Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **17** (2023), 047–056.
https://doi.org/10.2298/AADM200510005K

A NOTE ON DEGENERATE MULTI-POLY-BERNOULLI NUMBERS AND POLYNOMIALS

Taekyun Kim and Dae San Kim*

In this paper, we consider the degenerate multi-poly-Bernoulli numbers and polynomials which are defined by means of the multiple polylogarithms and degenerate versions of the multi-poly-Bernoulli numbers and polynomials. We investigate some properties for those numbers and polynomials. In addition, we give some identities and relations for the degenerate multi-poly-Bernoulli numbers and polynomials.

1. INTRODUCTION

In recent years, we have witnessed that explorations for degenerate versions of some special polynomials and transcendental functions have been very rewarding and fruitful. Indeed, the study of degenerate versions has applications to differential equations, identities of symmetry and probability theory as well as to number theory and combinatorics. Indeed, infinitely many families of linear and non-linear ordinary differential equations, satisfied by the generating functions of some degenerate special polynomials and numbers, were found with the purpose of discovering some new combinatorial identities for those polynomials and numbers (see [13]). As to identities of symmetry, abundant identities of symmetry for various degenerate versions of many special polynomials have been investigated by using p-adic integrals and p-adic and q-integrals (see [17]). For probability theory, some identities connecting some special numbers and moments of random variables were derived from the generating functions of the moments of certain random variables (see [19]). It is noteworthy that study of degenerate versions of some polynomials and numbers is not only limited to special polynomials and numbers but also extended to transcendental functions like gamma functions (see [16]).

*Corresponding author. Taekyun Kim
2020 Mathematics Subject Classification. 11B83, 05A19
Keywords and Phrases. Degenerate multi-poly-Bernoulli polynomials, Multiple poly-logarithm, Stirling numbers, Bernoulli polynomials and numbers

In this paper, we consider the degenerate multi-poly-Bernoulli numbers and polynomials (see (15)) which are defined by means of the multiple polylogarithms and degenerate versions of the multi-poly-Bernoulli numbers and polynomials studied earlier in the literature (see [2]). We investigate some properties for those numbers and polynomials. In addition, we give some identities and relations for the degenerate multi-poly-Bernoulli numbers and polynomials. For the rest of this section, we recall the facts that needed throughout this paper.

For any $0 \neq \lambda \in \mathbb{R}$, Carlitz considered the higher-order degenerate Bernoulli polynomials given by

$$(1) \qquad \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(r)}(x)\frac{t^{n}}{n!}, \quad \text{(see [3]).}$$

When $x=0$, $\beta_{n,\lambda}^{(r)}=\beta_{n,\lambda}^{(r)}(0)$ are called the higher-order degenerate Bernoulli numbers.

For $k \in \mathbb{Z}$, the polylogarithm function is defined by

$$(2) \qquad \mathrm{Li}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}, \quad \text{(see [1,12,23]).}$$

Note that $\mathrm{Li}_1(x)=\sum_{n=1}^{\infty}\frac{x^n}{n!}=-\log(1-x)$.
As is known, the poly-Bernoulli polynomials are defined by

$$(3) \qquad \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}, \quad \text{(see [12]).}$$

When $x=0$, $B_n^{(k)}=B_n^{(k)}(0)$ are called the poly-Bernoulli numbers. Note that $B_n^{(1)}=B_n(x)$, where $B_n(x)$ are the ordinary Bernoulli polynomials given by

$$\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}=\frac{t}{e^t-1}e^{xt}, \quad \text{(see [1-3,6-12,14-16,18,20-27]).}$$

The polyexponential functions were first studied by Hardy [4,5] and reconsidered by Kim as an inverse to the polylogarithm functions which were introduced by Kanako [8]. The degenerate exponential functions are defined by

$$(4) \qquad e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t)=e_\lambda^1(t)=(1+\lambda t)^{\frac{1}{\lambda}}, \quad \text{(see [11]).}$$

Here we observe that

$$(5) \qquad e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad \text{(see [11]),}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)\cdots(x-(n-1)\lambda)$, $(n\geq 1)$.

It is well known that the Stirling numbers of the second kind are defined by

$$\frac{1}{k!}(e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},\quad (n\geq 0),\quad (\text{see } [11,12,15,20,21]). \tag{6}$$

Also, the signed Stirling numbers of the first kind are given by

$$\frac{1}{k!}(\log(1+t))^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!},\quad (n\geq 0). \tag{7}$$

For $k_1,k_2,\ldots,k_r\in\mathbb{Z}$, the multiple polylogarithm is defined by

$$\mathrm{Li}_{k_1,k_2,\ldots,k_r}(x)=\sum_{0<n_1<n_2<\cdots<n_r}\frac{x^{n_r}}{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}},\quad (\text{see } [26]), \tag{8}$$

where the sum is over all integers $n_1,n_2,\ldots,n_r$ satisfying $0<n_1<n_2<\cdots<n_r$. Note here that (8) reduces to (2) for $r=1$.
About twenty years ago, the first author introduced the generalized Bernoulli numbers $B_n^{(k_1,k_2,\ldots,k_r)}$ of order r (see [14]) which are given by

$$\frac{r!\mathrm{Li}_{k_1,k_2,\ldots,k_r}(1-e^{-t})}{(e^t-1)^r}=\sum_{n=0}^{\infty}B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^n}{n!}. \tag{9}$$

Actually, the $r!$ in (9) does not appear in [14]. However, the present definition is more convenient, since $B_n^{(1,1,\ldots,1)}=B_n^{(r)}$ are the Bernoulli numbers of order r (see (14)). These numbers would have been called the multi-poly-Bernoulli numbers, since it is a multiple version of poly-Bernoulli numbers (see (3)). Furthermore, we may consider the multi-poly-Bernoulli polynomials, which are natural extensions of the multi-poly-Bernoulli numbers, given by

$$\frac{r!\mathrm{Li}_{k_1,k_2,\ldots,k_r}(1-e^{-t})}{(e^t-1)^r}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k_1,k_2,\ldots,k_r)}(x)\frac{t^n}{n!}. \tag{10}$$

The multi-poly-Bernoulli polynomials are multiple versions of the poly-Bernoulli polynomials in (3). We let the interested reader refer to [1] for the detailed properties on those polynomials.

2. DEGENERATE MULTI-POLY-BERNOULLI NUMBERS AND POLYNOMIALS

From (8), we note that

$$\begin{aligned}\frac{d}{dx}\mathrm{Li}_{k_1,k_2,\ldots,k_r}(x) &= \frac{d}{dx}\sum_{0<n_1<n_2<\cdots<n_r}\frac{x^{n_r}}{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}}\\ &= \frac{1}{x}\sum_{0<n_1<n_2<\cdots<n_r}\frac{x^{n_r}}{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r-1}}\\ &= \frac{1}{x}\mathrm{Li}_{k_1,k_2,\ldots,k_r-1}(x).\end{aligned} \tag{11}$$

Let us take $k_r = 1$. Then, from (11), we have

$$\begin{aligned} \frac{d}{dx}\mathrm{Li}_{k_1,\dots,k_{r-1},1}(x) &= \sum_{0<n_1<n_2<\cdots<n_{r-1}} \frac{1}{n_1^{k_1}n_2^{k_2}\cdots n_{r-1}^{k_{r-1}}} \sum_{n_r=n_{r-1}+1}^{\infty} x^{n_r-1}. \\ &= \frac{1}{1-x} \sum_{0<n_1<n_2<\cdots<n_{r-1}} \frac{x^{n_{r-1}}}{n_1^{k_1}n_2^{k_2}\cdots n_{r-1}^{k_{r-1}}} \\ &= \frac{1}{1-x}\mathrm{Li}_{k_1,k_2,\dots,k_{r-1}}(x) \end{aligned} \tag{12}$$

Thus, by (12), we get

$$\mathrm{Li}_{k_1,1}(x) = \int \frac{1}{1-x}\mathrm{Li}_{k_1}(x)dx. \tag{13}$$

By integration by parts, from (13), we note that

$$\mathrm{Li}_{1,1}(x) = \int \frac{1}{1-x}(-\log(1-x))dx = \frac{1}{2!}\big(-\log(1-x)\big)^2.$$

By induction, we get

$$\begin{aligned} \mathrm{Li}_{\underbrace{1,1,\dots,1}_{r-times}}(x) &= \frac{(-1)^r}{r!}\big(\log(1-x)\big)^r, \quad (r\in\mathbb{N}), \\ &= \sum_{l=r}^{\infty} S_1(l,r)(-1)^{l-r}\frac{x^l}{l!} \\ &= \sum_{l=r}^{\infty} |S_1(l,r)|\frac{x^l}{l!} \end{aligned} \tag{14}$$

where $S_1(l,r)$ (respectively, $|S_1(l,r)| = S_1(l,r)(-1)^{l-r}$) are the signed (respectively, unsigned) Stirling numbers of the first kind (see (7)).

Now, we consider the degenerate multi-poly-Bernoulli polynomials which are degenerate versions of the multi-poly-Bernoulli polynomials in (10) and given by

$$\frac{r!\mathrm{Li}_{k_1,k_2,\dots,k_r}(1-e^{-t})}{\big(e_\lambda(t)-1\big)^r}e_\lambda^x(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k_1,k_2,\dots,k_r)}(x)\frac{t^n}{n!}, \tag{15}$$

(see (4)). When $x=0$, $\beta_{n,\lambda}^{(k_1,k_2,\dots,k_r)} = \beta_{n,\lambda}^{(k_1,k_2,\dots,k_r)}(0)$ are called the degenerate multi-poly-Bernoulli numbers.

From (15) and recalling (5), we note that

$$\begin{aligned}(16)\qquad \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k_1,k_2,\dots,k_r)}(x)\frac{t^n}{n!} &= \frac{r!\mathrm{Li}_{k_1,k_2,\dots,k_r}(1-e^{-t})}{\big(e_\lambda(t)-1\big)^r}e_\lambda^x(t)\\ &= \sum_{l=0}^{\infty}\beta_{l,\lambda}^{(k_1,k_2,\dots,k_r)}\frac{t^l}{l!}\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{t^m}{m!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(x)_{n-l,\lambda}\beta_{l,\lambda}^{(k_1,k_2,\dots,k_r)}\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, by (16), we get

$$(17)\qquad \beta_{n,\lambda}^{(k_1,k_2,\dots,k_r)}(x)=\sum_{l=0}^{n}\binom{n}{l}(x)_{n-l,\lambda}\beta_{l,\lambda}^{(k_1,k_2,\dots,k_r)},\quad (n\geq 0).$$

From (1), (14) and (15), we note that

$$(18)\qquad \beta_{n,\lambda}^{\overbrace{(1,1,\dots,1)}^{r-times}}(x)=\beta_{n,\lambda}^{(r)}(x),\quad (n\geq 0).$$

Thus, from (17) and (18), we obtain the following result.

Proposition 1. *For $n\geq 0$, we have*

$$\beta_{n,\lambda}^{(k_1,k_2,\dots,k_r)}(x)=\sum_{l=0}^{n}\binom{n}{l}(x)_{n-l,\lambda}\beta_{l,\lambda}^{(k_1,k_2,\dots,k_r)},$$

and

$$\beta_{n,\lambda}^{\overbrace{(1,1,\dots,1)}^{r-times}}(x)=\beta_{n,\lambda}^{(r)}(x).$$

From (15) and recalling (6), we note that

$$\begin{aligned}(19)\qquad \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} &= \frac{r!}{\big(e_\lambda(t)-1\big)^r}\mathrm{Li}_{k_1,\dots,k_r}\big(1-e^{-t}\big)e_\lambda^x(t)\\ &= \frac{r!}{\big(e_\lambda(t)-1\big)^r}\sum_{0<n_1<\cdots<n_{r-1}}\frac{1}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}}\sum_{n_r=n_{r-1}+1}^{\infty}\frac{\big(1-e^{-t}\big)^{n_r}}{n_r^{k_r}}e_\lambda^x(t)\\ &= \frac{r!}{\big(e_\lambda(t)-1\big)^r}\sum_{0<n_1<\cdots<n_{r-1}}\frac{\big(1-e^{-t}\big)^{n_{r-1}}}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}}\sum_{n_r=1}^{\infty}\frac{\big(1-e^{-t}\big)^{n_r}}{(n_r+n_{r-1})^{k_r}}e_\lambda^x(t)\\ &= \frac{r!e_\lambda^x(t)}{\big(e_\lambda(t)-1\big)^r}\sum_{0<n_1<\cdots<n_{r-1}}\frac{\big(1-e^{-t}\big)^{n_{r-1}}}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}}\\ &\quad\times\sum_{n_r=1}^{\infty}\frac{n_r!}{(n_r+n_{r-1})^{k_r}}\sum_{l=n_r}^{\infty}(-1)^{l-n_r}S_2(l,n_r)\frac{t^l}{l!}\end{aligned}$$

Now, from (19), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!} &= \frac{rt}{e_\lambda(t)-1}\left(\frac{(r-1)!e_\lambda^x(t)}{\big(e_\lambda(t)-1\big)^{r-1}}\sum_{0<n_1<\cdots<n_{r-1}}\frac{\big(1-e^{-t}\big)^{n_{r-1}}}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}-m}}\right)\\
&\quad\times\frac{1}{t}\sum_{l=1}^{\infty}\left(\sum_{n_r=1}^{l}n_r!(-1)^{l-n_r}S_2(l,n_r)\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m n_r^{-k_r-m}\right)\frac{t^l}{l!}\\
&= \frac{rt}{e_\lambda(t)-1}\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k_1,\dots,k_{r-1}-m)}(x)\frac{t^j}{j!}\\
&\quad\times\sum_{l=0}^{\infty}\left(\sum_{n_r=1}^{l+1}\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)}{l+1}n_r^{-k_r-m}\right)\frac{t^l}{l!}\\
&= r\sum_{p=0}^{\infty}\beta_{p,\lambda}\frac{t^p}{p!}\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\right.\\
&\quad\left.\times\binom{k}{l}\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)}{l+1}n_r^{-k_r-m}\beta_{k-l,\lambda}^{(k_1,k_2,\dots,k_{r-1}-m)}\right)\frac{t^k}{k!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}r\binom{k_r+m-1}{m}\binom{k}{l}\binom{n}{k}(-1)^m\right.\\
&\quad\left.\times\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)}{l+1}n_r^{-k_r-m}\beta_{k-l,\lambda}^{(k_1,k_2,\dots,k_{r-1}-m)}(x)\beta_{n-k,\lambda}\right)\frac{t^n}{n!},\end{aligned}\tag{20}$$

where $\beta_{n,\lambda}$ are the Carlitz's degenerate Bernoulli numbers with $\beta_{n,\lambda}^{(1)}=\beta_{n,\lambda}$. Therefore, by (20), we obtain the following theorem.

Theorem 2. *For $k_1,k_2,\dots,k_r\in\mathbb{Z}$ and $n\geq 0$, we have*

$$\begin{aligned}\beta_{n,\lambda}^{(k_1,k_2,\dots,k_r)}(x) &= r\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}\binom{k}{l}\binom{n}{k}(-1)^m\\
&\quad\times\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)}{l+1}n_r^{-k_r-m}\beta_{n-k,\lambda}\beta_{k-l,\lambda}^{(k_1,k_2,\dots,k_{r-1}-m)}(x).\end{aligned}\tag{21}$$

Replacing k_r by $-k_r$ in (21), we obtain the following corollary.

Corollary 3. *For $k_1,k_2,\dots,k_r\in\mathbb{Z}$ and $n\geq 0$, we have*

$$\begin{aligned}\beta_{n,\lambda}^{(k_1,k_2,\dots,-k_r)}(x) &= r\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}\binom{k_r}{m}\binom{k}{l}\binom{n}{k}\\
&\quad\times\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)}{l+1}n_r^{k_r-m}\beta_{n-k,\lambda}\beta_{k-l,\lambda}^{(k_1,k_2,\dots,k_{r-1}-m)}(x).\end{aligned}$$

From (15), we have

$$
\begin{aligned}
(22)\quad & \sum_{n=0}^{\infty}\Big(\beta_{n,\lambda}^{(k_1,\ldots,k_r)}(x+1)-\beta_{n,\lambda}^{(k_1,\ldots,k_r)}(x)\Big)\frac{t^n}{n!} \\
&= \frac{r!}{\big(e_\lambda(t)-1\big)^{r-1}}e_\lambda^x(t)\sum_{0<n_1<\cdots<n_{r-1}}\frac{\big(1-e^{-t}\big)^{n_{r-1}}}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}}\sum_{n_r=1}^{\infty}\frac{\big(1-e^{-t}\big)^{n_r}}{(n_r+n_{r-1})^{k_r}} \\
&= r\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\frac{(r-1)!e_\lambda^x(t)}{\big(e_\lambda(t)-1\big)^{r-1}}\mathrm{Li}_{k_1,\ldots,k_{r-1}-m}\big(1-e^{-t}\big) \\
&\quad\times\sum_{n_r=1}^{\infty}n_r^{-k_r-m}n_r!\sum_{l=n_r}^{\infty}(-1)^{l-n_r}S_2(l,n_r)\frac{t^l}{l!} \\
&= r\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\sum_{j=0}^{\infty}\beta_{j,\lambda}^{(k_1,\ldots,k_{r-2},k_{r-1}-m)}(x)\frac{t^j}{j!} \\
&\quad\times\sum_{l=1}^{\infty}\Big(\sum_{n_r=1}^{l}n_r^{-k_r-m}n_r!(-1)^{l-n_r}S_2(l,n_r)\Big)\frac{t^l}{l!} \\
&= \sum_{n=1}^{\infty}\Big(r\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\sum_{l=1}^{n}\sum_{n_r=1}^{l}\binom{n}{l}n_r^{-k_r-m}n_r! \\
&\quad\times(-1)^{l-n_r}S_2(l,n_r)\beta_{n-l,\lambda}^{(k_1,k_2,\ldots,k_{r-2},k_{r-1}-m)}(x)\Big)\frac{t^n}{n!}.
\end{aligned}
$$

Therefore, from (22) we obtain the following theorem.

Theorem 4. *For $n,r\geq 1$ and $k_1,k_2,\ldots,k_r\in\mathbb{Z}$, we have*

$$
\begin{aligned}
&\frac{1}{r}\Big(\beta_{n,\lambda}^{(k_1,\ldots,k_r)}(x+1)-\beta_{n,\lambda}^{(k_1,\ldots,k_r)}(x)\Big) \\
&\qquad= \sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\sum_{l=1}^{n}\sum_{n_r=1}^{l}\binom{n}{l}n_r^{-k_r-m} \\
&\qquad\quad\times n_r!(-1)^{l-n_r}S_2(l,n_r)\beta_{n-l,\lambda}^{(k_1,\ldots,k_{r-2},k_{r-1}-m)}(x).
\end{aligned}
$$

By the definition of degenerate multi-poly-Bernoulli polynomials, we get

$$
\begin{aligned}
(23)\quad \sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k_1,k_2,\ldots,k_r)}(x+y)\frac{t^n}{n!} &= \frac{r!}{\big(e_\lambda(t)-1\big)^r}\mathrm{Li}_{k_1,k_2,\ldots,k_r}(1-e^{-t})e_\lambda^x(t)e_\lambda^y(t) \\
&= \sum_{l=0}^{\infty}\beta_{l,\lambda}^{(k_1,k_2,\ldots,k_r)}(x)\frac{t^l}{l!}\sum_{m=0}^{\infty}(y)_{m,\lambda}\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}^{(k_1,\ldots,k_r)}(x)(y)_{n-l,\lambda}\Big)\frac{t^n}{n!}.
\end{aligned}
$$

Thus, from (23) we obtain

$$\beta_{n,\lambda}^{(k_1,\ldots,k_r)}(x+y)=\sum_{l=0}^{n}\binom{n}{l}\beta_{l,\lambda}^{(k_1,\ldots,k_r)}(x)(y)_{n-l,\lambda},\quad (n\geq 0).$$

3. CONCLUSION

In [3], Carlitz initiated study of degenerate versions of Bernoulli and Euler polynomials, namely the degenerate Bernoulli and Euler polynomials. In recent years, some mathematicians intensively studied various versions of many special numbers and polynomials and quite a few interesting results were found about them (see [9,10,12,15,16,18,17-22,24,26]). As we mentioned in the Introduction, study of degenerate versions is not limited only to polynomials but can be extended also to transcendental functions like gamma functions (see [16]).

In this paper, we considered the degenerate multi-poly-Bernoulli numbers and polynomials which are defined by means of the multiple polylogarithms. They are degenerate versions of the multi-poly-Bernoulli numbers and polynomials, and multiple versions of the degenerate poly-Bernoulli numbers and polynomials ($r=1$ case of (15)). We investigated some properties for those numbers and polynomials. In fact, among other things, we derived some explicit expressions of the degenerate multi-poly-Bernoulli numbers and polynomials.

It is one of our future projects to continue this line of research and find applications not only in mathematics but also in science and engineering.

Acknowledgements. We would like to thank the referees for their helpful suggestions and comments that helped improve the original manuscript in its present form.

REFERENCES

1. Askey, R., *Book Review: Polylogarithms and associated functions,* Bull. Amer. Math. Soc.(N.S.) **6** (1982), no. 2, 248–251.
2. Bayad, A.; Hamahata, Y., *Multiple polylogarithms and multi-poly-Bernoulli polynomials,* Funct. Approx. Comment. Math. **46** (2012), part 1, 45–61.
3. Carlitz, L., *Degenerate Stirling, Bernoulli and Eulerian numbers,* Utilitas Math. **15** (1979), 51–88.
4. Hardy, G. H., *On the zeroes of certain classes of integral Taylor series. Part II.—On the integral function formula and other similar functions,* Proc. London Math. Soc. (2) **2** (1905), 401–431.
5. Hardy, G. H. *On the zeroes certain classes of integral Taylor series. Part I.—On the integral function formula,* Proc. London Math. Soc. (2) **2** (1905), 332–339.
6. He, Y.; Araci, S., *Sums of products of Apostol-Bernoulli and Apostol-Euler polynomials,* Adv. Difference Equ. 2014, 2014:155.

7. He, Y.; Zhang, W., *A three-term reciprocity formula for Bernoulli polynomials,* Util. Math. **100** (2016), 23–31.

8. Kaneko, M. *poly-Bernoulli numbers,* J.Theor. Nombres Bordeaux **9** (1997), no. 1, 221–228.

9. Khan, W. A., *A new class of degenerate Frobenius-Euler-Hermite polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **28** (2018), no. 4, 567–576.

10. Khan, W.A.; Ahmad, M., *Partially degenerate poly-Bernoulli polynomials associated with Hermite polynomials,* Adv. Stud. Contemp. Math. (Kyungshang) **28** (2018), no. 3, 487–496.

11. Kim, D. S.; Kim, T., *A note on polyexponential and unipoly functions,* Russ. J. Math. Phys. **26** (2019), no. 1, 40–49.

12. Kim, D. S.; Kim, T., *A note on degenerate poly-Bernoulli numbers and polynomials.* Adv. Difference Equ. 2015, 2015:258.

13. Kim, T.; Kim, D. S.; Dolgy, D. V.; Park, J.-W., *On the type 2 poly-Bernoulli polynomials associated with umbral calculus.* Open Math. **19** (2021), no. 1, 878–887.

14. Kim, M.-S.; Kim, T., *An explicit formula on the generalized Bernoulli number with order n,* Indian J. Pure Appl. Math. **31** (2000), no. 11, 1455–1461.

15. Kim, T.; Kim, D. S., *Degenerate polyexponential functions and degenerate Bell polynomials,* J. Math. Anal. Appl. **487** (2020), no. 2, 124017.

16. Kim, T.; Kim, D. S., *Degenerate Laplace transform and degenerate gamma function,* Russ. J. Math. Phys., **24** (2017), no. 2, 241–248.

17. Kim, D. S.; Kim, H. K.; Kim, T.; Lee, H.; Park, S., *Multi-Lah numbers and multi-Stirling numbers of the first kind.* Adv. Difference Equ. 2021, Paper No. 411, 9 pp.

18. Kim, T.; Kim, D. S.; Dolgy, D. V.; Kwon, J., *Some identities on generalized degenerate Genocchi and Euler numbers,* Informatica **31** (2020), no.4, 42–51.

19. Kim, T.; Kim, D. S.; Kim, H. Y.; Kwon, J., *Degenerate Stirling polynomials of the second kind and some applications,* Symmetry **11** (2019), no.8, Art.1046, 11pp.

20. Kim, T.; Kim, D. S.; Kwon, J.; Kim, H. Y., *A note on degenerate Genocchi and poly-Genocchi numbers and polynomials,* J. Inequal. Appl. 2020, 110 (2020).

21. Kim, T.; Kim, D. S.; Kwon, J.; Lee, H., *Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials,* Adv. Difference Equ. 2020, 2020:168.

22. Kim, T.; Kim, D. S.; Lee, H.; Kwon, J., *Degenerate binomial coefficients and degenerate hypergeometric functions,* Adv. Difference Equ. 2020, 2020:115.

23. Kim, D. S.; Kim, T. , *A note on a new type of degenerate Bernoulli numbers.* Russ. J. Math. Phys. **27** (2020), no. 2, 227–235.

24. Lewin, L., *Polylogarithms and associated functions, With a foreword by A. J. Van der Poorten,* North-Holland Publishing Co., New York-Amsterdam, 1981.

25. Lim, D.; Kwon, J., *A note on poly-Daehee numbers and polynomials,* Proc. Jangjeon Math. Soc. **19** (2016), no. 2, 219–224.

26. Simsek, Y., *On the Hermit based Genocchi polynomials,* Adv. Stud. Contemp. Math.(Kyungshang) **23** (2013), no. 1, 13–17.

27. Simsek, Y., *Identities and relations related to combinatorial numbers and polynomials,* Proc. Jangjeon Math. Soc. **20** (2017), no. 1, 127–135.

(Received 10. 05. 2020.)
(Revised 26. 03. 2022.)

Taekyun Kim
Department of Mathematics,
Kwangwoon University,
Seoul 139-701, Republic of Korea,
E-mail: *tkkim@kw.ac.kr*

Dae San Kim
Department of Mathematics,
Sogang University,
Seoul 121-742, Republic of Korea,
E-mail: *dskim@sogang.ac.kr*