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GENERATING FUNCTIONS FOR GENERALIZATION SIMSEK NUMBERS AND THEIR APPLICATIONS

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Our purpose in this work is the complete the study of Simsek numbers. We give answer to some open problems concerning polynomial representations and associated generating function. At the end of the study we investigate a new generalization of these numbers and obtain useful identities which connect Simsek numbers and Stirling numbers of second kind.

1. INTRODUCTION

The Stirling numbers of the second kind $S_2(n,k)$ are given by means of the generating function

$$\frac{1}{k!}\left(e^t-1\right)^k=\sum_{n\geq 0}S_2(n,k)\frac{t^n}{n!},$$

It is obvious to prove that

$$S_2(n,k)=\frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}(-1)^{k-j}j^n.$$

Just writing

$$\frac{1}{k!}\left(e^t-1\right)^k=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}e^{jt}$$

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and then

$$\frac{1}{k!}\left(e^t-1\right)^k=\frac{1}{k!}\sum_{n\geq 0}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j^n\frac{t^n}{n!}.$$

$S_2(n,k)$ are special case of Bell polynomials of second kind $B_{n,k}$ so called exponential partial Bell polynomials (see [**1**]) these are defined by

$$\frac{1}{k!}\left(\sum_{m\geq 1}x_m\frac{t^m}{m!}\right)^k=\sum_{n\geq k}B_{n,k}(x_1,\cdots,x_{n-k+1})\frac{t^n}{n!}.$$

Let the multinomial coefficient

$$\binom{\alpha}{\widehat{k}}=\frac{(\alpha)_k}{k_1!\cdots k_n!},\ \text{with } k_1+\cdots+k_n=k$$

where $(\alpha)_k=\alpha(\alpha-1)\cdots(\alpha-k+1)$ is the falling number. Then the explicit formula of $B_{n,k}$ is

$$B_{n,k}(x_1,\cdots,x_{n-k+1})=\frac{n!}{k!}\sum_{s_n(k)}\binom{k}{\widehat{k}}\prod_{r=1}^{n-k+1}\left(\frac{x_r}{r!}\right)^{k_r},$$

where $s_n(k)$ is the set of all $(n-k+1)$-uplet $(k_1,\cdots,k_{n-k+1})$ satisfying the conditions $k_1+\cdots+k_{n-k+1}=k$ and $k_1+2k_2+\cdots+(n-k+1)\,k_{n-k+1}=n$. Some well-known explicit formulae of Bell polynomials (see [**1**]) are

$$B_{n,k}(1,\cdots,1)=S_2(n,k),$$

$$B_{n,k}(1!,\cdots,(n-k+1)!)=\binom{n-1}{k-1}\frac{n!}{k!}.$$

Let λ a complex number, we consider the numbers $B\left(n,k,\lambda\right)$ defined by the following generating function (see [**7**])

$$\left(\lambda e^t+1\right)^k=\sum_{n=0}^{\infty}B\left(n,k,\lambda\right)\frac{t^n}{n!},$$

where $y_1\left(n,k,\lambda\right)=\frac{1}{k!}B\left(n,k,\lambda\right)$ are Simsek numbers. Explicitly from [**7**, Theorem 1] we get

$$B\left(n,k,\lambda\right)=\sum_{j=0}^{k}\binom{k}{j}j^n\lambda^j.$$

For more informations about these numbers, we refer to works [**6, 8, 9**] and references theirs in. Let $B\left(n,k\right)=B\left(n,k,1\right)$, It is shown in the works [**10, 3**] that the conjecture of Y. Simsek (see [**7**]) is true. We remember that

$$B(n,k)=\left(k^n+x_1k^{n-1}+\cdots+x_{n-2}k^2+x_{n-1}k\right)2^{k-n}$$

where $x_1, \cdots, x_{n-1}$ are integers. But the following question still open
- We assume that for $|x| < r$

$$\sum_{k=0}^{\infty} B(n,k)x^k = f_n(x)$$

Is it possible to find $f_n(x)$?

In our recent work [**3**] we have extended the problem to numbers $B(n,k,\lambda)$ and provide a positive answer. Introducing the notion of generating function of functions, we have proved that the expression of the generating function

$$f_n(\lambda,x) = \sum_{k=0}^{\infty} B(n,k,\lambda)x^k$$

is given by successive derivatives of the function $\frac{1}{1-(\lambda e^t+1)x}$. More precisely we have

$$f_n(\lambda,x) = \frac{\partial^n}{\partial t^n}\frac{1}{1-(\lambda e^t+1)x}|_{t=0}.$$

For $\lambda = 1$; $B(n,k,1) = B(n,k)$ and $f_n(1,x) = f_n(x)$. In this case we have provided that $f_n(x)$ takes the rational form

$$f_n(x) = \frac{P_n(x)}{(1-2x)^{n+1}},$$

where $P_n(x)$ is a polynomial in $\mathbb{Z}[x]$ of degree less than n. In this paper we use techniques from advanced algebra where the composition law (see [**5**]) and the Cauchy product of functions play an important role, to give another polynomial representation of $B(n,k,\lambda)$ and compute explicitly the function $f_n(\lambda,x)$ and obtain the corresponding polynomial $P_{n,\lambda}(x)$ such that

$$f_n(\lambda,x) = \frac{P_{n,\lambda}(x)}{(1-(1+\lambda)x)^{n+1}}.$$

2. EXPLICIT FORMULA OF SIMSEK NUMBERS

The numbers $B(n,k,\lambda)$ are expressed on different ways as polynomials on k with coefficients depend on the variable k. In the work [**3**], we have two expressions of $B(n,k,\lambda)$. The first is

$$B(n,k,\lambda) = k! \sum_{j=1}^{n} a_{n,k}(\lambda, j) k^j,$$

with $a_{n,k}(\lambda, n) = \frac{(1+\lambda)^k}{k!}$. Then

$$B(n,k,\lambda) = \left(k^n + a_{n,k}(\lambda, n-1)k^{n-1} + \cdots + a_{n,k}(\lambda, 1)k\right)(1+\lambda)^k.$$

The seconde is

$$B(n,k,\lambda) = k!\,(\lambda+1)^{k-n} \sum_{j=1}^{n} x_j\,(\lambda, n, k)\, k^j.$$

All the coefficients $a_{n,k}(\lambda, j)$ and $x_j\,(\lambda, n, k)$ are given by recursive formulae. Let the auxiliary sum

$$S_{k,l}(a) = \sum_{i=0}^{k} \binom{k}{i} a^i (-i)^l.$$

One explicit formula of the numbers $B(n,k,\lambda)$ as a polynomial on k is established in the following theorem.

Theorem 1. *Let n be a nonnegative integers. Then we have*

$$B(n,k,\lambda) = \lambda^k \sum_{j=0}^{n} \binom{n}{j} S_{k,n-j}\left(\frac{1}{\lambda}\right) k^j.$$

Proof. The proof depends on the way we develop the expression $(\lambda e^t + 1)^k$ on combinatorial sum. We can write

$$\left(\lambda e^t + 1\right)^k = \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} e^{(k-i)t}.$$

But

$$e^{(k-i)t} = e^{kt} e^{-it} = \left(\sum_{n\geq 0} k^n \frac{t^n}{n!}\right)\left(\sum_{n\geq 0} (-i)^n \frac{t^n}{n!}\right).$$

Using Cauchy product of generating functions (see [**2**]) we will have

$$e^{(k-i)t} = \sum_{n\geq 0} \sum_{j=0}^{n} \binom{n}{j} k^{n-j} (-i)^j \frac{t^n}{n!}.$$

Then

$$\left(\lambda e^t+1\right)^k=\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i=0}^{k}\binom{k}{i}\lambda^{k-i}\binom{n}{j}k^{n-j}(-i)^j\frac{t^n}{n!}.$$

Finally

$$B(n,k,\lambda)=\sum_{j=0}^{n}\sum_{i=0}^{k}\binom{k}{i}\binom{n}{j}\lambda^{k-i}(-i)^j k^{n-j}$$

□

Remark 1. *Let*

$$y_j=\binom{n}{j}\lambda^k S_{k,j}\left(\frac{1}{\lambda}\right)$$

and

$$S_{k,l}(a)=\sum_{i=0}^{k}\binom{k}{i}a^i(-i)^l.$$

Then

$$B(n,k,\lambda)=\sum_{j=0}^{n}y_j k^{n-j}.$$

But

$$y_0=\lambda^k S_{k,0}\left(\frac{1}{\lambda}\right)=\lambda^k\sum_{i=0}^{k}\binom{k}{i}\left(\frac{1}{\lambda}\right)^i=\lambda^k\left(1+\frac{1}{\lambda}\right)^k=(\lambda+1)^k,$$

furthermore, if $\lambda+1\neq 0$ *we have*

$$B(n,k,\lambda)=\left(k^n+z_1k^{n-1}+\cdots+z_n\right)(\lambda+1)^k$$

with

$$z_j=\left(\frac{\lambda}{\lambda+1}\right)^k\binom{n}{j}S_{k,j}\left(\frac{1}{\lambda}\right).$$

These results become better if one can compute the combinatorial sum $S_{k,j}\left(\frac{1}{\lambda}\right)$.

3. EXPLICIT FORMULA OF THE GENERATING FUNCTION

According to Stirling numbers $S_2(n,k)$, the expression of $f_n(\lambda,x)$ is given by the following theorem

Theorem 2. *For n a fixed positive integer, the generating function for the numbers $B(n,k,\lambda)$ is given by the expression*

$$f_n(\lambda,x)=\sum_{j=1}^{n}\frac{j!\lambda^j S_2(n,j)x^j}{(1-(\lambda+1)x)^{j+1}},\ n\geq 1. \tag{1}$$

with the initial term $f_0(\lambda,x)=\frac{1}{1-(\lambda+1)x}$.

Using the following well known formula

$$j!S_2(n,j) = \sum_{i=1}^{j} \binom{j}{i} (-1)^{j-i} i^n$$

we have

$$f_n(\lambda,x)(t) = \sum_{j=1}^{n} \sum_{i=1}^{j} \binom{j}{i} (-1)^{j-i} i^n \frac{(\lambda x)^j}{(1-(\lambda+1)x)^{j+1}}.$$

According to analytic approach of the function $f_n(\lambda,x)$ established in [**3**, Theorem 3.3], we have already computed the successive derivatives of the function $\frac{1}{1-(\lambda e^t+1)x}$ which is given by the following expression

$$\frac{\partial^n}{\partial t^n} \frac{1}{1-(\lambda e^t+1)\,x}|_{t=0} = n! \sum_{j=1}^{n} \frac{j!\lambda^j S_2(n,j) x^j}{(1-(\lambda+1)x)^{j+1}}.$$

Therefore, we get the following result:

Corollary 1. *$f_n(\lambda,x)$ is a rational function and takes the form*

$$f_n(\lambda,x) = \frac{P_{n,\lambda}(x)}{(1-(\lambda+1)x)^{n+1}}$$

where $P_0(x)=1$ and

$$P_{n,\lambda}(x) = \sum_{j=1}^{n} j!\lambda^j S_2(n,j)\,(1-(\lambda+1)x)^{n-j}\, x^j,\ \ n \geq 1.$$

Since $S_2(1,1) = S_2(2,1) = S_2(2,2) = 1$, we have

$$P_{1,\lambda}(x) = \frac{\lambda x}{(1-(\lambda+1)\,x)^2}$$

and

$$P_{2,\lambda}(x) = \frac{\lambda x + \left(\lambda^2-\lambda\right) x^2}{(1-(\lambda+1)\,x)^3}.$$

Consequently the answer to open question concerning the polynomial $P_n(x)$ is that $P_n(x) = P_{n,1}(x)$ and then

$$P_n(x) = \sum_{j=1}^{n} j!S_2(n,j)\,(1-2x)^{n-j}\, x^j.$$

The first few values of $f_n(x)$ are

$$f_0(x) = \frac{1}{1-2x},\ \ f_1(x) = \frac{x}{(1-2x)^2} \text{ and } f_2(x) = \frac{x}{(1-2x)^3}$$

identic to these founded in the work [**3**].

3.1 Proof of Theorem 2

Let the formal generating function $h(t) = \sum_{n\geq 0} a_n t^n$ with $a_0 \neq 0$ and $\alpha \neq 0$ a complex number. In our recent work (see [**4**, Lemma 2.3]) we have provided the following expression of the function h^α:

$$h^\alpha(t) = a_0^\alpha + \sum_{n\geq 1}\sum_{k=1}^{n}\sum_{s_n(k)} \binom{\alpha}{k}\binom{k}{k_1,\cdots,k_n} a_0^{\alpha-k}\prod_{r=1}^{n} a_r^{k_r} t^n.$$

Since $k_r = 0$ for $r > n-k+1$, then

$$h^\alpha(t) = a_0^\alpha + \sum_{n\geq 1}\sum_{k=1}^{n}(\alpha)_k a_0^{\alpha-k} B_{n,k}\left(1!a_1,\cdots,(n-k+1)!a_{n-k+1}\right)\frac{t^n}{n!}.$$

In the special case $\alpha = -1$, we have

$$(2)\quad \frac{1}{h(t)} = a_0^{-1} + \sum_{n\geq 1}\sum_{k=1}^{n}(-1)^k k! a_0^{-1-k} B_{n,k}\left(1!a_1,\cdots,(n-k+1)!a_{n-k+1}\right)\frac{t^n}{n!}.$$

Now returning back to the proof of Theorem 2, we have

$$\sum_{k\geq 0}\sum_{n\geq 0} B(n,k) x^k \frac{t^n}{n!} = \sum_{k\geq 0}\left(\lambda e^t + 1\right)^k x^k = \frac{1}{1 - x\left(\lambda e^t + 1\right)}.$$

Since

$$1 - x\left(\lambda e^t + 1\right) = 1 - (\lambda+1)x - \lambda x \sum_{n\geq 1}\frac{t^n}{n!}.$$

Using (2), we obtain

$$\frac{1}{1 - x\left(\lambda e^t + 1\right)} = \sum_{j=1}^{n}(-1)_j\left(1 - (\lambda+1)x\right)^{-1-j} B_{n,j}\left(-\lambda x,\cdots,-\lambda x\right)\frac{t^n}{n!}.$$

Since we have

$$(-1)_j B_{n,j}\left(-\lambda x,\cdots,-\lambda x\right) = (-1)^j j!(-\lambda x)^j S_2(n,j),$$

the identity (1) Theorem 2 follows.

4. SIMSEK NUMBERS OF HIGHER ORDER

Let $\alpha \neq 0$ a complex number, we consider numbers $B_\alpha(n,\lambda)$ and functions $f_k^{(\alpha)}(\lambda, x)$ defined respectively by means of the generating functions

$$\left(\lambda e^t + 1\right)^\alpha = \sum_{n\geq 0} B_\alpha(n,\lambda)\frac{t^n}{n!}$$

and

$$\left(\frac{1}{1-(\lambda e^t+1)\,x}\right)^{\alpha} = \sum_{n\geq 0} f_n^{(\alpha)}(\lambda,x)\,\frac{t^n}{n!}.$$

According to identity [**4**, Identity 14, Lemma 2.3] the following theorem gives explicit formulae of $B_\alpha(n,\lambda)$ and $f_k^{(\alpha)}(\lambda,x)$, we left the proof as exercise for the reader.

Theorem 3.

$$B_\alpha(n,\lambda) = \sum_{j=1}^{n} (\alpha)_j\,(\lambda+1)^{\alpha-j}\,\lambda^j S_2(n,j) \tag{3}$$

and

$$f_n^{(\alpha)}(\lambda,x) = \sum_{j=1}^{n} (-\alpha)_j\,(-\lambda)^j\,S_2(n,j)\,(1-(\lambda+1)\,x)^{-\alpha-j}\,x^j$$

with initials terms $B_\alpha(0,\lambda) = (\lambda+1)^\alpha$ *and* $f_0^{(\alpha)}(\lambda,x) = \frac{1}{(1-2x)^\alpha}$.

Using the well-known identity

$$(1+x)^\alpha = \sum_{i\geq 0}\binom{\alpha}{i}x^i, \text{ where } |x|<1$$

we will have

$$f_n^{(\alpha)}(\lambda,x) = \sum_{i\geq 0}\sum_{j=1}^{n}(-\alpha)_j\binom{-\alpha-j}{i}\lambda^j(-1)^{i+j}S_2(n,j)\,(\lambda+1)^i\,x^{i+j}$$

$f_n^{(\alpha)}(\lambda,x)$ becomes a polynomial if and only if $\alpha=-k$ with k positive integer and we have

$$f_n^{(-k)}(\lambda,x) = \sum_{j=1}^{\min\{k,n\}}(k)_j(-\lambda)^j S_2(n,j)\,(1-(\lambda+1)\,x)^{k-j}\,x^j$$

or

$$f_n^{(-k)}(\lambda,x) = \sum_{j=1}^{n}\sum_{i=0}^{k-j}(k)_j\binom{k-j}{i}\lambda^j(-1)^{i+j}S_2(n,j)\,(\lambda+1)^i\,x^{i+j}.$$

Another reformulation of polynomials $B(n,k,\lambda)$ by means of Stirling numbers of the second kind $S_2(n,j)$ is given by the following corollary.

Corollary 2.

$$B(n,k,\lambda) = \sum_{j=1}^{n}\frac{k!}{(k-j)!}\,(\lambda+1)^{k-j}\,S_2(n,j) \tag{4}$$

Proof. The result is immediate from identity (3) Theorem 3, by taking $\alpha = k$ a positive integer and remarking that $(k)_j = \frac{k!}{(k-j)!}$. To be sure, here another proof. We have

$$\left(\lambda e^t + 1\right)^k = \left(\lambda\left(e^t - 1\right) + 1 + \lambda\right)^k = \sum_{j=0}^{k} \binom{k}{j} \lambda^j \left(e^t - 1\right)^j (1+\lambda)^{k-j}.$$

Furthermore

$$\left(\lambda e^t + 1\right)^k = \sum_{n\geq 0} \sum_{j=0}^{k} \frac{k!}{(k-j)!} \lambda^j (1+\lambda)^{k-j} S_2(n,j) \frac{t^n}{n!}.$$

□

According to Theorem 3 and if $(1 - (\lambda+1)x)^{k-j}(-x)^j = (\lambda+1)^{k-j}$, we conclude that $f_n^{(-k)}(\lambda, x) = B(n,k,\lambda)$. On general the connection between numbers $B(n,k,\lambda)$ and polynomials $f_k^{(\alpha)}(\lambda, x)$ is given by the following theorem.

Theorem 4.

$$f_n^{(\alpha)}(\lambda, x) = \sum_{j\geq 1} \binom{-\alpha}{j} (-1)^j B(n,j,\lambda) x^j, \; n \geq 1. \tag{5}$$

Proof. We have

$$\left(\frac{1}{1 - (\lambda e^t + 1)x}\right)^{\alpha} = \sum_{n\geq 0} f_n^{(\alpha)}(\lambda, x) \frac{t^n}{n!}.$$

But

$$\left(\frac{1}{1 - (\lambda e^t + 1)x}\right)^{\alpha} = \sum_{j\geq 0} \binom{-\alpha}{j} (-x)^j \left(\lambda e^t + 1\right)^j$$

and then

$$\left(\frac{1}{1 - (\lambda e^t + 1)x}\right)^{\alpha} = \sum_{n\geq 0} \sum_{j\geq 0} \binom{-\alpha}{j} (-x)^j B(n,j,\lambda) \frac{t^n}{n!}.$$

Since $B(n,0,\lambda) = 1$ for $n = 0$ and 0 otherwise; the (5) Theorem 4 follows. □

As we have seen for $\alpha = -k$, $f_n^{(-k)}(\lambda, x)$ is a polynomial and takes the form

$$f_n^{(-k)}(\lambda, x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j B(n,j,\lambda) x^j.$$

Corollary 3. *We have*

$$\sum_{j=1}^{\min\{n,l\}} (-\alpha)_j \binom{-\alpha - j}{l - j} \lambda^j S_2(n,j) (\lambda+1)^{l-j} = \binom{-\alpha}{l} B(n,l,\lambda)$$

and then for $k \in \mathbb{N}$;

$$\sum_{j=1}^{\min\{n,l\}} \frac{k!}{(k-j)!}\binom{k-j}{l-j}\lambda^j S_2(n,j)(\lambda+1)^{l-j} = \binom{k}{l} B(n,l,\lambda).$$

Proof. Since we have

$$f_n^{(\alpha)}(\lambda,x) = \sum_{i\geq 0}\sum_{j=1}^{n}(-\alpha)_j\binom{-\alpha-j}{i}\lambda^j(-1)^{i+j}S_2(n,j)(\lambda+1)^i x^{i+j}.$$

Then

$$f_n^{(\alpha)}(\lambda,x) = \sum_{j=1}^{n}\sum_{l\geq j}(-\alpha)_j\binom{-\alpha-j}{l-j}\lambda^j(-1)^l S_2(n,j)(\lambda+1)^{l-j} x^l$$

and

$$f_n^{(\alpha)}(\lambda,x) = \sum_{l\geq 1}\sum_{j=1}^{\min\{n,l\}}(-\alpha)_j\binom{-\alpha-j}{l-j}\lambda^j(-1)^l S_2(n,j)(\lambda+1)^{l-j} x^l.$$

Comparing with the identity (5) we obtain

$$\sum_{j=1}^{\min\{n,l\}}(-\alpha)_j\binom{-\alpha-j}{l-j}\lambda^j S_2(n,j)(\lambda+1)^{l-j} = \binom{-\alpha}{l}B(n,l,\lambda).$$

□

If $\lambda = -1$, we have $B(n,l,1) = l!S_2(n,l)$ which is a special case of identiy (4) Corollary 2.

REFERENCES

1. L. Comtet, L: *Advanced Combinatorics: The Art of Finite and Infinite Expansions.* Dordrecht, Boston (1974).
2. M. Goubi: *Successive Derivatives of Fibonacci Type Polynomials of Higher Order in Two Variables.* Filomat, 32:14 (2018), 5149–5159
3. M. Goubi: *An affirmative answer to two questions concerning special case of Simsek numbers and open problems.* Appl. Anal. Discrete Math., **14** (2020), 094–105.
4. M. Goubi: *A new class of generalized polynomials associated with Hermite-Bernoulli polynomials.* J. Appl. Math. & Informatics, **38**, 3-4 (2020), 211–220.
5. M. Goubi: *On Composition of Generating Functions.* Caspian Journal of Mathematical Sciences, **9**, 2 (2020), 256–265.

6. D. Kim, D. Simsek and J. S. So: *Identities and Computation Formulas for Combinatorial Numbers Including Negative Order Changhee Polynomials.* Symmetry **12**, 9, (2020),

7. Y. Simsek: *New families of special numbers for computing negative order Euler numbers and related numbers and polynomials.* Appl. Anal. Discrete Math., **12** (2018), 001–035.

8. Y. Simsek: *Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers.* J. Math. Anal. Appl., **477**,2, (2019), 1328–1352.

9. B. Simsek: *Formulas derived from moment generating functions and Berstein polynomials.* Appl. Anal. Discrete Math., **13**, 3 (2019), 839–848.

10. A. Xu: *On an open problem of Simsek concerning the computation of a family of special numbers.* Appl. Anal. Discrete Math., **13** (1) (2019) 61–72.

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