41) *and* (42)*, this yields the following expansions.*

Corollary 19. *Let $\gamma\in\mathbb{R}, \gamma>1$, and $c_\gamma := -2^\gamma(1-2^{1-\gamma})^{-1}$. Then*

$$\zeta(\gamma) = c_\gamma \operatorname{Re}\left[Li_\gamma(i)\right]$$
$$= c_\gamma 2^{(\gamma-1)/2}\sum_{n=0}^{\infty}(-1)^n\left\{\sin\frac{(\gamma-1)\pi}{4}E(-\gamma,2n) - \cos\frac{(\gamma-1)\pi}{4}E(-\gamma,2n+1)\right\},$$
$$\beta(\gamma) = \operatorname{Im}\left[Li_\gamma(i)\right]$$
$$= 2^{(\gamma-1)/2}\sum_{n=0}^{\infty}(-1)^n\left\{\cos\frac{(\gamma-1)\pi}{4}E(-\gamma,2n) + \sin\frac{(\gamma-1)\pi}{4}E(-\gamma,2n+1)\right\}.$$

Next we express the occurring zeta and beta functions in terms of the generalized Bernoulli and Euler functions and their conjugates, which were defined in (15), (16) and (18), (19), respectively.

Proposition 20. (Theorem 5.1 a), b) of [**2**]). *Let $\gamma\in\mathbb{C}, \operatorname{Re}\gamma>1$. There hold*

$$\text{a)}\qquad \zeta(\gamma) = -\sec\left(\frac{\gamma\pi}{2}\right)\frac{(2\pi)^\gamma B_\gamma}{2\Gamma(\gamma+1)} \quad (\gamma\neq 2m+1, m\in\mathbb{N}), \tag{43}$$

$$\text{b)}\qquad \zeta(\gamma) = \csc\left(\frac{\gamma\pi}{2}\right)\frac{(2\pi)^\gamma B_\gamma^\sim}{2\Gamma(\gamma+1)} \quad (\gamma\neq 2m, m\in\mathbb{N}). \tag{44}$$

By choosing $\gamma = 2m$ in identity (43)*, Euler's classical formula* (11) *is recovered, while in the case $\gamma = 2m+1$ identity* (44) *reduces to the formula* (17).

Proposition 21. *Let $\gamma\in\mathbb{C}, \operatorname{Re}\gamma>1$. There hold*

$$\text{a)}\qquad \beta(\gamma) = \csc\left(\frac{\gamma\pi}{2}\right)\left(\frac{\pi}{2}\right)^\gamma\frac{E_{\gamma-1}}{2\Gamma(\gamma)} \quad (\gamma\neq 2m,\ m\in\mathbb{N}), \tag{45}$$

(46) *b)* $$\beta(\gamma) = \sec\left(\frac{\gamma\pi}{2}\right)\left(\frac{\pi}{2}\right)^{\gamma}\frac{E^{\sim}_{\gamma-1}}{2\Gamma(\gamma)} \quad (\gamma \neq 2m+1,\, m \in \mathbb{N}).$$

Choosing $\gamma = 2m+1$ *in identity* (45) *yields Euler's classical formula* (12)*, while formula* (20) *is the particular case* $\gamma = 2m$ *of identity* (46)*.*

Proof. Part a) is established in Theorem 3.12 of [**3**]. Concerning part b), it follows from Lemma 2.4 of [**3**] that

$$\begin{aligned} E^{\sim}_{\gamma-1} =& 2^{\gamma-1}E^{\sim}_{\gamma-1}\left(\frac{1}{2}\right) \\ =& 2^{\gamma}\Gamma(\gamma)\left\{\cos\frac{\gamma\pi}{2}\psi_{\gamma-1}\left(\frac{1}{2}\right) - \sin\frac{\gamma\pi}{2}\phi_{\gamma-1}\left(\frac{1}{2}\right)\right\}, \end{aligned}$$

where $\psi_{\gamma-1}$ and $\phi_{\gamma-1}$ take the values (cf. Proposition 2.3 of [**3**])

$$\begin{aligned} \psi_{\gamma-1}\left(\frac{1}{2}\right) :=& \frac{2}{\pi^{\gamma}}\sum_{k\geq 0}\frac{\sin((2k+1)\pi/2)}{(2k+1)^{\gamma}} \\ =& \frac{2}{\pi^{\gamma}}\sum_{k\geq 0}\frac{(-1)^k}{(2k+1)^{\gamma}} = \frac{2}{\pi^{\gamma}}\beta(\gamma), \\ \phi_{\gamma-1}\left(\frac{1}{2}\right) :=& \frac{2}{\pi^{\gamma}}\sum_{k\geq 0}\frac{\cos((2k+1)\pi/2)}{(2k+1)^{\gamma}} = 0. \end{aligned}$$

Hence, $E^{\sim}_{\gamma-1} = 2^{\gamma}\Gamma(\gamma)\cos\frac{\gamma\pi}{2}\frac{2}{\pi^{\gamma}}\beta(\gamma)$, which implies the identity (46). □

Finally, we apply the Propositions 20 and 21 to the relationship (10) to get the following representations of the polylogarithm at $z = i$.

Theorem 22. *Let* $\gamma \in \mathbb{C}, Re\,\gamma > 1$*. Then*

a) $$Li_{\gamma}(i) = \frac{\sec(\gamma\pi/2)}{2\Gamma(\gamma+1)}\left(\frac{\pi}{2}\right)^{\gamma}\left\{2(2^{\gamma-1}-1)B_{\gamma} + i\gamma E^{\sim}_{\gamma-1}\right\}$$

$(\gamma \neq 2m+1, m \in \mathbb{N})$,

b) $$Li_{\gamma}(i) = \frac{\csc(\gamma\pi/2)}{2\Gamma(\gamma+1)}\left(\frac{\pi}{2}\right)^{\gamma}\left\{2(1-2^{\gamma-1})B^{\sim}_{\gamma} + i\gamma E_{\gamma-1}\right\}$$

$(\gamma \neq 2m, m \in \mathbb{N})$,

c) $$Li_{2m}(i) = \frac{(-1)^m}{(2m)!}\left(\frac{\pi}{2}\right)^{2m}\left\{(2^{2m-1}-1)B_{2m} + imE^{\sim}_{2m-1}\right\}$$

$(m \in \mathbb{N})$,

d) $$Li_{2m+1}(i) = \frac{(-1)^m}{2(2m+1)!}\left(\frac{\pi}{2}\right)^{2m+1}\left\{2(1-2^{2m})B^{\sim}_{2m+1} + i(2m+1)E_{2m}\right\}$$

$(m \in \mathbb{N}_0)$.

Proof. For $\gamma \neq 2m+1, m \in \mathbb{N}$, we insert (43) and (46) into (10) to get

$$\begin{aligned} Li_\gamma(i) &= -2^{-\gamma}(1-2^{1-\gamma})\zeta(\gamma) + i\beta(\gamma) \\ &= 2^{-\gamma}(1-2^{1-\gamma})\sec\left(\frac{\gamma\pi}{2}\right)\frac{(2\pi)^\gamma B_\gamma}{2\Gamma(\gamma+1)} + i\sec\left(\frac{\gamma\pi}{2}\right)\left(\frac{\pi}{2}\right)^\gamma \frac{E^{\sim}_{\gamma-1}}{2\Gamma(\gamma)}. \end{aligned}$$

This yields part a) and, by substituting $\gamma = 2m, m \in \mathbb{N}$, also part c). Similarly, for $\gamma \neq 2m, m \in \mathbb{N}$, we use (44) and (45) to arrive at part b) and, if $\gamma = 2m+1, m \in \mathbb{N}_0$, at part d). □

In Proposition 21, the (generalized) Euler numbers and their conjugates can be substituted in view of the two identities in (21) as follows.

Corollary 23. *Let $\gamma \in \mathbb{C}, Re\,\gamma > 1$. There hold*

a) $$Li_\gamma(i) = \frac{\sec(\gamma\pi/2)}{\Gamma(\gamma+1)}\left(\frac{\pi}{2}\right)^\gamma \left\{(2^{\gamma-1}-1)B_\gamma + i2^{2\gamma-2}\left[B^{\sim}_\gamma\left(\frac{3}{4}\right) - B^{\sim}_\gamma\left(\frac{1}{4}\right)\right]\right\}$$

$(\gamma \neq 2m+1, m \in \mathbb{N})$,

b) $$Li_\gamma(i) = \frac{\csc(\gamma\pi/2)}{\Gamma(\gamma+1)}\left(\frac{\pi}{2}\right)^\gamma \left\{(1-2^{\gamma-1})B^{\sim}_\gamma + i2^{2\gamma-2}\left[B_\gamma\left(\frac{3}{4}\right) - B_\gamma\left(\frac{1}{4}\right)\right]\right\}$$

$(\gamma \neq 2m, m \in \mathbb{N})$.

Choosing $\gamma = 2m$ in part a) and $\gamma = 2m+1$ in part b), we immediately obtain counterparts of Theorem 22, parts c) and d), respectively.

Of course, the representations of $Li_\gamma(i)$ stated in Theorems 16, 17, and 22 can also be combined to end up with the following identities.

Corollary 24. *For $\gamma \in \mathbb{C}, Re\,\gamma > 1$, there holds*

$$\begin{aligned} &Li_\gamma(i) \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{2n+1}(-1)^{n-k}\binom{\gamma-1}{2n+1-k}E(-\gamma,k)\right\} \\ &+ i\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{2n}(-1)^{n-k}\binom{\gamma-1}{2n-k}E(-\gamma,k)\right\} \\ &= \sum_{n=1}^{\infty}(-1)^n\left\{\sum_{k=1}^{2n}\binom{2n}{k}k!S(-\gamma,k)\right\} + i\sum_{n=0}^{\infty}(-1)^n\left\{\sum_{k=1}^{2n+1}\binom{2n+1}{k}k!S(-\gamma,k)\right\} \\ &= \begin{cases} \dfrac{\pi^\gamma\sec[\gamma\pi/2]}{2^{\gamma+1}\Gamma(\gamma+1)}\left\{2(2^{\gamma-1}-1)B_\gamma + i\gamma E^{\sim}_{\gamma-1}\right\}, & \gamma \neq 2m+1, m \in \mathbb{N}, \\ \dfrac{\pi^\gamma\csc[\gamma\pi/2]}{2^{\gamma+1}\Gamma(\gamma+1)}\left\{2(1-2^{\gamma-1})B^{\sim}_\gamma + i\gamma E_{\gamma-1}\right\}, & \gamma \neq 2m, m \in \mathbb{N}. \end{cases} \end{aligned}$$

For $m \in \mathbb{N}$, the above identities reduce to

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{2n+1}(-1)^{n-k}\binom{2m-1}{2n+1-k}E(-2m,k)\right\}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\left\{\sum_{k=1}^{2n}\binom{2n}{k}k!S(-2m,k)\right\} = (-1)^m(2^{2m-1}-1)\left(\frac{\pi}{2}\right)^{2m}\frac{B_{2m}}{(2m)!},$$

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{2n}(-1)^{n-k}\binom{2m}{2n-k}E(-2m-1,k)\right\}$$

$$= \sum_{n=0}^{\infty}(-1)^{n}\left\{\sum_{k=1}^{2n+1}\binom{2n+1}{k}k!S(-2m-1,k)\right\} = (-1)^m\left(\frac{\pi}{2}\right)^{2m+1}\frac{E_{2m}}{2(2m)!}.$$

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