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THE RELATIONSHIP BETWEEN HUYGENS' AND WILKER'S INEQUALITIES AND FURTHER REMARKS

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The first aim of this paper is to show how the Huygens' and Wilker's inequalities are related. In this sense, we establish and prove a class of inequalities depending on a parameter n, where Huygens' and Wilker's inequalities are obtained when $n = 1$ and $n = 2$, respectively. By exploiting the above idea, we introduce other classes of inequalities depending on a parameter, extending an inequality of Wilker type and also the classical Cusa inequality. Finally, some open problems are posed.

1. INTRODUCTION

In the recent past, many researchers have given new results on the inequalities due to Huygens-Wilker-Cusa.

First we make considerations on Huygens' inequality [**5**]:

$$2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}>3 \qquad \left(0<x<\frac{\pi}{2}\right) \tag{1}$$

and Wilker's inequality [**14**]:

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2 \qquad \left(0<x<\frac{\pi}{2}\right). \tag{2}$$

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The following inequality

$$\frac{\sin x}{x} < \frac{2}{3} + \frac{1}{3}\cos x \qquad \left(0 < x < \frac{\pi}{2}\right), \tag{3}$$

now known as Cusa's inequality, was established by the German philosopher Nicolaus de Cusa (1401-1464), while a first rigorous proof was provided by Huygens in [**5**].

Undoubtedly, the above inequalities (1)-(3) have remained of high interest through the researchers because of their simple and nice form.

Inspired by Wilker's inequality (2), Wu and Srivastava [**15**] have introduced a similar inequality of Wilker's type:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2 \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{4}$$

These inequalities were widely extended to cases related to Bessel functions [**2**], Lemniscate functions [**3**], mixed trigonometric-polynomial problems [**6**], [**7**], [**11**], asymptotic expansions [**8**]-[**10**], generalized hyperbolic functions [**12**], [**13**], [**16**], [**21**], weighted and exponential functions [**15**], and other elementary extensions [**1**], [**4**], [**17**], [**18**]-[**20**].

It is true that all these new extensions are increasingly accurate but they have complicated form. As a consequence, they were provided with a sacrifice of simplicity and beauty.

2. THE RESULTS

Being preoccupied to give some extensions of simple form of Wilker's inequality (2), the author of this work has ultimately found the inequalities presented in Theorem 1.

Fortunately, these inequalities have an intrinsec importance, as they show us that Huygens' and Wilker's inequalities are closely related. More precisely, these inequalities are enclosed in a family of inequalities depending on an integral parameter $n \geq 1$. Until now no proofs hint this fact.

We are in a position to give the following

Theorem 1. *For every integer* $n \geq 1$ *and* $0 < x < \frac{\pi}{2}$, *we have:*

$$\left(\frac{\sin x}{x}\right)^n + \frac{n}{2}\frac{\tan x}{x} > \frac{n+2}{2}. \tag{5}$$

As we announced before, inequality (5) becomes Huygens' and Wilker's inequality for $n = 1$ and $n = 2$, respectively.

We give a similar result related to inequality (4).

Theorem 2. *For every integer* $n \geq 1$ *and* $0 < x < \frac{\pi}{2}$, *we have:*

$$\left(\frac{x}{\sin x}\right)^n + \frac{n}{2}\frac{x}{\tan x} > \frac{n+2}{2}. \tag{6}$$

The inequality (4) is obtained for $n = 2$.

Note that inequality (6) for $n = 1$ can be read as

$$\frac{x}{\sin x} + \frac{1}{2}\frac{x}{\tan x} > \frac{3}{2}.$$

Interesting fact, this new inequality can be viewed as the dual of (4), in the same way as Huygens' inequality would be considered the dual of Wilker's inequality.

The associated result to Cusa's inequality (3) is the following

Theorem 3. *For every integer $n \geq 2$ and $0 < x < \frac{\pi}{2}$, we have:*

$$\left(\frac{\sin x}{x}\right)^n - \frac{n}{3}\cos x > -\frac{n-3}{3} \tag{7}$$

As an interesting fact, note that the case $n = 1$ in (7) is Cusa's inequality, but reversed. Then obviously, (7) is not true for $n = 1$. Theorem 3 asserts that a reversed inequality of Cusa's type holds true, for every integer $n \geq 2$.

In a sense explained above, inequality (7) for $n = 2$:

$$\left(\frac{\sin x}{x}\right)^2 > \frac{1}{3} + \frac{2}{3}\cos x$$

can be viewed as the dual of Cusa's inequality (3).

3. THE PROOFS

Initially, the results listed in the previous section were obtained by the author of this work by investigating some asymptotic series. However, in the sequel, we choose to present different proofs that are much easier.

The Proof of Theorem 1. We proceed by induction with respect to $n \geq 1$. If $n = 1$, then inequality (5) is Huygens' inequality (1), so it holds true in this case. Note that inequality (5) is also true in case $n = 2$, being Wilker's inequality (2). We assume now that inequality (5) is true for a value $n \geq 2$. We have

$$\begin{aligned}
&\left(\frac{\sin x}{x}\right)^{n+1} + \frac{n+1}{2}\frac{\tan x}{x} - \frac{n+3}{2} \\
&= \frac{\sin x}{x}\left(\frac{\sin x}{x}\right)^n + \frac{n+1}{2}\frac{\tan x}{x} - \frac{n+3}{2} \\
&> \frac{\sin x}{x}\left(\frac{n+2}{2} - \frac{n}{2}\frac{\tan x}{x}\right) + \frac{n+1}{2}\frac{\tan x}{x} - \frac{n+3}{2} \\
&= \frac{n}{2}\left(\frac{\tan x}{x} - 1\right)\left(1 - \frac{\sin x}{x}\right) + \left(\frac{\sin x}{x} + \frac{\tan x}{2x} - \frac{3}{2}\right) \\
&= \frac{n}{2}u(x) + v(x) \\
&> 0,
\end{aligned}$$

where

$$u(x) = \left(\frac{\tan x}{x} - 1\right)\left(1 - \frac{\sin x}{x}\right) > 0,$$
$$v(x) = \frac{\sin x}{x} + \frac{\tan x}{2x} - \frac{3}{2} > 0.$$

Note that $v(x) > 0$ is Huygens' inequality. The proof is now completed. □

The Proof of Theorem 2. We proceed by induction with respect to $n \geq 1$.
We prove the case $n = 1$:

$$\frac{x}{\sin x} + \frac{x}{2\tan x} > \frac{3}{2}, \tag{8}$$

which, after multiplying by $2\sin x$, is equivalent to $w(x) > 0$, where

$$w(x) = 2x + x\cos x - 3\sin x.$$

We have $w'(x) = 2 - x\sin x - 2\cos x$. Then $w''(x) = \sin x - x\cos x > 0$, so w' is strictly increasing. We get $w'(x) > w'(0) = 0$, so w is strictly increasing. Finally, $w(x) > w(0) = 0$.
We assume now that inequality (6) is true for a value $n \geq 1$. We have

$$\begin{aligned}
&\left(\frac{x}{\sin x}\right)^{n+1} + \frac{n+1}{2}\frac{x}{\tan x} - \frac{n+3}{2} \\
&= \frac{x}{\sin x}\left(\frac{x}{\sin x}\right)^{n} + \frac{n+1}{2}\frac{x}{\tan x} - \frac{n+3}{2} \\
&> \frac{x}{\sin x}\left(\frac{n+2}{2} - \frac{n}{2}\frac{x}{\tan x}\right) + \frac{n+1}{2}\frac{x}{\tan x} - \frac{n+3}{2} \\
&= \frac{n}{2}\left(\frac{x}{\sin x} - 1\right)\left(1 - \frac{x}{\tan x}\right) + \left(\frac{x}{\sin x} + \frac{x}{2\tan x} - \frac{3}{2}\right) \\
&= \frac{n}{2}s(x) + \frac{1}{2\sin x}w(x) \\
&> 0,
\end{aligned}$$

where

$$s(x) = \left(\frac{x}{\sin x} - 1\right)\left(1 - \frac{x}{\tan x}\right) > 0$$

and $w(x) > 0$. The proof is now completed by induction. □

The Proof of Theorem 3. In order to proceed by induction, we prove first the initial case $n = 2$:

$$\left(\frac{\sin x}{x}\right)^{2} - \frac{2}{3}\cos x > \frac{1}{3}. \tag{9}$$

That is $m(x) > 0$, where

$$m(x) = \left(\frac{\sin x}{x}\right)^2 - \frac{2}{3}\cos x - \frac{1}{3}.$$

Let us denote $u(x) = \frac{x^3}{\sin x} m'(x)$, namely

$$u(x) = 2x\cos x - 2\sin x + \frac{2}{3}x^3.$$

Then $u'(x) = 2x(x - \sin x) > 0$, so u is strictly increasing. Thus $u(x) > u(0) = 0$ and then $m'(x) > 0$. It follows that m is strictly increasing. Now $m(x) > \lim_{x\to 0} m(x) = 0$ and inequality (9) is proved.
We assume now that inequality (7) is true for a value $n \geq 2$. We have

$$\begin{aligned} &\left(\frac{\sin x}{x}\right)^{n+1} - \frac{n+1}{3}\cos x + \frac{n-2}{3} \\ &= \frac{\sin x}{x}\left(\frac{\sin x}{x}\right)^{n} - \frac{n+1}{3}\cos x + \frac{n-2}{3} \\ &> \frac{\sin x}{x}\left(\frac{n}{3}\cos x - \frac{n-3}{3}\right) - \frac{n+1}{3}\cos x + \frac{n-2}{3} \\ &= \frac{n}{3}(1-\cos x)\left(1 - \frac{\sin x}{x}\right) + \left(\frac{\sin x}{x} - \frac{\cos x}{3} - \frac{2}{3}\right) \\ &= \frac{n}{3}a(x) + b(x) \\ &> 0, \end{aligned} \tag{10}$$

where

$$a(x) = (1-\cos x)\left(1 - \frac{\sin x}{x}\right) > 0,$$
$$b(x) = \frac{\sin x}{x} - \frac{\cos x}{3} - \frac{2}{3} < 0.$$

Note that $b(x) < 0$, as this is Cusa's inequality. Hence the prove of the remaining inequality in (10):

$$\frac{n}{3}a(x) + b(x) > 0, \qquad n \geq 2 \tag{11}$$

should suffer a modification compared to the previous proofs.

In this sense, remark that it suffices to prove the case $n = 2$, namely

$$\frac{2}{3}a(x) + b(x) > 0. \tag{12}$$

Let

$$\left(\frac{2}{3}a\left(x\right)+b\left(x\right)\right)\frac{3x}{\cos x}=q\left(x\right),$$

where $q\left(x\right)=\tan x-3x+2\sin x$. We have $q'\left(x\right)=\tan^{2}x+2\cos x-2$ and

$$q''\left(x\right)=2\left(\frac{1}{\cos^{3}x}-1\right)\sin x>0.$$

Thus q' is strictly increasing. Then $q'\left(x\right)>q'\left(0\right)=0$, so q is strictly increasing. Finally, $q\left(x\right)>q\left(0\right)=0$.

The inequalities (12) and (11) follow and the proof is completed. □

4. CONCLUSION

The ideas in this paper could be extended to other interesting results. For example, we propose to the reader the study of the above families of inequalities depending on a (positive) real parameter n. More precisely, we suggest the finding of real parameters a, b, c for which the following inequalities holds true, for every $x\in\left(0,\frac{\pi}{2}\right)$:

- $\left(\frac{\sin x}{x}\right)^{a}+\frac{a}{2}\frac{\tan x}{x}>\frac{a+2}{2}$
- $\left(\frac{x}{\sin x}\right)^{b}+\frac{b}{2}\frac{x}{\tan x}>\frac{b+2}{2}$
- $\left(\frac{\sin x}{x}\right)^{c}-\frac{c}{3}\cos x>-\frac{c-3}{3}$

Moreover, similar results could be obtained by replacing the trigonometric functions by hyperbolic functions or other functions.

We are convinced that other interesting results can be presented.

Theorem 4. *For* $0<x<\pi/2$ *and* $a>0$*,*

$$\left(\frac{\sin x}{x}\right)^{a}+\frac{a}{2}\frac{\tan x}{x}>\frac{a+2}{2}. \tag{13}$$

Proof. Write (13) as

$$\frac{2}{a+2}\left(\frac{\sin x}{x}\right)^{a}+\frac{a}{a+2}\frac{\tan x}{x}>1. \tag{14}$$

Using weighted arithmetic-geometric mean inequality, we have

$$\begin{aligned}\frac{2}{a+2}\left(\frac{\sin x}{x}\right)^{a}+\frac{a}{a+2}\frac{\tan x}{x}&>\left(\frac{\sin x}{x}\right)^{\frac{2a}{a+2}}\left(\frac{\tan x}{x}\right)^{\frac{a}{a+2}}\\&=\left[\left(\frac{\sin x}{x}\right)^{2}\left(\frac{\tan x}{x}\right)\right]^{\frac{a}{a+2}}.\end{aligned}$$

In order to prove (14), it suffices to show that

$$\left(\frac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right) > 1, \tag{15}$$

i.e.,

$$(\cos x)^{1/3} < \frac{\sin x}{x}. \tag{16}$$

The inequality (16) holds (Lazarevi'c-type inequality). The proof of Theorem 4 is complete. □

Theorem 4 $\Longrightarrow$ Theorem 1.

Theorem 5. *Let $0 < x < \pi/2$. Then, for $b \geq 0.65$,*

$$\left(\frac{x}{\sin x}\right)^b + \frac{b}{2}\frac{x}{\tan x} > \frac{b+2}{2}. \tag{17}$$

Proof. Write (17) as

$$\frac{2}{b+2}\left(\frac{x}{\sin x}\right)^b + \frac{b}{b+2}\frac{x}{\tan x} > 1. \tag{18}$$

Using the power series expansion of $\csc x$,

$$\csc x = \frac{1}{x} + \sum_{j=1}^{\infty}\frac{(2^{2j}-2)|B_{2j}|}{(2j)!}x^{2j-1}, \qquad |x| < \pi,$$

where B_n the Bernoulli number, we obtain

$$\frac{x}{\sin x} = x\csc x = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + \frac{127}{604800}x^8 + \dots,$$

and then, we have

$$\begin{aligned}
\frac{2}{b+2}\left(\frac{x}{\sin x}\right)^b &= \frac{2}{b+2}\left(1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + \frac{127}{604800}x^8 + \dots,\right)^b \\
&= \frac{2}{b+2} + \frac{b}{3(b+2)}x^2 + \frac{b(2+5b)}{180(b+2)}x^4 + \frac{b(16+42b+35b^2)}{22680(b+2)}x^6 + \dots \\
&> \frac{2}{b+2} + \frac{b}{3(b+2)}x^2 + \frac{b(2+5b)}{180(b+2)}x^4.
\end{aligned}$$

In order to prove (18), it suffices to show that

$$\frac{2}{b+2} + \frac{b}{3(b+2)}x^2 + \frac{b(2+5b)}{180(b+2)}x^4 + \frac{b}{b+2}\frac{x}{\tan x} > 1,$$

it suffices to show that

$$b > \frac{2}{5}\left(-\frac{30}{x^2} - 1 + \frac{90}{x^4} - \frac{90}{x^3 \tan x}\right) =: f(x). \tag{19}$$

It is easy to show that $f(x)$ is strictly increasing in $(0, \pi/2)$ (we here omit the proof), and we have

$$\frac{2}{5} < f(x) < \frac{-2\pi^4 - 240\pi^2 + 2880}{5\pi^4} = 0.64978896\cdots < b.$$

Hence, the inequality (19) holds. The proof of Theorem 5 is complete. □

Theorem 5 $\Longrightarrow$ Theorem 2.

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