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NEW SHARP INEQUALITIES OF MITRINOVIĆ-ADAMOVIĆ TYPE

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In this paper, new sharp Mitrinović-Adamović inequalities for circular functions are established.

1. INTRODUCTION

The following inequality

$$\frac{\sin x}{x} < \frac{2+\cos x}{3}, \qquad 0 < x < \frac{\pi}{2}. \tag{1}$$

is known as the Cusa-Huygens inequality. This inequality has been extended and sharpened in many different ways. In this regard, we may refer to [**3, 4**]. For example, in [**4**], the authors find the necessary and sufficient conditions such that the inequalities

$$\frac{\sin x}{x} > a + b\cos^c x, x \in (0, T)$$

and

$$\frac{\sin x}{x} < a + b\cos^c x, x \in (0, T)$$

where $a, b, c \in \mathbb{R}$ and $T \in (0, \pi/2]$.

Recently, Bagul et al. [**5**] drew two conclusions about the improvement of inequality (1).

$$-\left(\frac{2}{3} - \frac{2}{\pi}\right)\Phi_1(x) < \frac{\sin x}{x} - \frac{2+\cos x}{3} < -\left(\frac{2}{3} - \frac{2}{\pi}\right)\Phi_2(x),$$

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where $\Phi_1(x) = (\pi/2-1)^{-1}(x-\sin x)$ and $\Phi_2(x) = (\pi/2-1)^{-2}(x-\sin x)^2$.

$$-\left(\frac{2}{3}-\frac{2}{\pi}\right)\Psi_1(x) < \frac{\sin x}{x} - \frac{2+\cos x}{3} < -\left(\frac{2}{3}-\frac{2}{\pi}\right)\Psi_2(x),$$

where $\Psi_1(x) = (\sin x - x\cos x)$ and $\Psi_2(x) = (\sin x - x\cos x)^2$.

In literature, the inequality

$$\cos x < \left(\frac{\sin x}{x}\right)^3, \qquad 0 < x < \frac{\pi}{2}.$$

is known as Mitrinović-Adamović inequality (see[**17, 18**]).

In recent years, many researchers have paid attention to the improvements of the above-mentioned inequality.

In [**11, 21, 26**], a better lower bound for $\left(\frac{\sin x}{x}\right)^3$ was given as follows:

$$\cos^4\frac{x}{2} < \left(\frac{\sin x}{x}\right)^3, \qquad 0 < x < \frac{\pi}{2}.$$

Mortici[**19**] gave the following double inequality

$$\cos x + \frac{1}{15}x^4 - \frac{23}{1890}x^6 < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{1}{15}x^4, \quad 0 < x < \frac{\pi}{2}. \tag{2}$$

Chouikha [**10**] proved the following double inequality

$$\cos x + x^3\left(1-\frac{x^2}{63}\right)\frac{\sin x}{15} < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{x^3\sin x}{15}, 0 < x < \frac{\pi}{2}. \tag{3}$$

In [**33**, Lemma 7], Zhu proved the following results: the double inequality

$$1 - \frac{\pi^3-8}{\pi^3}\sin^2 x < \left(\frac{\sin x}{x}\right)^3 < 1 - \frac{1}{2}\sin^2 x. \tag{4}$$

holds for all $x \in (0, \pi/2)$, the constants $(\pi^3-8)/\pi^3$ and $1/2$ are best possible.

For more information on this topic, please see [**7, 8, 9, 21, 22, 12, 24, 20, 25, 27, 28, 34, 29, 30, 31, 32**] and closely related references therein.

The aim of this paper is to obtain some new upper and lower bounds of $\left(\frac{\sin x}{x}\right)^3$, which improve several known results.

Our main results can be formulated in details as the following theorems.

Theorem 1. *The function*

$$F(x) = \frac{1}{x^2}\left[\frac{x^3\sin x}{\left(\frac{\sin x}{x}\right)^3 - \cos x} - 15\right]$$

is decreasing on $(0,\pi)$. In paprticular, we have

(i)The double inequality

$$\cos x+\frac{x^3\sin x}{15+\frac{5}{21}x^2}<\left(\frac{\sin x}{x}\right)^3<\cos x+\frac{x^3\sin x}{15+\frac{\pi^6-960}{16\pi^2}x^2}. \tag{5}$$

holds for all $x\in(0,\frac{\pi}{2})$, the constants $\frac{5}{21}$ and $\frac{\pi^6-960}{16\pi^2}$ are the best possible.

(ii)The double inequality

$$\cos x+\frac{21}{5}\frac{x^3\sin x}{63+x^2}<\left(\frac{\sin x}{x}\right)^3<\cos x+\frac{\pi^2}{15}\frac{x^3\sin x}{\pi^2-x^2}. \tag{6}$$

holds for all $x\in(0,\pi)$, and the constants $\frac{21}{5}$ and $\frac{\pi^2}{15}$ are the best possible.

Theorem 2. *For $x\in(0,\pi/2)$, we have*

$$1-\left(\frac{1}{2}+\frac{\pi^3-16}{\pi^4}x\sin x\right)\sin^2 x<\left(\frac{\sin x}{x}\right)^3<1-\left(\frac{1}{2}+\frac{7}{120}x\sin x\right)\sin^2 x. \tag{7}$$

The constants $\frac{7}{120}$ and $\frac{\pi^3-16}{\pi^4}$ are the best possible.

2. LEMMAS

In order to establish our main results we need several lemmas, which we present in this section.

Lemma 3. *For $|x|<\pi$, B_{2n} be the even-indexed Bernoulli number. Then we have the following power series formulas*

$$\frac{x}{\sin x}=1+\sum_{n=1}^{\infty}\frac{2\big(2^{2n-1}-1\big)|B_{2n}|}{(2n)!}x^{2n}. \tag{8}$$

$$\cot x=\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}. \tag{9}$$

$$\frac{1}{\sin^2 x}=\frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!}x^{2(n-1)}. \tag{10}$$

$$\frac{\cos x}{\sin^2 x}=\frac{1}{x^2}-\sum_{n=1}^{\infty}\frac{2(2n-1)\big(2^{2n-1}-1\big)|B_{2n}|}{(2n)!}x^{2(n-1)}. \tag{11}$$

$$\begin{aligned}\frac{1}{\sin^3 x} =& \frac{1}{x^3} + \frac{1}{2}\sum_{n=2}^{\infty}\frac{(2^{2n}-2)(2n-1)(2n-2)}{(2n)!}|B_{2n}|x^{2n-3} \\ &+ \frac{1}{2x} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-1}.\end{aligned} \tag{12}$$

$$\frac{\cos x}{\sin^3 x} = \frac{1}{x^3} - \sum_{n=2}^{\infty}\frac{(2n-1)(n-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-3}. \tag{13}$$

Proof. The power series formulas (8) and (9) can be found in [**1**, p. 75, 4.3.68] and [**1**, p. 75, 4.3.70], and the power series formulas (10) and (11) can be obtained from (8) and (9) together with the facts that

$$\frac{1}{\sin^2 x} = \csc^2 x = -(\cot x)',$$

and

$$\frac{\cos x}{\sin^2 x} = -\left(\frac{1}{\sin x}\right)'.$$

(12) can be obtained from (8) and (11) together with

$$\frac{1}{\sin^3 x} = \frac{1}{2\sin x} - \frac{1}{2}\left(\frac{\cos x}{\sin^2 x}\right)'.$$

(13) can be obtained from (10) together with the facts that

$$\frac{\cos x}{\sin^3 x} = -\frac{1}{2}\left(\frac{1}{\sin^2 x}\right)'.$$

□

Lemma 4. *[**2, 6**] Let a_n and b_n $(n = 0, 1, 2, \ldots)$ be real numbers, and let the power series $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t| < R$. If $b_n > 0$ for $n = 0, 1, 2, \ldots,$ and if $\frac{a_n}{b_n}$ is strictly increasing (or decreasing) for $n = 0, 1, 2, \ldots$, then the function $\frac{A(t)}{B(t)}$ is strictly increasing (or decreasing) on $(0, R)$.*

The next lemma gives the sharp lower and upper bounds for a ratio involving absolute Bernoulli numbers, which was established in [**23**].

Lemma 5. *For $n \in \mathbb{N}$, the Bernoulli numbers satisfy*

$$\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2} < \frac{|B_{2n+2}|}{|B_{2n}|} < \frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+2)(2n+1)}{\pi^2}.$$

3. PROOF OF THEOREM 1

Proof. Consider

$$\begin{aligned}F(x) &= \frac{1}{x^2}\left[\frac{x^3\sin x}{\left(\frac{\sin x}{x}\right)^3-\cos x}-15\right]\\ &= \frac{x^6\frac{1}{\sin^2 x}+15x^3\frac{\cos x}{\sin^3 x}-15}{x^2(1-x^3\frac{\cos x}{\sin^3 x})}\\ &:= \frac{f_1(x)}{f_2(x)}, \quad 0<x<\pi.\end{aligned}$$

where

$$\begin{aligned}f_1(x) &= x^6\frac{1}{\sin^2 x}+15x^3\frac{\cos x}{\sin^3 x}-15\\ f_2(x) &= x^2(1-x^3\frac{\cos x}{\sin^3 x})\end{aligned}$$

By (10) and (13), we have

$$\begin{aligned}f_1(x) &= x^6\left(\frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!}x^{2(n-1)}\right)\\ &+15x^3\left(\frac{1}{x^3}-\sum_{n=2}^{\infty}\frac{(2n-1)(n-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-3}\right)-15\\ &= \sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!}x^{2n+4}-\sum_{n=3}^{\infty}\frac{15(2n-1)(n-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n}\Bigg)\\ &= \sum_{n=2}^{\infty}\left[\frac{2^{2n-2}(2n-3)2n(2n-1)|B_{2n-2}|}{(2n)!}-\frac{15(2n+1)n2^{2n+2}|B_{2n+2}|}{(2n+2)!}\right]x^{2n+2}\\ &=: \sum_{n=2}^{\infty}a_n x^{2n+2},\end{aligned}$$

and

$$\begin{aligned}f_2(x) &= x^2-x^5\left(\frac{1}{x^3}-\sum_{n=2}^{\infty}\frac{(2n-1)(n-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n-3}\right)\\ &= \sum_{n=2}^{\infty}\frac{(2n-1)(n-1)2^{2n}|B_{2n}|}{(2n)!}x^{2n+2}\\ &=: \sum_{n=2}^{\infty}b_n x^{2n+2},\end{aligned}$$

where

$$a_n = \frac{2^{2n-2}(2n-3)2n(2n-1)|B_{2n-2}|}{(2n)!} - \frac{15(2n+1)n2^{2n+2}|B_{2n+2}|}{(2n+2)!}$$
$$b_n = \frac{(2n-1)(n-1)2^{2n}|B_{2n}|}{(2n)!}$$

Let

$$c_n = \frac{a_n}{b_n} = \frac{n(2n-3)}{2(n-1)}\frac{|B_{2n-2}|}{|B_{2n}|} - \frac{30n}{(2n-1)(n^2-1)}\frac{|B_{2n+2}|}{|B_{2n}|}, \quad n \geq 2.$$

A direct computation gives $c_2 = \frac{5}{21}, c_3 = 0$, and therefore, $c_2 - c_3 > 0$.

For $n \geq 3$, using Lemma 5 yields

$$c_n > p'_n\pi^2 - \frac{q'_n}{\pi^2},$$
$$c_{n+1} < p''_n\pi^2 - \frac{q''_n}{\pi^2},$$

where

$$p'_n = \frac{2n-3}{4(n-1)(2n-1)}\frac{2^{2n}-1}{2^{2n-2}-1}, q'_n = \frac{30n(2n+1)(2n+2)}{(2n-1)(n^2-1)}\frac{2^{2n}-1}{2^{2n+2}-1},$$
$$p''_n = \frac{(n+1)(2n-1)}{2n(2n+1)(2n+2)}\frac{2^{2n+1}-1}{2^{2n-1}-1}, q''_n = \frac{30(n+1)(2n+3)(2n+4)}{n(2n+1)(n+2)}\frac{2^{2n+1}-1}{2^{2n+3}-1}.$$

Then

$$c_n - c_{n+1} > (p'_n - p'_n)\pi^2 - \frac{q'_n - q''_n}{\pi^2}.$$

Since

$$\frac{2^{2n}-1}{2^{2n-2}-1} - \frac{2^{2n+1}-1}{2^{2n-1}-1} = \frac{3 \times 2^{2n-2}}{(2^{2n-1}-1)(2^{2n-2}-1)} > 0,$$
$$\frac{2^{2n}-1}{2^{2n+2}-1} - \frac{2^{2n+1}-1}{2^{2n+3}-1} = -\frac{3 \times 2^{2n}}{(2^{2n+3}-1)(2^{2n+2}-1)} < 0,$$

we have

$$\begin{aligned} p'_n - p''_n &= \frac{2n-3}{4(n-1)(2n-1)}\frac{2^{2n}-1}{2^{2n-2}-1} - \frac{(n+1)(2n-1)}{2n(2n+1)(2n+2)}\frac{2^{2n+1}-1}{2^{2n-1}-1} \\ &> \left[\frac{2n-3}{4(n-1)(2n-1)} - \frac{(n+1)(2n-1)}{2n(2n+1)(2n+2)}\right]\frac{2^{2n+1}-1}{2^{2n-1}-1} \\ &= \frac{4n^2-8n+1}{4n(2n-1)(2n+1)(n-1)}\frac{2^{2n+1}-1}{2^{2n-1}-1}, \end{aligned}$$

$$
\begin{aligned}
q_n' - q_n'' &= \frac{30n(2n+1)(2n+2)}{(2n-1)(n^2-1)}\frac{2^{2n}-1}{2^{2n+2}-1} - \frac{30(n+1)(2n+3)(2n+4)}{n(2n+1)(n+2)}\frac{2^{2n+1}-1}{2^{2n+3}-1}\\
&> \left[\frac{30n(2n+1)(2n+2)}{(2n-1)(n^2-1)} - \frac{30(n+1)(2n+3)(2n+4)}{n(2n+1)(n+2)}\right]\frac{2^{2n+1}-1}{2^{2n+3}-1}\\
&= \frac{60(8n^2+4n-3)}{n(2n-1)(2n+1)(n-1)}\frac{2^{2n+1}-1}{2^{2n+3}-1}.
\end{aligned}
$$

In view of $\pi^4/60 = 1.623\cdots > 8/5$, it follows that

$$
\begin{aligned}
&c_n - c_{n+1} > (p_n' - p_n')\pi^2 - \frac{q_n' - q_n''}{\pi^2}\\
&> \frac{4n^2-8n+1}{4n(2n-1)(2n+1)(n-1)}\frac{2^{2n+1}-1}{2^{2n-1}-1}\pi^2 - \frac{60}{\pi^2}\frac{(8n^2+4n-3)}{n(2n-1)(2n+1)(n-1)}\frac{2^{2n+1}-1}{2^{2n+3}-1}\\
&> \frac{60}{\pi^2}\frac{2^{2n+1}-1}{n(n-1)(2n-1)(2n+1)}\left(\frac{4n^2-8n+1}{4}\frac{1}{2^{2n-1}-1}\frac{8}{5} - \frac{8n^2+4n-3}{2^{2n+3}-1}\right)\\
&= \frac{12}{\pi^2}\frac{2^{2n+1}-1}{n(n-1)(2n-1)(2n+1)}\frac{(88n^2-276n+47)2^{2n}+(64n^2+72n-34)}{(2^{2n+3}-1)(2^{2n}-2)} > 0
\end{aligned}
$$

for $n \geq 3$, where the inequality holds due to the coefficient of 2^{2n}

$$
88n^2 - 276n + 47 = 88(n-3)^2 + 252(n-3) + 11 > 0
$$

for $n \geq 3$. which means that the sequence c_n is decreasing. By Lemma 4, we deduce the function $F(x)$ is decreasing on $(0, \pi)$. Moreover, it is easy to obtain

$$
\lim_{x\to 0^+} F(x) = c_2 = \frac{5}{21}, \lim_{x\to\pi/2^-} F(x) = \frac{\pi^6-960}{16\pi^2}, \lim_{x\to\pi^-} F(x) = -\frac{15}{\pi^2}.
$$

The proof of Theorem 1 is thus completed. □

4. PROOF OF THEOREM 2

Proof. Consider the function

$$
\begin{aligned}
f(x) &= \frac{(\frac{\sin x}{x})^3 + \frac{1}{2}\sin^2 x - 1}{x\sin^3 x}\\
&= \frac{1}{x^4} - \frac{1}{x\sin^3 x} + \frac{1}{2x\sin x}.
\end{aligned}
$$

By (8) and (12)

$$
\begin{aligned}
f(x) = \frac{1}{x^4} - \Bigg(& \frac{1}{x^4} + \frac{1}{2}\sum_{n=2}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|(2n-1)(2n-2)x^{2n-4} \\
& + \frac{1}{2x^2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-2} \Bigg) \\
& + \frac{1}{2x^2} + \sum_{n=1}^{\infty} \frac{\left(2^{2n-1}-1\right)|B_{2n}|}{(2n)!}x^{2n-2} \\
& = -\frac{1}{2}\sum_{n=2}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|(2n-1)(2n-2)x^{2n-4} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-2} \\
& + \sum_{n=1}^{\infty} \frac{\left(2^{2n-1}-1\right)|B_{2n}|}{(2n)!}x^{2n-2} \\
& = -\frac{1}{2}\sum_{n=2}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|(2n-1)(2n-2)x^{2n-4} \\
& = \sum_{n=2}^{\infty} a_n x^{2n-4},
\end{aligned}
$$

where

$$
a_n = -\frac{(2^{2n-1}-1)(2n-1)(2n-2)|B_{2n}|}{(2n)!}, \quad n \geq 2. \tag{14}
$$

It follows from (14) that the function $f(x)$ is strictly decreasing on $(0, \pi/2)$. Moreover, it is not difficult to obtain $\lim_{x\to 0^+} f(x) = a_2 = -\frac{7}{120}$ and $\lim_{x\to \frac{\pi}{2}^-} f(x) = \frac{16-\pi^3}{\pi^4}$. the proof of Theorem 2 is thus completes. □

5. REMARKS

Remark 6. *The left-hand side of* (5) *is stronger than the one of* (2). *since*

$$
\begin{aligned}
& \frac{x^3 \sin x}{15 + \frac{5x^2}{21}} - \left(\frac{x^4}{15} - \frac{23x^6}{1890}\right) \\
& = \frac{x^3(7983 \sin x - 7938x + 1323x^3 + 23x^5)}{1890(63 + x^2)} \\
& > \frac{x^3(7983(x - \frac{x^3}{6}) - 7938x + 1323x^3 + 23x^5)}{1890(63 + x^2)} \\
& = \frac{23x^8}{1890(63 + x^2)} > 0.
\end{aligned}
$$

Remark 7. *The left-hand side of* (5) *is stronger than the one of* (3). *since*

$$\frac{x^3\sin x}{15+\frac{5}{21}x^2}-x^3\left(1-\frac{x^2}{63}\right)\frac{\sin x}{15}$$
$$=\frac{x^7\sin x}{945(63+x^2)}>0.$$

Remark 8. *The left-hand side of* (7) *is better than the right-hand side inequality of* (4). *since*

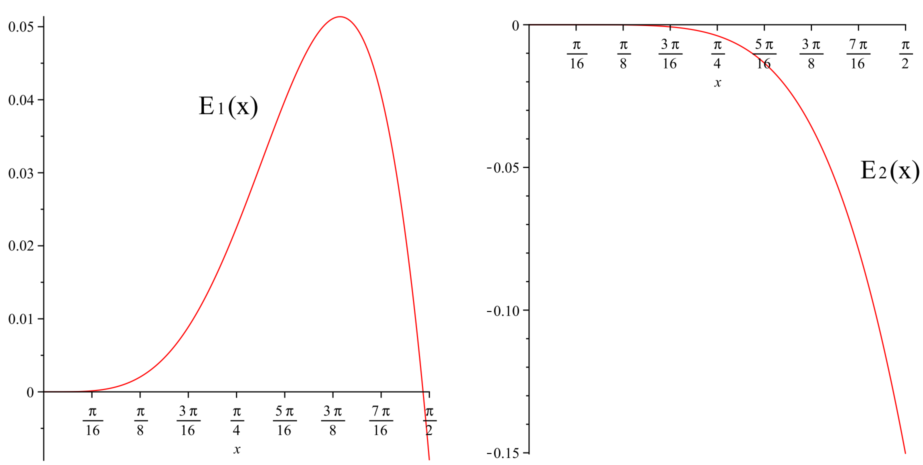
$$1-\left(\frac{1}{2}+\frac{16-\pi^3}{\pi^4}x\sin x\right)\sin^2 x-\left(1-\frac{\pi^3-8}{\pi^3}\sin^2 x\right)$$
$$=\frac{\sin^2 x(\pi^4-16\pi-32+2\pi^3)}{2\pi^4}>0.$$

Remark 9. *Now let us compare graphically the lower and upper bounds of* $\left(\frac{\sin x}{x}\right)^3$ *given in 5 and 7 on the same interval* $(0,\pi/2)$ *, respectively. Consider the functions* E_1, E_2 *defined by*

$$E_1(x)=\cos x+\frac{x^3\sin x}{15+\frac{5}{21}x^2}-\left[1-\left(\frac{1}{2}+\frac{\pi^3-16}{\pi^4}x\sin x\right)\sin^2 x\right],$$

$$E_2(x)=\cos x+\frac{x^3\sin x}{15+\frac{\pi^6-960}{16\pi^2}x^2}-\left[1-\left(\frac{1}{2}+\frac{7}{120}x\sin x\right)\sin^2 x\right].$$

The plots of E_1 *and* E_2 *are shown in figure1(a) and figure1(b), respectively.*



(a) The graph of the functions $E_1(x)$ (b) The graph of the function $E_2(x)$

Figure 1: The graph of the functions $E_1(x)$ and $E_2(x)$

Based on Figure 1(a) and a numerical analysis, we see that, for all $x \in (0, \delta_*)$*, where* $\delta_* \approx 1.5451$*, the lower bound in* (5) *is stronger than the lower bound in* (7)*. It is weaker for* $x \in (\delta_*, \pi/2)$*.*

Also, based on Figure1(b), it shows that for all $x \in (0, \pi/2)$*, the upper bound in* (5) *is stronger than the upper bound in* (7)*.*

6. FUTHER IMPROVEMENTS

Using a method developed and applied in [**13, 14, 15, 16**], the results of Theorem 1 and Theorem 2 can be futher improved. For this, in the following, we will present an overview of the results related to double-sided Taylor's approximations.

Let us consider a real function $f : (a, b) \longrightarrow \mathbb{R}$, such that there exist finite limits $f^{(k)}(a+) = \lim\limits_{x \to a+} f^{(k)}(x)$, for $k = 0, 1, \ldots, n$.
Taylor's polynomial

$$T_n^{f,\,a+}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k,\ n \in \mathbb{N}_0,$$

and the polynomial

$$\mathbb{T}_n^{f;\,a+,\,b-}(x) = \begin{cases} T_{n-1}^{f,\,a+}(x) + \dfrac{1}{(b-a)^n} R_n^{f,\,a+}(b-)(x-a)^n & , \quad n \geq 1 \\ f(b-) & , \quad n = 0, \end{cases}$$

are called the *first* Taylor*'s approximation for the function* f *in the right neighborhood of* a, and the *second* Taylor*'s approximation for the function* f *in the right neighborhood of* a, respectively.

Also, the following functions:

$$R_n^{f,\,a+}(x) = f(x) - T_{n-1}^{f,\,a+}(x), \quad n \in \mathbb{N}$$

and

$$\mathbb{R}_n^{f;\,a+,\,b-}(x) = f(x) - \mathbb{T}_{n-1}^{f;\,a+,\,b-}(x), \quad n \in \mathbb{N}$$

are called the *remainder of the first* Taylor*'s approximation in the right neighborhood of* a, and the *remainder of the second* Taylor*'s approximation in the right neighborhood of* a, respectively.

In our applications, of special interest is the following theorem:

Theorem 10. ([**13**], Theorem 4) *Consider the real analytic functions* $f : (a, b) \longrightarrow \mathbb{R}$*:*

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k,$$

where $c_k \in \mathbb{R}$ *and* $c_k \geq 0$ *for all* $k \in \mathbb{N}_0$. *Then,*

$$\begin{gathered} T_0^{f,\,a+}(x) \leq \ldots \leq T_n^{f,\,a+}(x) \leq T_{n+1}^{f,\,a+}(x) \leq \ldots \\ \ldots \leq f(x) \leq \ldots \\ \ldots \leq \mathbb{T}_{n+1}^{f;\,a+,\,b-}(x) \leq \mathbb{T}_n^{f;\,a+,\,b-}(x) \leq \ldots \leq \mathbb{T}_0^{f;\,a+,\,b-}(x), \end{gathered}$$

for all $x \in (a,b)$. *If* $c_k \in \mathbb{R}$ *and* $c_k \leq 0$ *for all* $k \in \mathbb{N}_0$, *then the reversed inequalities hold.*

Based on theorem 10, we have

Theorem 11. *For every* $x \in (0,b), 0 < b \leq \frac{\pi}{2}$ *and* $m \in \mathbb{N}, m \geq 4$, *the following inequalities hold:*

$$\begin{gathered} T_6^{f,\,0+,\,b-}(x) \leq \ldots \leq T_{2m-4}^{f,\,0+,\,b-}(x) \leq T_{2m-2}^{f,\,0+,\,b-}(x) \leq \ldots \\ \ldots \leq f(x) \leq \ldots \\ \ldots \leq \mathbb{T}_{2m-2}^{f;\,0+}(x) \leq \mathbb{T}_{2m-4}^{f;\,0+}(x) \leq \ldots \leq \mathbb{T}_6^{f;\,0+}(x), \end{gathered} \tag{15}$$

where

$$f(x) = \frac{x^3 \sin x}{\left(\frac{\sin x}{x}\right)^3 - \cos x} - 15 - \frac{5}{21}x^2,$$

$$T_{2m-2}^{f,\,0+}(x) = \sum_{k=4}^{m} A(k) x^{2k-4}$$

$$\mathbb{T}_{2m-2}^{f;\,0+,\,b-}(x) = \sum_{k=4}^{m-1} A(k) x^{2k-2} + \frac{1}{b^{2m}} \left(f(b) - \sum_{k=4}^{m-1} A(k) b^{2k-2} \right) x^{2m-4}$$

where $A_2 = 15, A_3 = \frac{5}{21}$ *and* $A_n (n \geq 4)$ *satisfies the recurrence relation*

$$\sum_{k=2}^{n} A_k C_{n-k+2} = 0,$$

and

$$C_k = \frac{2^{2k}|B_{2k}|}{(2k)!} + \frac{(-1)^k 2^{2k+1}}{(2k+2)!}$$

Proof. Let $g(x) = \frac{\left(\frac{\sin x}{x}\right)^3 - \cos x}{x^3 \sin x}$, using power series expansions , for $0 < x < \frac{\pi}{2}$, we have

$$\begin{aligned}
g(x) &= \frac{\left(\frac{\sin x}{x}\right)^3 - \cos x}{x^3 \sin x} = \frac{1-\cos 2x}{2x^6} - \frac{1}{x^3}\cot x \\
&= \frac{1}{2x^6}\left[1 - \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{(2k)!} x^{2k}\right] - \frac{1}{x^3}\left[\frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1}\right] \\
&= -\sum_{k=2}^{\infty} \frac{(-1)^k 2^{2k-1}}{(2k)!} x^{2k-6} + \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-4} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+2)!} x^{2k-4} + \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-4} \\
&= \sum_{k=2}^{\infty} \left[\frac{2^{2k}|B_{2k}|}{(2k)!} + \frac{(-1)^k 2^{2k+1}}{(2k+2)!}\right] x^{2k-4} \\
&= \frac{1}{15} - \frac{1}{945}x^2 + \sum_{k=4}^{\infty} C_k x^{2k-4},
\end{aligned}$$

where $C_2 = \frac{1}{15}, C_3 = -\frac{1}{945}$ and

$$C_k = \frac{2^{2k}|B_{2k}|}{(2k)!} + \frac{(-1)^k 2^{2k+1}}{(2k+2)!}, \quad k \geq 4.$$

It is well known [**1**, p.805] that Bernoulli numbers with even indexes satisfy the following double inequality

$$\frac{2(2k)!}{(2\pi)^{2k}} < |B_{2k}| < \frac{2(2k)!}{(2\pi)^{2k}(1-2^{1-2k})}. \tag{16}$$

By (16), we find that for $k \geq 4$

$$\frac{2^{2k}|B_{2k}|}{(2k)!} - \frac{2^{2k+1}}{(2k+2)!} > \frac{2^{2k}}{(2k)!}\frac{2(2k)!}{(2\pi)^{2k}} - \frac{2^{2k+1}}{(2k+2)!} = \frac{2^{2k+1}\left[(2k+2)! - (2\pi)^{2k}\right]}{(2\pi)^{2k}(2k+2)!}$$

By induction on k, it is easy to see that

$$(2k+2)! > (2\pi)^{2k}, \quad k \geq 4.$$

Hence $C_k > 0$ for $k \geq 4$.

Let

$$f(x) = \frac{x^3 \sin x}{\left(\frac{\sin x}{x}\right)^3 - \cos x},$$

then

$$\begin{aligned}
f(x) &= \frac{1}{g(x)} \\
&= \frac{1}{\frac{1}{15} - \frac{1}{945}x^2 + \sum_{k=4}^{\infty} C_k x^{2k-4}} = \sum_{k=2}^{\infty} A_k x^{2k-4},
\end{aligned}$$

thus we have

$$\left(\frac{1}{15}-\frac{1}{945}x^2+\sum_{k=4}^{\infty}C_kx^{2k-4}\right)\left(15+\frac{5}{21}x^2+\sum_{k=4}^{\infty}A_kx^{2k-4}\right)=1$$

which lead to the conclusion that $A_2=15, A_3=\frac{5}{21}$ and $A_n(n\geq 4)$ satisfies the recurrence relation

$$\sum_{k=2}^{n}A_kC_{n-k+2}=0, \tag{17}$$

Since $C_k>0$ for $k\geq 4$, by the monotonicity of $g(x)$, we can conclude that $A_k<0$ for all $k\geq 4$. □

Based on theorem 10 and theorem 2, we have

Theorem 12. *For every $x\in(0,b), 0<b\leq\frac{\pi}{2}$ and $m\in\mathbb{N}, m\geq 4$, the following inequalities hold:*

$$\begin{gathered}T_2^{f,\,0+,\,b-}(x)\leq\ldots\leq T_{2m-4}^{f,\,0+,\,b-}(x)\leq T_{2m-2}^{f,\,0+,\,b-}(x)\leq\ldots\\ \ldots\leq f(x)\leq\ldots\\ \ldots\leq \mathbb{T}_{2m-2}^{f;\,0+}(x)\leq\mathbb{T}_{2m-4}^{f;\,0+}(x)\leq\ldots\leq\mathbb{T}_2^{f;\,0+}(x),\end{gathered} \tag{18}$$

where

$$f(x)=\frac{(\frac{\sin x}{x})^3+\frac{1}{2}\sin^2 x-1}{x\sin^3 x},$$

$$T_{2m-2}^{f,\,0+}(x)=\sum_{k=2}^{m}D(k)x^{2k-4}$$

$$\mathbb{T}_{2m-2}^{f;\,0+,\,b-}(x)=\sum_{k=2}^{m-1}D(k)x^{2k-2}+\frac{1}{b^{2m}}\left(f(b)-\sum_{k=2}^{m-1}D(k)b^{2k-2}\right)x^{2m-4}$$

where

$$D_n=-\frac{(2^{2n-1}-1)(2n-1)(2n-2)|B_{2n}|}{(2n)!},\quad n\geq 2.$$

Let $m=4$ and $b=\frac{\pi}{2}$ in Theorem 11, we have

$$\cos x+\frac{x^3\sin x}{15+\frac{5}{21}x^2-\frac{100}{1323}x^4}<\left(\frac{\sin x}{x}\right)^3<\cos x+\frac{x^3\sin x}{15+\frac{5}{21}x^2-\lambda x^4}. \tag{19}$$

where

$$\lambda = \frac{20160 + 80\pi^2 - 21\pi^6}{84\pi^4}.$$

Let $m = 4, b = \frac{\pi}{2}$ in Theorem 12, we have

(20)

$$\left(\mu x^2 - \frac{7}{120}\right) x \sin^3 x < \left(\frac{\sin x}{x}\right)^3 - 1 + \frac{1}{2}\sin^2 x < -\left(\frac{7}{120} + \frac{31}{1512}x^2\right) x \sin^3 x$$

where

$$\mu = \frac{1920 - 120\pi^3 + 7\pi^4}{30\pi^6}.$$

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