Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **17** (2023), 057–075.
https://doi.org/10.2298/AADM210112001B

NEW SERIES WITH CAUCHY AND STIRLING NUMBERS, PART 2

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We evaluate in closed form several series involving products of Cauchy numbers with other special numbers (harmonic, skew-harmonic, hyperharmonic, and central binomial). Similar results are obtained with series involving Stirling numbers of the first kind. We focus on several particular cases which give new closed forms for Euler sums of hyperharmonic numbers and products of hyperharmonic and harmonic numbers.

1. INTRODUCTION

The Cauchy numbers c_n are defined by the generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n \quad (|x| < 1)$$

(see [**7, 12, 19**]). They are called Cauchy numbers of the first kind by Comtet [**12**]. The numbers $c_n/n!$ are also known as the Bernoulli numbers of the second kind (see the comments in [**7**]). The Cauchy numbers have the important representation

(1) $$c_n = \int_0^1 z(z-1)\cdots(z-n+1)\,dz$$

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2020 Mathematics Subject Classification. 11B65, 05A19, 33B99, 30B10.
Keywords and Phrases. Cauchy number, Harmonic number, Hyperharmonic number, Central binomial coefficient, Series with special numbers.

The Stirling numbers of the first kind $s(n,k)$ are defined by the ordinary generating function

$$z(z-1)\cdots(z-n+1)=\sum_{k=0}^{n}s(n,k)\,z^k$$

or, equivalently

$$n!\binom{z}{n}=\sum_{k=0}^{n}s(n,k)\,z^k \tag{2}$$

and together with the Cauchy numbers play a major role in this paper. Integrating equation (1) and considering (2) we see that the numbers c_n can be expressed in terms of $s(n,k)$ in the following way

$$c_n=\sum_{k=0}^{n}\frac{s(n,k)}{k+1}. \tag{3}$$

The Stirling numbers of the first kind are very popular numbers in mathematics and have various important applications (see the comments and references in [**7, 12**]). In the recent paper [**7**] the first author stated the following two propositions:

Proposition A: *Let $f(z)$ be a function analytic in a region of the form* $\mathrm{Re}(z)>\lambda$ *for some $\lambda<0$ and with moderate growth in that region. Then we have the representation*

$$\int_0^1 f(x)dx=\sum_{n=0}^{\infty}\frac{(-1)^n c_n}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)\right\}. \tag{4}$$

Proposition B: *Under the same assumptions on the function f(z) as in Proposition A, for every $m\geq 0$ we have the representation*

$$\frac{f^{(m)}(0)}{m!}=\sum_{n=0}^{\infty}\frac{(-1)^n s(n,m)}{n!}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)\right\} \tag{5}$$

and in particular,

$$f'(0)=\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}f(k)\right\}.$$

(The summation in (5) de facto starts from $n=m$ since $s(n,m)=0$ for $n<m$.)

For details see [**7**]. In that paper various series identities were proved based on these two propositions by applying them to appropriate functions. In order to show that both Propositions are efficient method to evalaute infinite and finite series involving Cauchy numbers and Stirling numbers of the first kind, we continue this project. Then we present further results in this direction. We construct new generating functions for $c_n\binom{n}{q}$ and $s(n,m)\binom{n}{q}$. We establish a new explicit formula for Cauchy numbers and a general binomial identity for generalized harmonic number (see Propositions 4 and 5). Moreover, we prove new series identities involving Cauchy numbers and Stirling numbers of the first kind with binomial and reciprocal binomial coefficients. One of these results reduce to the answer of the Conjecture 4 in [**11**], which was also answered in [**26**, Corollary 2.4]. Finally, we evaluate various series involving products of Cauchy and skew-harmonic numbers, Cauchy and hyperharmonic numbers, Stirling numbers of the first kind and skew-harmonic numbers, Stirling numbers of the first kind and hyperharmonic numbers.

2. SERIES WITH CAUCHY NUMBERS AND BINOMIAL COEFFICIENTS

In this theorem we construct a generating function for the numbers $c_n\binom{n}{q}$ for any integer $q \geq 0$.

Theorem 1. *For any non-negative integer q and every $|z| < 1$ we have the representation*

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!}\binom{n}{q} z^n = (-1)^q \left(\frac{z}{1-z}\right)^q \int_0^1 \binom{x}{q}(1-z)^x dx \tag{6}$$

$$= (-z)^q \int_0^1 \binom{x}{q}(1-z)^{x-q} dx,$$

where

$$\int_0^1 \binom{x}{q}(1-z)^x dx = \frac{1}{q!}\sum_{k=0}^{q} s(q,k) A_k$$

with

$$A_k = k!\left(\frac{1}{(-\ln(1-z))^{k+1}} - (1-z)\sum_{j=0}^{k} \frac{1}{j!(-\ln(1-z))^{k-j+1}}\right).$$

In particular, with $z = 1/2$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n! 2^n}\binom{n}{q} = \frac{(-1)^q}{q!}\sum_{k=0}^{q} s(q,k) A_k$$

with

$$A_k = \int_0^1 x^k 2^{-x} dx = k!\left(\frac{1}{(\ln 2)^{k+1}} - \frac{1}{2}\sum_{j=0}^{k} \frac{1}{j!(\ln 2)^{k-j+1}}\right).$$

Proof. For the proof we use the binomial formula (see [**5**, Eq.(10.25)])

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{k}{q}\alpha^k = (-\alpha)^q(1-\alpha)^{n-q}\binom{n}{q}$$

$$= \left(\frac{-\alpha}{1-\alpha}\right)^q (1-\alpha)^n\binom{n}{q} \tag{7}$$

where $0 \leq \alpha \leq 1$. We take $f(x) = \binom{x}{q}\alpha^x$ to get from (4)

$$\int_0^1 \binom{x}{q}\alpha^x dx = \left(\frac{-\alpha}{1-\alpha}\right)^q \sum_{n=0}^{\infty}\frac{(-1)^n(1-\alpha)^n c_n}{n!}\binom{n}{q}. \tag{8}$$

When $\alpha = 1$ this becomes the familiar

$$c_q = q!\int_0^1 \binom{x}{q}dx.$$

Now equation (6) follows from (8) with $z = 1-\alpha$. For the evaluation of the integral we use equations (2) and

$$\int_0^u x^k e^{-\mu x}dx = \frac{k!}{\mu^k} - e^{-u\mu}\sum_{j=0}^{k}\frac{k!}{j!}\frac{u^j}{\mu^{k-j+1}},\ u > 0 \text{ and } \operatorname{Re}(\mu) > 0$$

[**17**, entry 3.352(1)]. □

Next, applying Proposition B to the same function $f(x) = \binom{x}{q}\alpha^x$ and using again the binomial identity (7) we come to a generating function for the numbers $s(n,m)\binom{n}{q}$.

Proposition 2. *For every non-negative integer* q *and every* $0 \leq z < 1$ *we have for* $m = 1, 2, ...$

$$\sum_{n=0}^{\infty}\frac{(-1)^n s(n,m)z^n}{n!}\binom{n}{q} = \frac{(-1)^q}{m!}\left(\frac{z}{1-z}\right)^q\left(\frac{d}{dx}\right)^m (1-z)^x\binom{x}{q}\Bigg|_{x=0}. \tag{9}$$

For $m = 1$ using the formula

$$\frac{d}{dx}\binom{x}{q} = \binom{x}{q}\sum_{j=0}^{q-1}\frac{1}{x-j} = \binom{x}{q}\frac{1}{x} + \binom{x}{q}\sum_{j=1}^{q-1}\frac{1}{x-j}$$

we compute

$$\lim_{x\to 0}\left\{\frac{d}{dx}(1-z)^x\binom{x}{q}\right\} = \frac{(-1)^{q-1}}{q}.$$

Also $s(n,1) = (-1)^{n-1}(n-1)!$ so that (9) takes the form

$$\sum_{n=0}^{\infty} \frac{z^n}{n} \binom{x}{q} = \frac{1}{q}\left(\frac{z}{1-z}\right)^q$$

which is equivalent to the well-known expansion

$$\sum_{n=0}^{\infty} \binom{n}{q} z^n = \frac{z^q}{(1-z)^{q+1}}.$$

In the next proposition, we use the central binomial coefficients $\binom{2n}{n}$. We start with the binomial formula [**5**, Eq. (10.35a)]

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{2k}{k} \frac{1}{4^k} = \binom{2n}{n} \frac{1}{4^n}.$$

So we apply (4) and (5) to the function

$$f(x) = \binom{2x}{x} \frac{1}{4^x} = \frac{\Gamma(2x+1)}{\Gamma^2(x+1)4^x}$$

to get the following:

Proposition 3.

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!4^n} \binom{2n}{n} = \int_0^1 \binom{2x}{x} \frac{1}{4^x} dx \approx 0.6703837612$$

and also for $m = 1, 2, ...$

$$\frac{1}{m!}\left(\frac{d}{dx}\right)^m \binom{2x}{x} \frac{1}{4^x}\bigg|_{x=0} = \sum_{n=1}^{\infty} \frac{(-1)^n s(n,m)}{n!4^n} \binom{2n}{n}.$$

For $m = 1, 2, 3$ we have correspondingly

(10)
$$\sum_{n=1}^{\infty} \frac{1}{n4^n} \binom{2n}{n} = \ln 4$$

$$\sum_{n=1}^{\infty} \frac{H_{n-1}}{n4^n} \binom{2n}{n} = \frac{\pi^2}{6} + 2\ln^2 2$$

$$\sum_{n=1}^{\infty} \frac{(H_{n-1}^2 - H_{n-1}^{(2)})}{n4^n} \binom{2n}{n} = 4\zeta(3) + \frac{8}{3}\ln^3(2) + \frac{2\pi^2}{3}\ln 2$$

as $s(n,2) = (-1)^{n-2}(n-1)!H_{n-1}$ and $s(n,3) = (-1)^{n-3}\frac{(n-1)!}{2}(H_{n-1}^2 - H_{n-1}^{(2)})$. Here

$$H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \quad (k \geq 1), \quad H_0 = 0$$

and

$$H_k^{(2)} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \quad (k \geq 1), \quad H_0^{(2)} = 0.$$

The first series (10) is known (see [**22**, Eq.(6)]). Moreover, evaluation of several infinite series involving products of harmonic numbers and central binomail coefficients can be found in [**11**, Corollary 9]. All these series are very slowly convergent.

The starting point is the binomial identity ([**5**, Eq.(10.9)])

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{p+k}{k} = (-1)^n \binom{p}{n},$$

where $p \geq 0$ is an integer. With the function $f(x) = \binom{p+x}{p}$ we find from (4) the identity

$$\int_0^1 \binom{p+x}{p} dx = \sum_{n=0}^{p} \frac{c_n}{n!} \binom{p}{n}.$$

We have

$$\binom{p+x}{p} = \frac{(-1)^p}{p!} \sum_{k=0}^{p+1} (-1)^{k-1} s(p+1,k)\, x^{k-1}.$$

Then (after changing the index $k = j+1$) we get

$$\begin{aligned} \int_0^1 \binom{p+x}{p} dx &= \frac{(-1)^p}{p!} \sum_{k=1}^{p+1} \frac{(-1)^{k-1} s(p+1,k)}{k} \\ &= \frac{(-1)^p}{p!} \sum_{j=0}^{p} \frac{(-1)^j s(p+1,j+1)}{j+1}. \end{aligned}$$

Using the explicit expression of the Cauchy polynomials of the second kind $\hat{c}_n(x)$ [**21**, Theorem 2]

$$\hat{c}_n(-r) = \sum_{k=0}^{n} s_r(n,k) \frac{(-1)^k}{k+1},$$

where $s_r(n,k)$ is the r-Stirling numbers of the first kind [**8**], and $s_1(n,k) = s(n+1,k+1)$, we obtain

$$\sum_{n=0}^{p} \frac{c_n}{n!} \binom{p}{n} = \frac{(-1)^p}{p!} \hat{c}_p(-1). \tag{11}$$

With the function $f(x) = \binom{p+x}{p}$, (5) implies

$$\frac{1}{m!} \left(\frac{d}{dx}\right)^m \binom{p+x}{p} \bigg|_{x=0} = \sum_{n=m}^{p} \frac{s(n,m)}{n!} \binom{p}{n}.$$

Moreover, it is known that [**23**, Theorem 2] ($p \geq m > 0$)

$$\left(\frac{d}{dx}\right)^m \binom{p+x}{r}\Big|_{x=0} = (-1)^m \binom{p}{r} Y_m \left(-0! \left(H_p - H_{p-r}\right), \ldots, -\left(m-1\right)! \left(H_p^{(m)} - H_{p-r}^{(m)}\right)\right),$$

where $H_n^{(m)}$ is the nth generalized harmonic number defined by

$$H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m} \quad (m \geq 1), \quad H_0^{(m)} = 0$$

and $Y_i\left(t_1, t_2, \ldots t_i\right)$ is the exponential complete Bell polynomial [**12**, Sect. 3.3]. Thus, we have

$$\text{(12)} \qquad \sum_{n=m}^{p} \frac{s\left(n,m\right)}{n!}\binom{p}{n} = \frac{(-1)^m}{m!} Y_m \left(-0! H_p, \ldots, -\left(m-1\right)! H_p^{(m)}\right).$$

The following proposition summarizes these results.

Proposition 4. *For every non-negative integers m and p, (11) and (12) are true.*

For $m = 1$ and $m = 2$, (12) reduce to [**5**, Eq. (9.2)]

$$\sum_{n=1}^{p} \binom{p}{n} \frac{(-1)^{n+1}}{n} = H_p$$

and [**10**, Eq. (2.8)]

$$\sum_{n=1}^{p} (-1)^{n+1} \binom{p+1}{n+1} \frac{H_n}{n+1} = \frac{H_{p+1}^2 - H_{p+1}^{(2)}}{2},$$

respectively. Moreover, for $m = 3$, we find that

$$\sum_{n=1}^{p} (-1)^{n-1} \binom{p+2}{n+2} \frac{H_{n+1}^2 - H_{n+1}^{(2)}}{n+2} = \frac{H_{p+2}^3 - 3H_{p+2}H_{p+2}^{(2)} + H_{p+2}^{(3)}}{3}.$$

Now, we give a new explicit formula, extending (3) and [**19**, Eq.(17)]. If $q > n$, $\binom{n}{q} = 0$. Then (6) can be rewritten as

$$\begin{aligned} q! \int_0^1 \binom{x}{q} (1-z)^x dx &= \sum_{n=q}^{\infty} \frac{(-1)^{n-q} c_n}{(n-q)!} z^{n-q} \left(1-z\right)^q \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n!} z^n \left(1-z\right)^q \end{aligned}$$

Setting $z \to 1 - e^{-t}$ in the above, we have

$$\sum_{n=0}^{\infty} c_{n+q} \frac{\left(e^{-t}-1\right)^n}{n!} e^{-tq} = q! \int_0^1 \binom{x}{q} e^{-xt} dx.$$

Using the generating function of the r-Stirling numbers of the second kind [**8**]

$$\frac{\left(e^{t}-1\right)^n}{n!} e^{tr} = \sum_{k=n}^{\infty} S_r\left(k,n\right) \frac{t^k}{k!}$$

gives

$$\sum_{n=0}^{\infty} c_{n+q} \frac{\left(e^{-t}-1\right)^n}{n!} e^{-tq} = \sum_{k=0}^{\infty} \frac{\left(-t\right)^k}{k!} \sum_{n=0}^{k} S_q\left(k,n\right) c_{n+q}.$$

On the other hand, using (2) and Maclaurin's expansion of e^x, we have

$$q! \int_0^1 \binom{x}{q} e^{-xt} dx = \sum_{k=0}^{\infty} \frac{\left(-t\right)^k}{k!} \sum_{m=0}^{q} \frac{s\left(q,m\right)}{k+m+1}.$$

Then comparing the coefficients of $\frac{(-t)^k}{k!}$ yields

$$\sum_{n=0}^{k} S_q\left(k,n\right) c_{n+q} = \sum_{m=0}^{q} \frac{s\left(q,m\right)}{k+m+1}.$$

Finally, employing the r-Stirling transform, given by

$$a_n = \sum_{k=0}^{n} S_r\left(n,k\right) b_k \ \ (n \geq 0) \text{ if and only if } b_n = \sum_{k=0}^{n} s_r\left(n,k\right) a_k \ \ (n \geq 0),$$

we obtain the following:

Proposition 5. *For any non-negative integers k and q*

$$c_{k+q} = \sum_{n=0}^{k} \sum_{m=0}^{q} \frac{s_q\left(k,n\right) s\left(q,m\right)}{n+m+1}.$$

Now, we evaluate an infinite series involving Cauchy numbers with shifted indices.

Proposition 6. *For every non-negative integer q*

$$\sum_{n=0}^{\infty} \frac{\left(-1\right)^n c_{n+q}}{n!\left(n+q+1\right)\cdots\left(n+2q+1\right)} = \sum_{j=0}^{q} \left(-1\right)^{q+j} \binom{q}{j} \binom{q+j}{j} \ln\left(\frac{j+2}{j+1}\right).$$

Proof. (6) can be rewritten as

$$z^q \int_0^1 \binom{x}{q} (1-z)^x dx = \sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n!q!} z^{n+q} (1-z)^q .$$

Integrating both sides of the above equation with respect to z from 0 to 1 and using well-known identity

$$B(p,q) = \int_0^1 z^{q-1} (1-z)^{p-1} dz = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n!(n+q+1)\cdots(n+2q+1)} = q! \int_0^1 \binom{x}{q} \frac{\Gamma(x+1)}{\Gamma(x+q+2)} dx.$$

With the use of (2), the properties $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(n+1) = n!$ $(n \in \mathbb{N})$ and

(13) $$\frac{1}{(x+1)\cdots(x+q+1)} = \frac{1}{(q+1)!} \sum_{j=1}^{q+1} (-1)^{j-1} \binom{q+1}{j} \frac{j}{x+j},$$

we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n!(n+q+1)\cdots(n+2q+1)}$$
$$= \frac{1}{(q+1)!} \sum_{j=1}^{q+1} \sum_{k=0}^{q} (-1)^{j-1} \binom{q+1}{j} s(q,k)\, j \int_0^1 \frac{x^k}{x+j} dx.$$

One can see that

$$\int_0^1 \frac{x^k}{x+j} dx = (-1)^k j^k \ln\left(\frac{j+1}{j}\right) + \sum_{m=1}^{k} \binom{k}{m} \frac{(-j)^{k-m}}{m} \left((j+1)^m - j^m\right).$$

Then we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_{n+q}}{n!(n+q+1)\cdots(n+2q+1)}$$
$$= \frac{1}{(q+1)!} \sum_{j=1}^{q+1} (-1)^{j-1} \binom{q+1}{j} \ln\left(\frac{j+1}{j}\right) \sum_{k=0}^{q} (-1)^{q-k} s(q,k)\, j^{k+1}$$
$$+ \frac{1}{(q+1)!} \sum_{j=1}^{q+1} \sum_{k=0}^{q} \sum_{m=1}^{k} (-1)^{j+m-1} \binom{q+1}{j} s(q,k) \binom{k}{m} \frac{j^{k-m+1}}{m}$$
$$\cdot \left((j+1)^m - j^m\right).$$

The second sum of the right-hand side is zero since [**5**, Eq. (4.15)]

$$\sum_{k=1}^{n}(-1)^{k}\binom{n}{k}k^{j}=0 \text{ for } j<n.$$

Moreover, using

$$\sum_{k=0}^{q}(-1)^{q-k}s(q,k)\,j^{k}=j(j+1)\cdots(j+q-1),$$

we come to the desired result. □

In the next proposition we use the reciprocal binomial coefficients $\binom{n+l}{l}^{-1}$, where l is any non-negative integer. We take $f(x)=\frac{1}{(x+l+1)\cdots(x+l+r)}$ and use [**20**, Proposition 2],

$$\frac{1}{(r-1)!\,(n+r+l)\binom{n+r+l-1}{l}}=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}}{(k+l+1)\cdots(k+l+r)}$$

and (13). Thus (4) and (5) imply the following result.

Proposition 7. *For any integer $r\geq 1$*

$$\sum_{n=0}^{\infty}\frac{(-1)^{n}c_{n}}{n!\,(n+r+l)\binom{n+r+l-1}{l}}=\sum_{j=1}^{r}(-1)^{j-1}\binom{r-1}{j-1}\ln\left(\frac{l+j+1}{l+j}\right),$$

$$\sum_{n=m}^{\infty}\frac{(-1)^{n-m}s(n,m)}{n!\,(n+r+l)\binom{n+r+l-1}{l}}=\sum_{j=1}^{r}\binom{r-1}{j-1}\frac{(-1)^{j-1}}{(l+j)^{m+1}}. \tag{14}$$

Setting $l=0$ in (14), we reach that

$$\sum_{n=m}^{\infty}\frac{(-1)^{n-m}s(n,m)}{n!\,(n+r)}=(-1)^{r+1}(r-1)!S(-m,r), \tag{15}$$

where $S(-n,r)$ is the Stirling numbers of the second kind with negative integral values, defined by [**3**]

$$\frac{(-1)^{r}}{r!}\sum_{j=1}^{r}\binom{r}{j}\frac{(-1)^{j}}{j^{n}}=S(-n,r). \tag{16}$$

With the use of [**23**, Theorem 4], we can list some special cases as follows:

$$S(0,r)=\frac{(-1)^{r+1}}{r!},\ S(-1,r)=\frac{(-1)^{r+1}}{r!}H_{r},\ S(-2,r)=\frac{(-1)^{r+1}}{2r!}\left(H_{r}^{2}+H_{r}^{(2)}\right).$$

It is good to note that by choosing appropriate m and r in (15) Corollary 3 and Conjecture 4 in [**11**] is obtained. This conjecture was also answered in [**26**, Corollary 2.4].

(See [**2, 18, 24, 25**] for more examples of series involving Stirling numbers of the first kind.)

3. SERIES WITH SKEW-HARMONIC NUMBERS

We work here with the skew-harmonic numbers

$$H_n^- = 1 - \frac{1}{2} + \frac{1}{3} + ... + \frac{(-1)^{n-1}}{n} \quad (n \geq 1), \quad H_0^- = 0.$$

Applying the binomial formula [**5**, Eq.(9.21)]

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k \frac{1-2^k}{k} = H_n^- \tag{17}$$

we use the (entire) function

$$f(x) = \frac{1-2^x}{x} = -\sum_{n=0}^{\infty} \frac{(\ln 2)^{n+1} x^n}{(n+1)!}$$

where $f(0) = -\ln 2$. With summation from $k = 0$ the binomial formula (17) takes the form

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = -\ln 2 + H_n^-.$$

Proposition A implies

$$\int_0^1 \frac{1-2^x}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \left\{ -\ln 2 + H_n^- \right\}.$$

The integral can be computed this way: with $t = x \ln 2$

$$\int_0^1 \frac{2^x - 1}{x} dx = \sum_{n=1}^{\infty} \frac{(\ln 2)^n}{n! n} = -Ein(-\ln 2)$$

where

$$Ein(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n! n}$$

is the entire exponential integral function. This gives the evaluation

Proposition 8.

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \left\{ H_n^- - \ln 2 \right\} = Ein(-\ln 2).$$

Note that

$$\lim_{n\to\infty} \left\{ H_n^- - \ln 2 \right\} = 0$$

since $\lim_{n\to\infty} H_n^- = \ln 2$.

4. SERIES WITH HYPERHARMONIC NUMBERS

In this section, we work with the hyperharmonic numbers which are defined by the equation [**13**]

$$h_n^{(r)} = \binom{n+r-1}{r-1} \left(H_{n+r-1} - H_{r-1} \right).$$

Please see [**4, 9, 13, 15, 16**] for more detail on hyperharmonic numbers.

Let $r \geq 1$ be an integer. We use the function $f(x) = \frac{1}{(x+1)^2 (x+2) \cdots (x+r)}$ together with the binomial identity [**20**, Theorem 3]

$$\frac{h_{n+1}^{(r)}}{(n+1)\cdots(n+r)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^2 (k+2) \cdots (k+r)}$$

in Proposition A and Proposition B to obtain the following proposition.

Proposition 9.

$$\sum_{n=0}^{\infty} \frac{(-1)^n c_n h_{n+1}^{(r)}}{(n+r)!} = \frac{1}{2(r-1)!} - \frac{r-1}{r!} \ln(2) + \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} \frac{(-1)^{j+1}}{j-1} \ln\left(\frac{2j^j}{(j+1)^j} \right) \tag{18}$$

and

$$\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n,m) h_{n+1}^{(r)}}{(n+r)!} = (-1)^{r+1} \sum_{k=0}^{m} S(-k,r). \tag{19}$$

Proof. One have

$$\frac{1}{(x+1)^2 (x+2) \cdots (x+r)} = \frac{1}{(r-1)!\,(x+1)^2} - \frac{r-1}{r!} \frac{1}{x+1} + \frac{1}{r!} \sum_{j=2}^{r} \binom{r}{j} \frac{(-1)^{j+1}}{j-1} \left(\frac{1}{x+1} - \frac{j}{x+j} \right). \tag{20}$$

Using Proposition A in the above equation yields (18).

For the proof of (19), we first use

$$\frac{1}{(1+x)^{\alpha}}=\sum_{m=0}^{\infty}(-1)^{m}\binom{\alpha+m-1}{m}x^{m}$$

in (20) to obtain

$$\begin{aligned}\sum_{n=m}^{\infty}\frac{(-1)^{n-m}s(n,m)\,h_{n+1}^{(r)}}{(n+r)!}&=\frac{rm+1}{r!}+\frac{1}{r!}\sum_{j=2}^{r}\binom{r}{j}\frac{(-1)^{j+1}}{j-1}\left(1-\frac{1}{j^{m}}\right)\\&=\frac{rm+1}{r!}+\frac{1}{r!}\sum_{j=2}^{r}\binom{r}{j}\frac{(-1)^{j+1}}{j^{m}}\left(\frac{j^{m}-1}{j-1}\right).\end{aligned}$$

Then using $\sum_{k=1}^{m}x^{k}=(1-x^{m})/(1-x)$ in the above equation we have

$$\begin{aligned}\sum_{n=m}^{\infty}\frac{(-1)^{n-m}s(n,m)\,h_{n+1}^{(r)}}{(n+r)!}&=\frac{rm+1}{r!}+\frac{1}{r!}\sum_{j=2}^{r}\binom{r}{j}\frac{(-1)^{j+1}}{j^{m}}\sum_{k=0}^{m-1}j^{k}\\&=\frac{rm+1}{r!}+\frac{1}{r!}\left\{-mr+\sum_{j=1}^{r}\binom{r}{j}\frac{(-1)^{j+1}}{j^{m}}\sum_{k=0}^{m-1}j^{k}\right\}\\&=\frac{1}{r!}+\frac{1}{r!}\sum_{k=0}^{m-1}\sum_{j=1}^{r}\binom{r}{j}\frac{(-1)^{j+1}}{j^{m-k}}.\end{aligned}$$

With the use of (16), we come to the desired result. □

It is good to note that the case $r=1$ of (18) and (19) are given in [**7**]. Moreover, setting $r=2$ in (18), $m=1$ and $m=2$ in (19) give

$$\sum_{n=0}^{\infty}\frac{(-1)^{n}c_{n}H_{n+2}}{(n+1)!}=\ln 3-\ln 2+\frac{1}{2},$$

$$\sum_{n=1}^{\infty}\frac{h_{n+1}^{(r)}}{n\binom{n+r}{r}}=1+H_{r},\qquad \sum_{n=2}^{\infty}\frac{H_{n-1}h_{n+1}^{(r)}}{n\binom{n+r}{r}}=1+H_{r}+\frac{H_{r}^{2}+H_{r}^{(2)}}{2},$$

respectively.

Now, we exploit the binomial formula [**20**, Theorem 9]

$$\frac{-h_{n}^{(r)}}{(n+1)\cdots(n+r)}=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{k}}{(k+1)\cdots(k+r)}H_{k},$$

and the function

$$f(x)=\frac{\psi(x+1)+\gamma}{(x+1)\cdots(x+r)}.$$

With the use of (13) and well-known identity [**1**, Eq. (6.3.14)]

$$\psi(x+1)+\gamma=\sum_{n=1}^{\infty}(-1)^{n+1}\zeta(n+1)x^n,\quad |x|<1. \tag{21}$$

we have

$$\frac{\psi\left(x+1\right)+\gamma}{\left(x+1\right)\cdots\left(x+r\right)}=\frac{1}{r!}\sum_{j=1}^{r}\left(-1\right)^{j-1}\binom{r}{j}\sum_{n=1}^{\infty}\left(-1\right)^{n+1}x^n\left(\sum_{k=1}^{n}\frac{\zeta\left(k+1\right)}{j^{n-k}}\right)$$

Then using (16) and (5) give the following evaluation.

Proposition 10.

$$\sum_{n=m}^{\infty}\frac{\left(-1\right)^{n-m}s\left(n,m\right)h_n^{(r)}}{\left(n+r\right)!}=\left(-1\right)^{r+1}\sum_{k=1}^{m}S\left(k-m,r\right)\zeta\left(k+1\right). \tag{22}$$

The case $r=1$ was discussed in [**7**]. Setting $m=1$ in (22) gives [**14**, Eq.(27)]. Moreover, $m=2$ and $m=3$ in (22) yield

$$\sum_{n=2}^{\infty}\frac{H_{n-1}h_n^{(r)}}{n\binom{n+r}{r}}=H_r\frac{\pi^2}{6}+\zeta\left(3\right),$$

$$\sum_{n=3}^{\infty}\frac{(H_{n-1}^2-H_{n-1}^{(2)})h_n^{(r)}}{n\binom{n+r}{r}}=\left(H_r^2+H_r^{(2)}\right)\frac{\pi^2}{6}+2H_r\zeta\left(3\right)+\frac{\pi^4}{45}$$

respectively.

Let take

$$f\left(x\right)=\frac{\psi\left(x+1\right)+\gamma}{\left(x+1\right)^2\left(x+2\right)\cdots\left(x+r\right)}$$

and use the identity [**20**, Theorem 11]

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\left(-1\right)^{k+1}}{\left(k+1\right)\left(k+1\right)^{(r)}}H_k$$
$$=\frac{1}{2\left(n+1\right)\left(r-1\right)!}\left\{\left(H_{n+r}-H_{r-1}\right)^2-\left(H_{n+r}^{(2)}-H_{r-1}^{(2)}\right)\right\}.$$

Using (20) and (21), we have

$$\begin{aligned}
&\frac{\psi(x+1)+\gamma}{(x+1)^2(x+2)\cdots(x+r)}\\
&\quad=\frac{1}{(r-1)!}\left[\sum_{m=1}^{\infty}(-1)^{m-1}x^m\sum_{k=1}^{m}(m-k+1)\zeta(k+1)\right]\\
&\quad+\frac{1-r}{r!}\left[\sum_{m=1}^{\infty}(-1)^{m-1}x^m\sum_{k=1}^{m}\zeta(k+1)\right]\\
&\quad+\frac{1}{r!}\sum_{j=2}^{r}\binom{r}{j}\frac{(-1)^{j+1}}{j-1}\left[\sum_{m=1}^{\infty}(-1)^{m-1}x^m\sum_{k=1}^{m}\zeta(k+1)\right]\\
&\quad+\frac{1}{r!}\sum_{j=2}^{r}\binom{r}{j}\frac{(-1)^{j}}{(j-1)}\left[\sum_{m=1}^{\infty}(-1)^{m-1}x^m\right]\left[\sum_{k=1}^{m}\frac{\zeta(k+1)}{j^{m-k}}\right].
\end{aligned}$$

From (5) and some arrangements we obtain

$$\begin{aligned}
&\sum_{n=m}^{\infty}\frac{(-1)^{n-m+1}s(n,m)\left[(H_{n+r}-H_{r-1})^2-\left(H_{n+r}^{(2)}-H_{r-1}^{(2)}\right)\right]}{2(n+1)!}\\
&\quad=\frac{1}{r}\sum_{k=1}^{m}\zeta(k+1)+\frac{1}{r}\sum_{k=1}^{m}\zeta(k+1)\sum_{j=1}^{r}\binom{r}{j}\frac{(-1)^{j+1}}{j^{m-k}}\left(1+j+\cdots+j^{m-k-1}\right).
\end{aligned}$$

Then using (16) yields the following:

Proposition 11.

$$\sum_{n=m}^{\infty}\frac{(-1)^{n-m+1}s(n,m)\left[(H_{n+r}-H_{r-1})^2-\left(H_{n+r}^{(2)}-H_{r-1}^{(2)}\right)\right]}{2(n+1)!}=\frac{1}{r}\sum_{k=0}^{m-1}A_k(r)\zeta(m-k+1) \tag{23}$$

where

$$A_k(r)=1+(-1)^{r+1}r!\sum_{l=1}^{k}S(-l,r).$$

Since $S(-l,1)=1$, for $r=1$ in (23), we have

$$\sum_{n=m}^{\infty}\frac{(-1)^{n-m}s(n,m)\left[H_{n+1}^{(2)}-H_{n+1}^2\right]}{2(n+1)!}=\sum_{k=1}^{m}(m-k+1)\zeta(k+1). \tag{24}$$

On the other hand, we have [**7**, Eq.(22)]

$$\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n,m) \left[H_{n+1}^2 + H_{n+1}^{(2)}\right]}{2(n+1)!} = (m+1)(m+2).$$

Combining this result with (24), we reach that

$$\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n,m) H_{n+1}^{(2)}}{2(n+1)!} = (m+1)(m+2) + \sum_{k=1}^{m} (m-k+1)\zeta(k+1),$$
$$\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n,m) H_{n+1}^{2}}{2(n+1)!} = (m+1)(m+2) - \sum_{k=1}^{m} (m-k+1)\zeta(k+1)$$

Setting $m = 1, 2, 3$ in the above give

$$\sum_{n=1}^{\infty} \frac{H_{n+1}^{(2)}}{n(n+1)} = 12 + \frac{\pi^2}{3}, \quad \sum_{n=1}^{\infty} \frac{H_{n+1}^{2}}{n(n+1)} = 12 - \frac{\pi^2}{3},$$
$$\sum_{n=2}^{\infty} \frac{H_{n-1}H_{n+1}^{(2)}}{n(n+1)} = 24 + \frac{2\pi^2}{3} + 2\zeta(3),$$
$$\sum_{n=2}^{\infty} \frac{H_{n-1}H_{n+1}^{2}}{n(n+1)} = 24 - \frac{2\pi^2}{3} - 2\zeta(3)$$
$$\sum_{n=3}^{\infty} \frac{(H_{n-1}^2 - H_{n-1}^{(2)})H_{n+1}^{(2)}}{n(n+1)} = 80 + 2\pi^2 + 8\zeta(3) + \frac{2\pi^4}{45},$$
$$\sum_{n=3}^{\infty} \frac{(H_{n-1}^2 - H_{n-1}^{(2)})H_{n+1}^{2}}{n(n+1)} = 80 - 2\pi^2 - 8\zeta(3) - \frac{2\pi^4}{45}.$$

Let $r > 1$ be an integer. Then we use the identity [**14**]

$$\frac{-h_n^{(r)}}{\binom{n+r-1}{n}^2} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k}{(k+r-1)^2}$$

together with the function $f(x) = \frac{x}{(x+r-1)^2}$. For $|x| < |r-1|$ we have the Taylor series

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (m+1)}{(r-1)^{m+2}} x^{m+1}.$$

From (4) and (5) we find the representations below:

Proposition 12. *Let $r > 1$ be an integer. Then we have*

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} c_n h_n^{(r)}}{n! \binom{n+r-1}{n}^2} = \ln\left(\frac{r}{r-1}\right) - \frac{1}{r},$$

$$\sum_{n=m}^{\infty} \frac{(-1)^{n-m} s(n,m) h_n^{(r)}}{n! \binom{n+r-1}{n}^2} = \frac{m}{(r-1)^{m+1}}. \tag{25}$$

For $m = 1$ and $m = 2$ in (25) we find that

$$\sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n \binom{n+r-1}{n}^2} = \frac{1}{(r-1)^2} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{H_{n-1} h_n^{(r)}}{n \binom{n+r-1}{n}^2} = \frac{2}{(r-1)^3}.$$

Remark 13. *At the end of this section we want to note that if we apply (4) to the functions and binomial identities given just above the Propositions 10 and 11, we come to very challenging integrals.*

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} c_n h_n^{(r)}}{(n+r)!} = \int_0^1 \frac{\psi(x+1) + \gamma}{(x+1)\cdots(x+r)} dx,$$

and

$$\frac{1}{(r-1)!} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} c_n \left\{ (H_{n+r} - H_{r-1})^2 - \left(H_{n+r}^{(2)} - H_{r-1}^{(2)} \right) \right\}}{(n+1)!\,(n+r)!}$$
$$= \int_0^1 \frac{\psi(x+1) + \gamma}{(x+1)^2 (x+2)\cdots(x+r)} dx,$$

respectively. The first integral when $r = 1$ is

$$\int_0^1 \frac{\psi(x+1) + \gamma}{x+1} dx.$$

*This integral also occurs in [**6**]:*

$$\int_0^1 \frac{\psi(x+1) + \gamma}{x+1} dx = \frac{1}{2} + \sum_{n=2}^{\infty} \left\{ \frac{(-1)^n}{n} \sum_{k=2}^{n} \zeta(k) - n \right\} \approx 0.3606201929,$$

and

$$\int_0^1 \frac{\psi(x+1) + \gamma}{x+1} dx = \frac{-1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{(n+1)!\,(n+1)}.$$

*Please see [**6**] for more represantations of this integral.*

Acknowledgements. The authors are grateful to the anonymous referees and the editor for their constructive and encouraging comments, which helped to improve this article.

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(Received 12. 01. 2021.)
(Revised 04. 02. 2023.)

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