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INTEGRAL CAYLEY GRAPHS OVER SEMI-DIHEDRAL GROUPS

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Classifying integral graphs is a hard problem that initiated by Harary and Schwenk in 1974. In this paper, with the help of character table, we treat the corresponding problem for Cayley graphs over the semi-dihedral group $\mathrm{SD}_{8n} = \langle a, b \mid a^{4n} = b^2 = 1, bab = a^{2n-1}\rangle$, $n \geq 2$. We present several necessary and sufficient conditions for the integrality of Cayley graphs over SD_{8n}, we also obtain some simple sufficient conditions for the integrality of Cayley graphs over SD_{8n} in terms of the Boolean algebra of $\langle a\rangle$. In particular, we give the sufficient conditions for the integrality of Cayley graphs over semi-dihedral groups SD_{2^n} $(n \geq 4)$ and SD_{8p} for a prime p, from which we determine several infinite classes of integral Cayley graphs over SD_{2^n} and SD_{8p}.

1. INTRODUCTION

We only consider simple undirected graphs throughout this paper. For a graph X, the *adjacency matrix* of a simple graph X of order n is $A(X) = (a_{ij})_{n\times n}$, whose entries satisfy $a_{ij} = 1$ if vertices i and j are adjacent in X, and $a_{ij} = 0$ otherwise. Since $A(X)$ is real and symmetric, all eigenvalues of X are real. The eigenvalues of $A(X)$ are called the *eigenvalues of* X, they attract tremendous amount of attention in the literature. For more details, one may refer [**18, 21, 25**] and the monograph [**35**].

We call a graph X *integral* if all eigenvalues of $A(X)$ are integers. The notion of integral graphs dates back to Harary and Schwenk in [**19**], and they proposed

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Problem 1. *Classifying all integral graphs.*

Recently, Ahmadi et al. [**3**] proved that only a fraction of $2^{-\Omega(n)}$ of the graphs on n vertices have an integral spectrum. Therefore this number is negligible compared to the total number of graphs. This further impulses people to study Problem 1. Although Problem 1 seems easy at first glance, it indeed is extremely difficult and still far away from being solved completely. Thus, mathematicians try to investigate special classes of graphs, such as trees [**34, 38, 39, 36, 37**], graphs with bounded degrees [**34**] and regular graphs [**24**]. In particular, Csikvári [**14**] constructed integral trees with arbitrarily large diameter. It has recently been discovered that integral graphs may be of interest for designing the network topology of perfect state transfer networks, see for example [**30**].

Given a finite group G and a subset $1 \notin S \subseteq G$ with $S = S^{-1}$, the *Cayley graph* $X(G, S)$ has vertex set G and two vertices a, b are adjacent if $a^{-1}b \in S$. If S generates G, then $X(G, S)$ is connected. The spectrum of Cayley graphs is concerned with the interplay between spectral graph theory and group representation theory, it is very fruitful for both areas recently, and leads to some new and emerging interdisciplinary field.

In the present paper, we treat integral Cayley graphs. From the outstanding paper of Babai [**7**], many mathematicians studied various classes of graphs over certain finite groups. Further, in 2009, Abdollahi and Vatandoost [**1**] proposed the following problem:

Problem 2. *Which Cayley graphs are integral?*

This problem was also studied extensively in the literature, such as cubic integral Cayley graphs [**1**], integral graphs over abelian groups [**8, 33**], normal Cayley graphs over symmetric groups [**11**]. In 2010, Klotz and Sander developed a Boolean algebra theory [**23**] in order to study integral graphs. They proved that, for an abelian group G, if the Cayley graph $X(G, S)$ is integral, then S belongs to the Boolean algebra $B(G)$ generated by the subgroups of G. Moreover, they conjectured that the converse is also true, which has been confirmed by Alperin and Peterson [**5**]. For more details in this area, one may see [**6, 4, 17, 28**] and the exhaustive survey paper [**26**].

It is unexpected that, although plenty of works were obtained for abelian groups, few papers were found for integral Cayley graphs over nonabelian groups, and there is no one unified approach to tackle this problem as far as we know. Therefore people have to deal with nonabelian groups one by one. From this point, Lu et al. [**27**] considered the problem for dihedral group of order $2n$ given by $\mathrm{D}_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, and obtained several elegant criteria for integral Cayley graphs over D_{2n}. Cheng et al. [**12, 13**] studied the integral Cayley graphs over dicyclic group $\mathrm{T}_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$ of order $4n$.

In this paper, we consider the integral Cayley graphs over the semi-dihedral group given by

$$\mathrm{SD}_{8n} = \langle a, b \mid a^{4n} = b^2 = 1, bab = a^{2n-1} \rangle,$$

for $n \geq 2$, which is a non-abelian group of order $8n$. This group is well studied in group theory [**9, 10**], and also attracts much attention by combinatorists [**2, 29**].

This paper is organized as follows. At first, by using the expression of spectra of Cayley graphs, we obtain a necessary and sufficient condition for the integrality of Cayley graphs over SD_{8n} (see Theorem 14). By using of atoms of Boolean algebra of cyclic group, we obtain a simple sufficient condition (see Theorem 21), we also obtain the necessary and sufficient conditions for the integrality of Cayley graphs over SD_{8n} (see Theorems 23 and 26). At last, we also give the sufficient condition for the integrality of Cayley graphs over two special semi-dihedral groups SD_{2^n} ($n \geq 4$) and SD_{8p} for a prime p (see Theorems 29 and 31), and determine some infinite families of connected integral Cayley graphs over SD_{2^n} and SD_{8p}.

2. PRELIMINARIES

In this section, we will present some basic knowledge in representation theory [**22**] and several lemmas which will be used later.

Let G be a finite group and V an n-dimensional vector space over the complex field $\mathbb{C}$. A *representation* of G on V is a group homomorphism $\rho : G \to GL(V)$, where $GL(V)$ denotes the group of automorphisms of V. The *degree* of ρ is the dimension of V. Two representations ρ_1 and ρ_2 of G on V_1 and V_2 respectively are *equivalent* if there is an isomorphism $T : V_1 \to V_2$ such that $T\rho_1(g) = \rho_2(g)T$ for all $g \in G$.

Let $\rho : G \to GL(V)$ be a representation. The *character* $\chi_\rho : G \to \mathbb{C}$ of ρ is defined by setting $\chi_\rho(g) = Tr(\rho(g))$ for $g \in G$, where $Tr(\rho(g))$ is the trace of the representation matrix of $\rho(g)$ with respect to some basis of V. The *degree* of the character χ_ρ is just the degree of ρ, which equals to $\chi_\rho(1)$. If W is a $\rho(g)$-invariant subspace of V for each $g \in G$, then we call W a $\rho(G)$-invariant subspace of V. If we restrict each $\rho(g)$ to W, we will get $\rho_W : G \to GL(W)$, which is a representation of G on W, called the *subrepresentation* of ρ on W. Obviously, $\{1\}$ and V are always G-invariant subspaces, which are *trivial*. If V has no non-trivial $\rho(G)$-invariant subspace, we call ρ an *irreducible representation* of G and the corresponding character χ_ρ an *irreducible character* of G. For a group G, we denote by $\mathrm{IRR}(G)$ and $\mathrm{Irr}(G)$ the complete set of non-equivalent irreducible representations of G and the complete set of non-equivalent irreducible characters of G, respectively.

The characters of cyclic group are needed in this paper.

Lemma 1 ([**22**]). *Let $C_n = \langle a \rangle$ be a cyclic group of order n. Then the irreducible representations of C_n are $\phi_j(a^k) = \varepsilon^{jk}$ $(j, k = 0, 1, 2, \ldots, n-1)$, where $\varepsilon = e^{\frac{2\pi i}{n}}$ is the primitive nth root of unity.*

We denote the *group algebra* of G over $\mathbb{C}$ by $\mathbb{C}G$. That is, $\mathbb{C}G$ is the vector space over $\mathbb{C}$ with basis G, and the multiplication is defined by extending the group multiplication linearly. Therefore, $\mathbb{C}G$ is the set of the forms $\sum_{g \in G} a_g g$, where $a_g \in \mathbb{C}$. We assume $1 \cdot g = g$. The multiplication of the elements in $\mathbb{C}G$ is followed by

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) = \sum_{g\in G}\sum_{h\in G} a_g b_h gh.$$

The *left regular representation* ρ_{reg} of G on $\mathbb{C}G$ is defined by

$$\rho_{reg} : G \to GL(\mathbb{C}G), \qquad \rho_{reg}(g)\left(\sum_{h\in G} a_h h\right) = \sum_{h\in G} a_h gh.$$

Then we have

Lemma 2 ([**32**]). *If* $\mathrm{IRR}(G) = \{\rho_1, \ldots, \rho_k\}$ *is the complete set of non-equivalent irreducible representations of* G*, then*

$$\rho_{reg} = \oplus_{i=1}^{k} m_i \rho_i,$$

where m_i *is the degree of* ρ_i*.*

By Lemma 2, we obtain the following lemma, which might be served as bridges between spectral graph theory, representations and characters of finite groups.

Lemma 3 ([**7**]). *Let* G *be a finite group of order* n*, let* $S \subseteq G \setminus \{1\}$ *be such that* $S = S^{-1}$*, and* $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_h\}$ *with* $\chi_i(1) = d_i (i = 1, \ldots, h)$*. Then the spectrum of the Cayley graph* $X(G, S)$ *can be arranged as*

$$Spec(X(G,S)) = \left\{[\lambda_{11}]^{d_1}, \ldots, [\lambda_{1d_1}]^{d_1}, \ldots, [\lambda_{h1}]^{d_h}, \ldots, [\lambda_{hd_h}]^{d_h}\right\}.$$

Furthermore, for any natural number t*, we have*

$$\lambda_{i1}^t + \lambda_{i2}^t + \cdots + \lambda_{id_i}^t = \sum_{s_1,\ldots,s_t \in S} \chi_i\left(\prod_{l=1}^{t} s_l\right).$$

Let G be a finite group, and $\mathcal{F}_G$ the set of all subgroups of G. Then the *Boolean algebra* $B(G)$ of G is the set whose elements are obtained by arbitrarily finite intersections, unions, and complements of the elements in $\mathcal{F}_G$. The minimal non-empty elements of $B(G)$ are called *atoms*, we denote the set of atoms of $B(G)$ by $[B(G)]$. Apparently, distinct atoms are disjoint. Alperin and Peterson [**5**] show that each element of $B(G)$ is the union of some atoms, and each atom of $B(G)$ has the form $[g] = \{x \mid \langle x\rangle = \langle g\rangle, x \in G\}$, where $g \in G$. For cyclic group $\langle a\rangle$ of order n, the atom of $B(\langle a\rangle)$ containing $a^d \in \langle a\rangle$ (where $d \mid n$) can be presented as $[a^d] = \{a^l \mid (l, n) = d\}$, where (l, n) is the greatest common divisor of l and n.

Lemma 4 ([**1**]). *Let* $G = \langle a\rangle$ *be the cyclic group of order* n*, and* $[a^d]$ *one atom of* $B\langle a\rangle$*. Then* $[a^d]^{-1} = [a^d]$*. Furthermore, if* $S \in B(\langle a\rangle)$*, then* $S = S^{-1}$*.*

We call a subset $S \subseteq G$ an *integral set* if $\chi(S) = \sum_{s\in S}\chi(s)$ is an integer for every character χ of G. From Lemma 3, it follows that S must be an integral set if the Cayley graph $X(G, S)$ is integral.

Using the integral sets and atoms of $B(G)$, Alperin and Peterson obtained the following criterion for integral Cayley graphs over an abelian group.

Lemma 5 ([**5, 16**])**.** *Let G be an abelian group. Then, $S \subseteq G$ is integral iff $S \in B(G)$ iff S is a union of atoms of $B(G)$ iff $X(G, S)$ is integral.*

Let S be a subset of G. A *multi-set* based on S, denoted by S^m, is defined by a multiplicity function $m_S : S \to \mathbb{N}$, where $m_S(s)$ counts how many times $s \in S$ appears in the multi-set. We further denote $m(s) = 0$ if $s \notin S$. The multi-set S^m is called *inverse closed* if $m_S(s) = m_S(s^{-1})$ for each $s \in S$, and S^m is called *integral* if $\chi(S^m) = \sum_{s\in S^m}\chi(s) = \sum_{s\in S} m_S(s)\chi(s)$ is an integer for each character χ of G.

For $S \in B(G)$, we have $S = [g_1] \cup [g_2] \cup \cdots \cup [g_k]$, denote by $S^{m_{g_1,g_2,\ldots,g_k}}$ the multi-set with multiplicity function $m_{g_1,g_2,\ldots,g_k}$, where $m_{g_1,g_2,\ldots,g_k}(s) = m_i \in \mathbb{N}$ for each $s \in [g_i]$, then $S^{m_{g_1,g_2,\ldots,g_k}} = m_1 * [g_1] \cup m_2 * [g_2] \cup \cdots \cup m_k * [g_k]$. We define $C(G) = \{S^{m_{g_1,g_2,\ldots,g_k}} \mid S = [g_1] \cup [g_2] \cup \cdots \cup [g_k], g_i \in G, k \in \mathbb{N}\}$ to be the collection of all multi-sets like $S^{m_{g_1,g_2,\ldots,g_k}}$, which is called the *integral cone* over $B(G)$. Following the above notations, it is obtained that

Lemma 6 ([**8**])**.** *Let G be an abelian group, and T^m be a multi-subset of G. Then T^m is integral if and only if $T^m \in C(G)$, where $C(G)$ is the integral cone over $B(G)$.*

Our focus in this paper is on the semi-dihedral group $\mathrm{SD}_{8n}=\langle a, b \mid a^{4n}=b^2=1, bab = a^{2n-1}\rangle$ for $n \geq 2$, all the $8n$ elements of SD_{8n} may be given by

$$\mathrm{SD}_{8n} = \{1, a, a^2, \ldots, a^{4n-1}, ba, ba^2, \ldots, ba^{4n-1}\}.$$

Lemma 7 ([**20**])**.** *For the semi-dihedral group* SD_{8n}*, $n \geq 2$, we have*

(1) $ba^k = a^{(2n-1)k}b$;

(2) $a^k b = ba^{(2n-1)k}$;

(3) $a^{-k} = a^{4n-k}$, $a^k = a^{4n+k}$, $b = b^{-1}$;

(4) $(ba^k)^{-1} = ba^{(2n+1)k}$.

In order to classify the conjugacy classes and the irreducible characters of SD_{8n}, we need the following definition.

Definition 8 ([**20**])**.** *We denote $C_1 := \{0, 2, 4, \ldots, 2n\}$. Let $C_2^{even} := \{1, 3, 5, \ldots, n-1\}$ and $C_3^{even} := \{2n+1, 2n+3, 2n+5, \ldots, 3n-1\}$ for even n; $C_2^{odd} := \{1, 3, 5, \ldots, n\}$ and $C_3^{odd} := \{2n+1, 2n+3, 2n+5, \ldots, 3n\}$ for odd n. Then we define*

- $C^{even} := C_1 \cup C_2^{even} \cup C_3^{even}$ *and* $C^{odd} := C_1 \cup C_2^{odd} \cup C_3^{odd}$;
- $C^{\dagger}_{even} := C_1 \backslash \{0, 2n\}$ *and* $C^{\dagger}_{odd} := C_2^{even} \cup C_3^{even}, C_{2,3}^{odd} = C_2^{odd} \cup C_3^{odd}$;
- $C_*^{even} := C^{even} \backslash \{0, 2n\}$ *and* $C_*^{odd} := C^{odd} \backslash \{0, n, 2n, 3n\}$.

Remark. From above, we have $|C^{\dagger}_{even}| = n-1$, $|C^{\dagger}_{odd}| = n$, and $|C_{2,3}^{odd}| = n+1$.

Lemma 9 ([**20**])**.** *For $n \geq 2$, the conjugacy classes of SD_{8n} are as follows:*

- *If n is even, there are $2n+3$ conjugacy classes. Precisely,*
  - *2 classes of sizes one being $\lfloor 1 \rfloor = \{1\}$ and $\lfloor a^{2n} \rfloor = \{a^{2n}\}$,*
  - *$2n-1$ classes of sizes two being $\lfloor a^r \rfloor = \{a^r, a^{(2n-1)r}\}$, where $r \in C_*^{even}$ and*
  - *2 classes of sizes $2n$ being $\lfloor b \rfloor = \{ba^{2t} \mid t = 0, 1, 2, \ldots, 2n-1\}$ and $\lfloor ba \rfloor = \{ba^{2t+1} \mid t = 0, 1, 2, \ldots, 2n-1\}$.*
- *If n is odd, there are $2n+6$ conjugacy classes. Precisely,*
  - *4 classes of sizes one being $\lfloor 1 \rfloor = \{1\}$, $\lfloor a^n \rfloor = \{a^n\}$, $\lfloor a^{2n} \rfloor = \{a^{2n}\}$ and $\lfloor a^{3n} \rfloor = \{a^{3n}\}$,*
  - *$2n-2$ classes of sizes two being $\lfloor a^r \rfloor = \{a^r, a^{(2n-1)r}\}$, where $r \in C_*^{odd}$ and*
  - *4 classes of sizes n being $\lfloor b \rfloor = \{ba^{4t} \mid t = 0, 1, 2, \ldots, n-1\}$, $\lfloor ba \rfloor = \{ba^{4t+1} \mid t = 0, 1, 2, \ldots, n-1\}$, $\lfloor ba^2 \rfloor = \{ba^{4t+2} \mid t = 0, 1, 2, \ldots, n-1\}$ and $\lfloor ba^3 \rfloor = \{ba^{4t+3} \mid t = 0, 1, 2, \ldots, n-1\}$.*

Now, with the aid of Definition 8 and Lemma 9, we tabulate the character table for SD_{8n}, $n \geq 2$ as follows.

Lemma 10 ([**20**])**.** *The character table of semi-dihedral group SD_{8n}, $n \geq 2$ is given in Table 1 if n is even, and in Table 2 if n is odd, χ_h's ($h = 1, 2, 3, 4$ if n is even and $h = 1, 2, \ldots, 8$ if n is odd) are irreducible characters of degree one, ς_j and ψ_l are irreducible characters of degree two, where $\omega = e^{\pi i/2n}$ and $i^2 = -1$.*

	$\lfloor a^r \rfloor$; $r \in C_1$	$\lfloor a^r \rfloor$; $r \in C^{\dagger}_{odd}$	$\lfloor b \rfloor$	$\lfloor ba \rfloor$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1
ς_j, $j \in C^{\dagger}_{even}$	$\omega^{jr} + \omega^{-jr}$	$\omega^{jr} + \omega^{-jr}$	0	0
ψ_l, $l \in C^{\dagger}_{odd}$	$\omega^{lr} + \omega^{-lr}$	$\omega^{lr} - \omega^{-lr}$	0	0

Table 1: Character table of SD_{8n} for n even.

	$\lfloor a^r \rfloor$; $r \in C_1$	$\lfloor a^r \rfloor$; $r \in C_{2,3}^{odd}$	$\lfloor b \rfloor$	$\lfloor ba \rfloor$	$\lfloor ba^2 \rfloor$	$\lfloor ba^3 \rfloor$
χ_1	1	1	1	1	1	1
χ_2	1	1	-1	-1	-1	-1
χ_3	1	-1	1	-1	1	-1
χ_4	1	-1	-1	1	-1	1
χ_5	$(-1)^{\frac{r}{2}}$	i^r	1	i	-1	$-i$
χ_6	$(-1)^{\frac{r}{2}}$	i^r	-1	$-i$	1	i
χ_7	$(-1)^{\frac{r}{2}}$	$(-i)^r$	1	$-i$	-1	i
χ_8	$(-1)^{\frac{r}{2}}$	$(-i)^r$	-1	i	1	$-i$
ς_j, $j \in C_{even}^{\dagger}$	$\omega^{jr} + \omega^{-jr}$	$\omega^{jr} + \omega^{-jr}$	0	0	0	0
ψ_l, $l \in C_{2,3}^{odd} \backslash \{n, 3n\}$	$\omega^{lr} + \omega^{-lr}$	$\omega^{lr} - \omega^{-lr}$	0	0	0	0

Table 2: Character table of SD_{8n} for n odd.

Lemma 11. *Let $\omega = e^{\pi i/2n}$, where $i^2 = -1$. Then, for all $0 < m < 4n$, we have*

(i) $\sum_{k=1}^{4n-1} \omega^k = -1$;

(ii) $\sum_{k=0}^{2n-1} \omega^{2km} = 0$;

(iii) $\sum_{k=0}^{2n-1} \omega^{(2k+1)m} = 0$.

Proof. The proof is straightforward. □

Lemma 12 ([**15**]). *A graph is bipartite if and only if the spectrum of its adjacency matrix is symmetric with respect to 0.*

3. THE SPECTRA OF CAYLEY GRAPHS OVER SD_{8n}

In light of Lemmas 3 and 10, we can get the spectrum of Cayley graphs over SD_{8n} immediately.

Theorem 13. *Let $\mathrm{SD}_{8n} (n \geq 2)$ be the semi-dihedral group and $S \subseteq \mathrm{SD}_{8n} \backslash \{1\}$ satisfying $S = S^{-1}$. Then*

$$Spec(X(\mathrm{SD}_{8n}, S)) \quad = \quad \left\{ [\lambda_h]^1; [\mu_{j1}]^2, [\mu_{j2}]^2; [\nu_{l1}]^2, [\nu_{l2}]^2 \right\},$$

and

$$\begin{cases} \lambda_h = \sum_{s \in S} \chi_h(s), \\ \mu_{j1} + \mu_{j2} = \sum_{s \in S} \varsigma_j(s), \\ \mu_{j1}^2 + \mu_{j2}^2 = \sum_{s_1, s_2 \in S} \varsigma_j(s_1 s_2), \\ \nu_{l1} + \nu_{l2} = \sum_{s \in S} \psi_l(s), \\ \nu_{l1}^2 + \nu_{l2}^2 = \sum_{s_1, s_2 \in S} \psi_l(s_1 s_2), \end{cases} \tag{1}$$

where $h = 1, 2, 3, 4$, $j \in C^{\dagger}_{even}$ *and* $l \in C^{\dagger}_{odd}$ *if* n *is even; or* $h = 1, 2, \ldots, 8$, $j \in C^{\dagger}_{even}$ *and* $l \in C^{odd}_{2,3} \backslash \{n, 3n\}$ *if* n *is odd.*

For convenience, we need some symbols. Let A, B be two subsets of a group G. For any character χ of G, we denote $\chi(A) = \sum_{a \in A} \chi(a)$ and $\chi(AB) = \sum_{a \in A, b \in B} \chi(ab)$. Particularly, $\chi(A^2) = \sum_{a_1, a_2 \in A} \chi(a_1 a_2)$. Then we have

Theorem 14. *Let* $\mathrm{SD}_{8n} (n \geq 2)$ *be the semi-dihedral group and let* $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{8n} \backslash \{1\}$ *be such that* $S = S^{-1}$, *where* $S_1 \subseteq \langle a \rangle$ *and* $S_2 \subseteq b\langle a \rangle$. *Then* $X(\mathrm{SD}_{8n}, S)$ *is integral if and only if the following conditions hold:*

(i) $\varsigma_j(S_1), \varsigma_j(S_1^2) + \varsigma_j(S_2^2)$ *are integers;*

(ii) $\Delta_j(S) = 2\left[\varsigma_j(S_1^2) + \varsigma_j(S_2^2)\right] - \left[\varsigma_j(S_1)\right]^2$ *is a square number;*

(iii) $\psi_l(S_1)$, $\psi_l(S_1^2) + \psi_l(S_2^2)$ *are integers;*

(iv) $\Delta_l(S) = 2\left[\psi_l(S_1^2) + \psi_l(S_2^2)\right] - \left[\psi_l(S_1)\right]^2$ *is a square number.*

Here $j \in C^{\dagger}_{even}$ *and* $l \in C^{\dagger}_{odd}$ *if* n *is even; or* $j \in C^{\dagger}_{even}$ *and* $l \in C^{odd}_{2,3} \backslash \{n, 3n\}$ *if* n *is odd.*

Proof. First we have that $S_1 S_2 = \{s_1 s_2 \mid s_1 \in S_1, s_2 \in S_2\} \subseteq b\langle a \rangle$ and $S_2 S_1 \subseteq b\langle a \rangle$, then by Lemma 10, $\varsigma_j(S_1 S_2) = 0 = \varsigma_j(S_2 S_1)$. Thus,

$$\begin{aligned} \varsigma_j(S) &= \sum_{s_1 \in S_1} \varsigma_j(s_1) + \sum_{s_2 \in S_2} \varsigma_j(s_2) = \varsigma_j(S_1), \\ \varsigma_j(S^2) &= \sum_{s_1, s_2 \in S} \varsigma_j(s_1 s_2) \\ &= \varsigma_j(S_1^2) + \varsigma_j(S_1 S_2) + \varsigma_j(S_2 S_1) + \varsigma_j(S_2^2) \\ &= \varsigma_j(S_1^2) + \varsigma_j(S_2^2). \end{aligned}$$

Similarly, we have

$$\psi_l(S) = \psi_l(S_1), \quad \psi_l(S^2) = \psi_l(S_1^2) + \psi_l(S_2^2).$$

Hence, the spectrum of $X(\mathrm{SD}_{8n}, S)$ presented in (1) should satisfy

$$\left\{ \begin{array}{l} \lambda_h = \sum_{s \in S} \chi_h(s) = \chi_h(S), \\ \mu_{j1} + \mu_{j2} = \varsigma_j(S_1), \\ \mu_{j1}^2 + \mu_{j2}^2 = \varsigma_j(S_1^2) + \varsigma_j(S_2^2), \\ \nu_{l1} + \nu_{l2} = \psi_l(S_1), \\ \nu_{l1}^2 + \nu_{l2}^2 = \psi_l(S_1^2) + \psi_l(S_2^2), \end{array} \right. \tag{2}$$

where $h = 1, 2, 3, 4$, $j \in C^{\dagger}_{even}$ and $l \in C^{\dagger}_{odd}$ if n is even; or $h = 1, 2, \ldots, 8$, $j \in C^{\dagger}_{even}$ and $l \in C^{odd}_{2,3} \backslash \{n, 3n\}$ if n is odd.

Now we suppose that $X(\mathrm{SD}_{8n}, S)$ is integral. Hence, by (2), $\varsigma_j(S_1)$ and $\varsigma_j(S_1^2)+\varsigma_j(S_2^2)$ must be integers, and thus (i) holds. Since μ_{j1} and μ_{j2} are integers, and they are also the roots of the following quadratic equation:

$$x^2 - \varsigma_j(S_1)x + \frac{1}{2}\left([\varsigma_j(S_1)]^2 - (\varsigma_j(S_1^2) + \varsigma_j(S_2^2))\right) = 0, \tag{3}$$

we have that the discriminant $\Delta_j(S) = 2\left[\varsigma_j(S_1^2) + \varsigma_j(S_2^2)\right] - [\varsigma_j(S_1)]^2$ must be a square number, thus (ii) follows. In the same way, we can get $(iii), (iv)$ hold.

Next we suppose that (i) and (ii) hold. Then, the solutions μ_{j1} and μ_{j2} of (3) must be rational. This implies that μ_{j1} and μ_{j2} must be integers because they are algebraic integers. And if $(iii), (iv)$ hold, we can get that ν_{l1} and ν_{l2} must be integers in the same way. We at last need to verify λ_i's are integers. Note that $\lambda_h = \chi_h(S) = \sum_{s\in S}\chi(s)$. Since $\chi_h(s) \in \{\pm 1, \pm i\}$, $\lambda_h = a + bi$ for some $a, b \in \mathbb{Z}$. It implies that $\lambda_h = a \in \mathbb{Z}$ since it is real. Thus the eigenvalues λ_h's are always integers too. Hence, $X(\mathrm{SD}_{8n}, S)$ is integral. □

By Theorems 13 and 14, we obtain the explicit formula of $X(\mathrm{SD}_{8n}, S)$ in the following way.

Corollary 15. *Let $\mathrm{SD}_{8n}(n \geq 2)$ be the semi-dihedral group and let $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{8n}\setminus\{1\}$ be such that $S = S^{-1}$. Then,*

$$Spec(X(\mathrm{SD}_{8n}, S)) \quad = \quad \left\{[\lambda_h]^1; [\mu_{j1}]^2, [\mu_{j2}]^2; [\nu_{l1}]^2, [\nu_{l2}]^2\right\},$$

where $\lambda_h = \sum_{s\in S}\chi_h(s)$, $\mu_{j1}, \mu_{j2} = \frac{\varsigma_j(S_1)\pm\sqrt{\Delta_j(S)}}{2}$ and $\nu_{l1}, \nu_{l2} = \frac{\psi_l(S_1)\pm\sqrt{\Delta_l(S)}}{2}$, for all $h = 1, 2, 3, 4$, $j \in C_{even}^{\dagger}$ and $l \in C_{odd}^{\dagger}$ if n is even; or $h = 1, 2, \ldots, 8$, $j \in C_{even}^{\dagger}$ and $l \in C_{2,3}^{odd}\setminus\{n, 3n\}$ if n is odd.

As an application of Theorem 14, we obtain a class of connected, integral, bipartite Cayley graphs over SD_{8n}.

Corollary 16. *Let $\mathrm{SD}_{8n}(n \geq 2)$ be the semi-dihedral group and let $S = S_1 \cup S_2 \subseteq SD_{8n}\setminus\{1\}$ be such that $S = S^{-1}$, where $S_1 = a\langle a^2\rangle$ and $S_2 = \{ba^{2m}\}$ for some $m \in \{0, 1, \ldots, 2n-1\}$. Then $X(\mathrm{SD}_{8n}, S)$ is connected, integral and bipartite.*

Proof. It is apparent that $S = S^{-1}$ generates SD_{8n}, so $X(\mathrm{SD}_{8n}, S)$ is connected.

Firstly we have $S_1^2 = 2n * \langle a^2\rangle$, $S_2^2 = \{1\}$. By Lemmas 10 and 11, we have

$$\varsigma_j(S_1) = \sum_{k=0}^{2n-1}\varsigma_j(a^{2k+1}) = \sum_{k=0}^{2n-1}(\omega^{j(2k+1)} + \omega^{-j(2k+1)}) = 0 + 0 = 0,$$

$$\varsigma_j(S_1^2) \quad = \quad 2n\sum_{k=0}^{2n-1}\varsigma_j(a^{2k}) = 2n\sum_{k=0}^{2n-1}(\omega^{2kj} + \omega^{-2kj}) = 2n(0+0) = 0,$$

and

$$\varsigma_j(S_2^2) = \varsigma_j(1) = 2.$$

Similarly, we have

$$\psi_l(S_1) = \sum_{k=0}^{2n-1} \psi_l(a^{2k+1}) = \sum_{k=0}^{2n-1} (\omega^{l(2k+1)} - \omega^{-l(2k+1)}) = 0 - 0 = 0,$$

$$\psi_l(S_1^2) \quad = \quad 2n \sum_{k=0}^{2n-1} \psi_l(a^{2k}) = 2n \sum_{k=0}^{2n-1} (\omega^{2kl} + \omega^{-2kl}) = 2n(0+0) = 0,$$

and

$$\psi_l(S_2^2) = \psi_l(1) = 2.$$

So we have that

$$\varsigma_j(S_1^2) + \varsigma_j(S_2^2) = 0 + 2 = 2, \qquad \psi_l(S_1^2) + \psi_l(S_2^2) = 0 + 2 = 2,$$

are integers. And

$$\Delta_j(S) = 2\left[\varsigma_j(S_1^2) + \varsigma_j(S_2^2)\right] - [\varsigma_j(S_1)]^2 = 2 \cdot (0+2) - 0 = 4,$$

$$\Delta_l(S) = 2\left[\psi_l(S_1^2) + \psi_l(S_2^2)\right] - [\psi_l(S_1)]^2 = 2 \cdot (0+2) - 0 = 4$$

are square numbers. So $X(\mathrm{SD}_{8n}, S)$ is integral by Theorem 14.

By Corollary 15, the spectrum of the graph $X(\mathrm{SD}_{8n}, S)$ can be obtained as

$$\begin{aligned} & Spec(X(\mathrm{SD}_{8n}, S)) \\ = \ & \{2n+1, 2n-1, -2n+1, -2n-1, [1]^{2n-2}, [-1]^{2n-2}[1]^{2n}, [-1]^{2n}\} \\ = \ & \{-2n-1, -2n+1, [-1]^{4n-2}, [1]^{4n-2}, 2n-1, 2n+1\} \end{aligned}$$

for even n, and

$$\begin{aligned} & Spec(X(\mathrm{SD}_{8n}, S)) \\ = \ & \{2n+1, 2n-1, -2n+1, -2n-1, -1, 1, -1, 1, [1]^{2n-2}, \\ & [-1]^{2n-2}[1]^{2n-2}, [-1]^{2n-2}\} \\ = \ & \{-2n-1, -2n+1, [-1]^{4n-2}, [1]^{4n-2}, 2n-1, 2n+1\} \end{aligned}$$

for odd n. Thus the graph $X(\mathrm{SD}_{8n}, S)$ is bipartite by Lemma 12. □

Corollary 16 implies

Corollary 17. *For any natural number $n \geq 2$, there is at least one connected $(2n+1)$-regular integral graph with $8n$ vertices.*

4. TESTING THE INTEGRALITY USING BOOLEAN ALGEBRA

In this section, we aim to simplify the result of Theorem 14 and provide infinite classes of integral Cayley graphs over $X(\mathrm{SD}_{8n}, S)$ in terms of Boolean algebra on cyclic groups.

Firstly, for cyclic group $\langle a\rangle$ of order $4n$, recall that the atom of $B\langle a\rangle$ has the form $[a^d] = \{a^l \mid (l, 4n) = d\}$, then $[a^d] \in [B(\langle a^2\rangle)] \subseteq \langle a^2\rangle$ if d is even, and $[a^d] \in [B(\langle a\rangle)]\backslash[B(\langle a^2\rangle)] \subseteq a\langle a^2\rangle$ if d is odd.

Let ϕ_h be the irreducible representations of $\langle a\rangle$, then $\phi_h(a^k) = \omega^{hk}$ for all $h, k = 0, 1, \ldots, 4n-1$ by Lemma 1. Using the irreducible characters of cyclic group of order $4n$, we have

Lemma 18. *Let* $\mathrm{SD}_{8n}(n \geq 2)$ *be the semi-dihedral group. If* $T \in B(\langle a\rangle)$*, then* $\varsigma_j(T)$ *is an integer and* $2\varsigma_j(T^2) = [\varsigma_j(T)]^2$ *is a square number for all* $j \in C^{\dagger}_{even}$.

Proof. Since $T \in B(\langle a\rangle)$, by Lemma 4, we have $T = T^{-1}$. So we may suppose that $T = \{a^d \mid d \in \Phi\}$, where $\Phi = -\Phi$ is a set of integers, then $\phi_h(T) = \sum_{d\in\Phi} \omega^{hd}$ is an integer by Lemma 5. It is easy to see $T^2 = \{a^{s+t} \mid s, t \in \Phi\}$. By Lemma 10, for each j, we have

$$\varsigma_j(T) = \sum_{d\in\Phi} \varsigma_j(a^d) = \sum_{d\in\Phi}(\omega^{jd} + \omega^{-jd}) = 2\sum_{d\in\Phi}\omega^{jd} = 2\sum_{d\in\Phi}\phi_j(a^d) = 2\phi_j(T).$$

Thus $\varsigma_j(T)$ is an integer since $\phi_j(T)$ is an integer. Therefore

$$\begin{aligned}
2\varsigma_j(T^2) &= 2\sum_{s,t\in\Phi}\varsigma_j(a^{s+t}) \\
&= 2\sum_{s,t\in\Phi}(\omega^{j(s+t)} + \omega^{-j(s+t)}) \\
&= 4\sum_{s,t\in\Phi}\omega^{j(s+t)} \\
&= \sum_{s\in\Phi}2\omega^{js}\sum_{t\in\Phi}2\omega^{jt} \\
&= \sum_{s\in\Phi}(\omega^{js} + \omega^{-js})\sum_{t\in\Phi}(\omega^{jt} + \omega^{-jt}) \\
&= \sum_{s\in\Phi}\varsigma_j(a^s)\sum_{t\in\Phi}\varsigma_j(a^t) \\
&= [\varsigma_j(T)]^2.
\end{aligned}$$

This completes the proof. □

Lemma 19. *Let* $\mathrm{SD}_{8n}(n \geq 2)$ *be the semi-dihedral group. If* $T \in B(\langle a^2\rangle)$*, then* $\psi_l(T)$ *is an integer and* $2\psi_l(T^2) = [\psi_l(T)]^2$ *is a square number; and if* $T \in B(\langle a\rangle)\backslash B(\langle a^2\rangle)$*, then* $\psi_l(T) = 0$ *and* $\psi_l(T^2)$ *is an integer for all* $l \in C^{\dagger}_{odd}$ *if* n *is even, or* $l \in C^{odd}_{2,3}\backslash\{n, 3n\}$ *if* n *is odd.*

Proof. If $T \in B(\langle a^2 \rangle)$, by the similar method as in Lemma 18, we have $\psi_l(T)$ is an integer and $2\psi_l(T^2) = [\psi_l(T)]^2$ is a square number.

If $T \in B(\langle a \rangle) \backslash B(\langle a^2 \rangle)$, then $T = T^{-1}$ by Lemma 4. We may suppose that $T = \{a^d \mid d \in \Psi\}$, where $\Psi = -\Psi$ is a set of integers. Thus, by Lemma 10, we have

$$\psi_l(T) = \sum_{d \in \Psi} \psi_l(a^d) = \sum_{d \in \Psi} (\omega^{ld} - \omega^{-ld}) = \frac{1}{2} \sum_{d, -d \in \Psi} \big((\omega^{ld} - \omega^{-ld}) + (\omega^{-ld} - \omega^{ld}) \big) = 0.$$

Note that $T^2 \in C(\langle a^2 \rangle)$, by Lemma 6, we have $\phi_l(T^2)$ is an integer. Since T is inverse-closed, T^2 is inverse-closed, i.e., there exists a multi-set $\Omega = -\Omega$ of integers such that $T^2 = \{a^p \mid p \in \Omega\}$. Therefore

$$\psi_l(T^2) = \sum_{p \in \Omega} \psi_l(a^p) = \sum_{p \in \Omega} (\omega^{lp} + \omega^{-lp}) = 2 \sum_{p \in \Omega} \omega^{lp} = 2 \sum_{p \in \Omega} \phi_l(a^p) = 2\phi_l(T^2),$$

which is an integer for each l. □

Lemma 20. *Let* $\mathrm{SD}_{8n} (n \geq 2)$ *be the semi-dihedral group, and* T^m *be an inverse-closed multi-set with* $T \subseteq \langle a^2 \rangle \subseteq \mathrm{SD}_{8n}$*. For all* $j \in C^{\dagger}_{even}$*,* $l \in C^{\dagger}_{odd}$ *if* n *is even, or* $j \in C^{\dagger}_{even}$*,* $l \in C^{odd}_{2,3} \backslash \{n, 3n\}$ *if* n *is odd, we have* $\varsigma_j(T^m)$ *and* $\psi_l(T^m)$ *are integers if and only if* $T^m \in C(\langle a^2 \rangle)$*. In particular,* $\varsigma_j(T)$ *and* $\psi_l(T)$ *are integers if and only if* $T \in B(\langle a^2 \rangle)$*.*

Proof. By Lemma 6, it suffices to show that $\varsigma_j(T^m)$ and $\psi_l(T^m)$ are integers if and only if $\phi_h(T^m)$ is an integer for $h \in \{0, 1, 2, \ldots, 2n-1\}$. Since T^m is inverse-closed, there exists a multi-set $U = -U$ of integers such that $T^m = \{a^{2u} \mid u \in U\}$.

If n is even, then we have

$$\{0, 1, 2, \ldots, 2n-1\} = \{0\} \cup C^{\dagger}_{even} \cup C^{even}_2 \cup \{4n - k \mid k \in C^{even}_3\}.$$

Note that $\phi_0(T^m)$ is always an integer.

For all $j \in C^{\dagger}_{even}$, we have

$$\varsigma_j(T^m) = \sum_{u \in U} \varsigma_j(a^{2u}) = \sum_{u \in U} (\omega^{2ju} + \omega^{-2ju}) = 2 \sum_{u \in U} \omega^{2ju} = 2 \sum_{u \in U} \phi_j(a^{2u}) = 2\phi_j(T^m).$$

For all $l \in C^{even}_2 \subseteq C^{\dagger}_{odd}$, we have

$$\psi_l(T^m) = \sum_{u \in U} \psi_l(a^{2u}) = \sum_{u \in U} (\omega^{2lu} + \omega^{-2lu}) = 2 \sum_{u \in U} \omega^{2lu} = 2 \sum_{u \in U} \phi_l(a^{2u}) = 2\phi_l(T^m).$$

For all $l \in C^{even}_3 \subseteq C^{\dagger}_{odd}$, we have $4n - l \in \{4n - k \mid k \in C^{even}_3\}$, therefore

$$\begin{aligned} \psi_l(T^m) &= \sum_{u \in U} \psi_l(a^{2u}) = \sum_{u \in U} (\omega^{2lu} + \omega^{-2lu}) = \sum_{u \in U} (\omega^{2(4n-l)u} + \omega^{-2(4n-l)u}) \\ &= 2 \sum_{u \in U} \omega^{2(4n-l)u} = 2 \sum_{u \in U} \phi_{4n-l}(a^{2u}) = 2\phi_{4n-l}(T^m). \end{aligned}$$

Notice that $\varsigma_j(T^m)$, $\psi_l(T^m)$ and $\phi_h(T^m)$ are algebraic integers, therefore the fact that $\varsigma_j(T^m)$, $\psi_l(T^m)$ are integers ($j \in C^\dagger_{even}$, $l \in C^\dagger_{odd}$) is equivalent to the fact $\phi_h(T^m)$ is an integer ($h \in \{0, 1, \ldots, 2n-1\}$).

If n is odd, in a similar way, the result follows. □

Equipped with Lemmas 18 and 19, we have

Theorem 21. *Let* $\mathrm{SD}_{8n}(n \geq 2)$ *be the semi-dihedral group, and let* $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{8n} \setminus \{1\}$ *be such that* $S = S^{-1}$, *where* $S_1 = H_{even} \cup H_{odd} \subseteq \langle a \rangle$, $H_{even} \subseteq \langle a^2 \rangle, H_{odd} \subseteq a\langle a^2 \rangle$ *and* $S_2 \subseteq b\langle a \rangle$. *If* $S_1 \in B(\langle a \rangle)$ *and* $2\varsigma_j(S_2^2)$, $2[\psi_l(H^2_{odd}) + \psi_l(S_2^2)]$ *are square numbers for all* $j \in C^\dagger_{even}$, $l \in C^\dagger_{odd}$ *if* n *is even, or* $j \in C^\dagger_{even}$, $l \in C^{odd}_{2,3} \setminus \{n, 3n\}$ *if* n *is odd, then* $X(\mathrm{SD}_{8n}, S)$ *is integral.*

Proof. Since $S = S^{-1}$, we have $S_1 = S_1^{-1}$, $H_{even} = H^{-1}_{even}$ and $H_{odd} = H^{-1}_{odd}$. Thus we have

$$S_1^2 = H^2_{even} \cup H_{even}H_{odd} \cup H_{odd}H_{even} \cup H^2_{odd},$$

where H^2_{even}, $H^2_{odd} \subseteq \langle a^2 \rangle$ and $H_{even}H_{odd} = H_{odd}H_{even} \subseteq a\langle a^2 \rangle$. By Lemma 19, we have

$$\psi_l(H_{odd}) = 0 \qquad \text{and} \qquad \psi_l(H_{even}H_{odd}) = \psi_l(H_{odd}H_{even}) = 0.$$

Therefore, we have

$$\psi_l(S_1) = \psi_l(H_{even}) + \psi_l(H_{odd}) = \psi_l(H_{even})$$

and

$$\psi_l(S_1^2) = \psi_l(H^2_{even}) + 2\psi_l(H_{even}H_{odd}) + \psi_l(H^2_{odd}) = \psi_l(H^2_{even}) + \psi_l(H^2_{odd}).$$

If $S_1 \in B(\langle a \rangle)$, then $H_{even} \in B(\langle a^2 \rangle)$ and $H_{odd} \in B(\langle a \rangle) \setminus B(\langle a^2 \rangle)$. By Lemma 18, both $\varsigma_j(S_1)$ and $2\varsigma_j(S_1^2)$ are integers for each j. By Lemma 19, $\psi_l(H_{even})$, $\psi_l(H^2_{even})$ and $\psi_l(H^2_{odd})$ are integers. Therefore, $\psi_l(S_1)$ and $\psi_l(S_1^2)$ are integers. And $\psi_l(S_2^2)$ is an integer for each l because $2[\psi_l(H^2_{odd}) + \psi_l(S_2^2)]$ is a square number and $\psi_l(H^2_{odd})$ is an integer. Note that $2\varsigma_j(S_1^2)$, $2\varsigma_j(S_2^2)$ are integers and $\varsigma_j(S_1^2)$, $\varsigma_j(S_2^2)$ are algebraic integers, we have $\varsigma_j(S_1^2)$, $\varsigma_j(S_2^2)$ must be integers. Therefore, $\varsigma_j(S_1^2) + \varsigma_j(S_2^2)$ is an integer. Similarly, we have $\psi_l(S_1^2) + \psi_l(S_2^2)$ is an integer. Since $S_1 \in B(\langle a \rangle)$, we have $2\varsigma_j(S_1^2) = [\varsigma_j(S_1)]^2$ by Lemma 18. Since $H_{even} \in B(\langle a^2 \rangle)$, we have $2\psi_l(H^2_{even}) = [\psi_l(H_{even})]^2 = [\psi_l(S_1)]^2$ by Lemma 19. Then, $\Delta_j(S) = 2\varsigma_j(S_2^2)$ and $\Delta_l(S) = 2[\psi_l(H^2_{odd}) + \psi_l(S_2^2)]$ are square numbers. Thus, $X(\mathrm{SD}_{8n}, S)$ is integral by Theorem 14. □

Theorem 21 provides a sufficient condition for the integrality of $X(\mathrm{SD}_{8n}, S)$. The following example reveals that this sufficient condition is not necessary.

Example 22. *Let* $\mathrm{SD}_{24} = \langle a, b | a^{12} = b^2 = 1, bab = a^5 \rangle$ *be the semi-dihedral group of order* 24 *and* $S = S_1 \cup S_2$ *where* $S_1 = H_{odd} = \{a, a^{11}\}$ *and* $S_2 = \{b\}$*. It is clear that* $X(\mathrm{SD}_{24}, S)$ *is connected and* $S = S^{-1}$*. By direct computation, we have* $S_1^2 = \{1, 1, a^2, a^{10}\}$ *and* $S_2^2 = \{1\}$*. Therefore, by Table 3, we have*

$$\varsigma_2(S_1) = 1 + 1 = 2, \quad \varsigma_4(S_1) = (-1) + (-1) = -2$$

and

$$\varsigma_j(S_1^2) + \varsigma_j(S_2^2) = 4 + (-1) + (-1) + 2 = 4$$

are integers, where $j = 2, 4$*. And*

$$\Delta_2(S) = 2 \cdot 4 - 2^2 = 4, \quad \Delta_4(S) = 2 \cdot 4 - (-2)^2 = 4$$

are square numbers. Similarly, we have $\psi_l(S_1) = 0$ *and* $\psi_l(S_1^2) + \psi_l(S_2^2) = 8$ *are integers,* $\Delta_l = 16$ *is a square number, where* $l = 1, 7$*. Therefore, by Theorem 14,* $X(\mathrm{SD}_{24}, S)$ *is integral, and the spectrum of the graph* $X(\mathrm{SD}_{24}, S)$ *is* $\{-3, [-2]^6, [-1]^3, [0]^4, [1]^3, [2]^6, 3\}$*, which is bipartite from Lemma 12. However,* $S_1 \notin B(\langle a \rangle)$*.*

	1	$\lfloor a^2 \rfloor$	$\lfloor a^4 \rfloor$	$\lfloor a^6 \rfloor$	$\lfloor a \rfloor$	$\lfloor a^3 \rfloor$	$\lfloor a^7 \rfloor$	$\lfloor a^9 \rfloor$	$\lfloor b \rfloor$	$\lfloor ba \rfloor$	$\lfloor ba^2 \rfloor$	$\lfloor ba^3 \rfloor$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_3	1	1	1	1	-1	-1	-1	-1	1	-1	1	-1
χ_4	1	1	1	1	-1	-1	-1	-1	-1	1	-1	1
χ_5	1	-1	1	-1	i	$-i$	$-i$	i	1	i	-1	$-i$
χ_6	1	-1	1	-1	i	$-i$	$-i$	i	-1	$-i$	1	i
χ_7	1	-1	1	-1	$-i$	i	i	$-i$	1	$-i$	-1	i
χ_8	1	-1	1	-1	$-i$	i	i	$-i$	-1	i	1	$-i$
ς_2	2	-1	-1	2	1	-2	1	-2	0	0	0	0
ς_4	2	-1	-1	2	-1	2	-1	2	0	0	0	0
ψ_1	2	1	-1	-2	i	$2i$	$-i$	$-2i$	0	0	0	0
ψ_7	2	1	-1	-2	$-i$	$-2i$	i	$2i$	0	0	0	0

Table 3: Character table of SD_{24}.

In the remainder of this section, we will provide two necessary and sufficient conditions for the integrality of $X(\mathrm{SD}_{8n}, S)$ by adding certain restrictions over S_1.

Theorem 23. *Let* $\mathrm{SD}_{8n} (n \geq 2)$ *be the semi-dihedral group, and let* $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{8n} \setminus \{1\}$ *be such that* $S = S^{-1}$*, where* $S_1 = H_{even} \cup H_{odd} \subseteq \langle a \rangle$*,* $H_{even} \subseteq \langle a^2 \rangle, H_{odd} \subseteq a\langle a^2 \rangle$ *and* $S_2 \subseteq b\langle a \rangle$*. If* $S_1 \in B(\langle a \rangle)$*, then* $X(\mathrm{SD}_{8n}, S)$ *is integral if and only if* $2\varsigma_j(S_2^2)$*,* $2[\psi_l(H_{odd}^2) + \psi_l(S_2^2)]$ *are square numbers, for all* $j \in C_{even}^{\dagger}$*,* $l \in C_{odd}^{\dagger}$ *if* n *is even, or* $j \in C_{even}^{\dagger}$*,* $l \in C_{2,3}^{odd} \setminus \{n, 3n\}$ *if* n *is odd.*

Proof. If $S_1 \in B(\langle a \rangle)$ and $2\varsigma_j(S_2^2)$, $2[\psi_l(H_{odd}^2)+\psi_l(S_2^2)]$ are square numbers for each j, l, by Theorem 21, $X(\mathrm{SD}_{8n}, S)$ is integral.

Conversely, assume that $S_1 \in B(\langle a \rangle)$ and $X(\mathrm{SD}_{8n}, S)$ is integral. In the proof of Theorem 21, we have $\Delta_j(S) = 2\varsigma_j(S_2^2)$ and $\Delta_l(S) = 2[\psi_l(H_{odd}^2)+\psi_l(S_2^2)]$. Thus, by Theorem 14, $2\varsigma_j(S_2^2)$ and $2[\psi_l(H_{odd}^2)+\psi_l(S_2^2)]$ are square numbers. □

Next we present another necessary condition for the integrality of $X(\mathrm{SD}_{8n}, S)$.

Corollary 24. *Let $\mathrm{SD}_{8n} (n \geq 2)$ be the semi-dihedral group, and let $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{8n} \setminus \{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a \rangle$, and $S_2 \subseteq b\langle a \rangle$. If $S_1 \in B(\langle a \rangle)$ and $X(\mathrm{SD}_{8n}, S)$ is integral, then $S_2^2 \in C(\langle a \rangle)$.*

Proof. If $S_1 \in B(\langle a \rangle)$ and $X(\mathrm{SD}_{8n}, S)$ is integral, then $2\varsigma_j(S_2^2)$ and $2[\psi_l(H_{odd}^2)+\psi_l(S_2^2)]$ are square numbers for all j, l stated Theorem 23. Then $\varsigma_j(S_2^2)$ and $\psi_l(S_2^2)$ must be rational numbers because $\psi_l(H_{odd}^2)$ is an integer by Lemma 19. Thus, we conclude that $\varsigma_j(S_2^2), \psi_l(S_2^2)$ are integers because $\varsigma_j(S_2^2)$ and $\psi_l(S_2^2)$ are algebraic integers. By Lemma 20, we get $S_2^2 \in C(\langle a \rangle)$. □

However, the following example reveals that the necessary condition given in Corollary 24 is not sufficient yet.

Example 25. *Let $\mathrm{SD}_{16} = \langle a, b \mid a^8 = b^2 = 1, bab = a^3 \rangle$ be the semi-dihedral group of order 16, and $S_1 = \emptyset$, $S = S_2 = \{b, ba, ba^4\}$. It is clear that $X(\mathrm{SD}_{16}, S)$ is connected. By direct computation, we have*

$$S^2 = \{1, 1, a^4, a^4, a^4, a, a^3, a^5, a^7\} = 2 * \{1\} \cup 3 * [a^4] \cup [a] \in C(\langle a \rangle).$$

Therefore, from Lemma 10, we have

$$2\varsigma_1(S_2^2) = 2\left(2\varsigma_1(1) + 3\varsigma_1(a^4) + \sum_{k=0}^{3} \varsigma_1(a^{2k+1})\right) = 2(2 \cdot 2 + 3 \cdot 2 + 0) = 20,$$

which is not a square number. By Theorem 14, $X(\mathrm{SD}_{8n}, S)$ is not integral.

Theorem 26. *Let $\mathrm{SD}_{8n} (n \geq 2)$ be the semi-dihedral group, and let $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{8n} \setminus \{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a^2 \rangle$, and $S_2 \subseteq b\langle a \rangle$. Then $X(\mathrm{SD}_{8n}, S)$ is integral if and only if $S_1 \in B(\langle a^2 \rangle)$ and $2\varsigma_j(S_2^2)$, $2\psi_l(S_2^2)$ are square numbers for all $j \in C_{even}^\dagger$, $l \in C_{odd}^\dagger$ if n is even, or $j \in C_{even}^\dagger$, $l \in C_{2,3}^{odd} \setminus \{n, 3n\}$ if n is odd.*

Proof. First we suppose that $X(\mathrm{SD}_{8n}, S)$ is integral. By Theorem 14, $\varsigma_j(S_1)$ and $\psi_l(S_1)$ are integers. By Lemma 20, we have $S_1 \in B(\langle a^2 \rangle)$. By Lemmas 18 and 19, we have $2\varsigma_j(S_1^2) = [\varsigma_j(S_1)]^2$ and $2\psi_l(S_1^2) = [\psi_l(S_1)]^2$. Therefore,

$$\Delta_j(S) = 2\left[\varsigma_j(S_1^2) + \varsigma_j(S_2^2)\right] - [\varsigma_j(S_1)]^2 = 2\varsigma_j(S_2^2)$$

and

$$\Delta_l(S) = 2\left[\psi_l(S_1^2) + \psi_l(S_2^2)\right] - [\psi_l(S_1)]^2 = 2\psi_l(S_2^2).$$

Again by Theorem 14, $2\varsigma_j(S_2^2)$ and $2\psi_l(S_2^2)$ are square numbers for each j, l.

Conversely, by Theorem 21, it is easy to see that $X(\mathrm{SD}_{8n}, S)$ is integral. $\square$

By Theorem 26, we give another class of integral Cayley graphs over semi-dihedral group SD_{8n}.

Corollary 27. *Let* $\mathrm{SD}_{8n} = \langle a, b | a^{4n} = b^2 = 1, bab = a^{2n-1}\rangle$ *be the semi-dihedral group and let* $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{8n} \setminus \{1\}$ *be such that* $S = S^{-1}$*, where* $S_1 = \{a^{2n}\}$ *and* $S_2 = ba\langle a^2\rangle \cup \{b\}$*. Then* $X(\mathrm{SD}_{8n}, S)$ *is connected and integral.*

Proof. It is easy to see that $S = S^{-1}$ generates SD_{8n}, and so $X(\mathrm{SD}_{8n}, S)$ is connected. Firstly, we have $S_1 = \{a^{2n}\} = [a^{2n}] \in B(\langle a^2\rangle)$, then $S_1^2 = \{1\}$. By direct calculation, we have $S_2^2 = 2n * \langle a^2\rangle \cup 2 * a\langle a^2\rangle \cup \{1\}$. By Lemmas 10 and 11, we have, for each j

$$\begin{aligned}
2\varsigma_j(S_2^2) &= 2\left(2n\sum_{k=0}^{2n-1}\varsigma_j(a^{2k}) + 2\sum_{k=0}^{2n-1}\varsigma_j(a^{2k+1}) + \varsigma_j(1)\right)\\
&= 2\left(2n\sum_{k=0}^{2n-1}(\omega^{2kj} + \omega^{-2kj}) + 2\sum_{k=0}^{2n-1}(\omega^{(2k+1)j} + \omega^{-(2k+1)j}) + 2\right)\\
&= 2\left(2n(0+0) + 2(0+0) + 2\right)\\
&= 4
\end{aligned}$$

is a square number. Similarly, we have $2\psi_l(S_2^2) = 4$ is a square number for each l. By Theorem 26, $X(\mathrm{SD}_{8n}, S)$ is integral.

Thus, we have $\Delta_j(S) = 2\varsigma_j(S_2^2) = 4$, $\Delta_l(S) = 2\psi_l(S_2^2) = 4$ and $\varsigma_j(S_1) = \varsigma_j(a^{2n}) = \omega^{2nj} + \omega^{-2nj} = 2$, $\psi_l(S_1) = \psi_l(a^{2n}) = -2$. Therefore, by Corollary 15, the spectrum of the graph $X(\mathrm{SD}_{8n}, S)$ can be obtained. If n is even, we have

$$\begin{aligned}
Spec(X(\mathrm{SD}_{8n}, S)) &= \left\{2n+2, -2n, -2n+2, 2n, [2]^{2n-2}, [0]^{2n-2}, [-2]^{2n}, [0]^{2n}\right\}\\
&= \left\{-2n, -2n+2, [-2]^{2n}, [0]^{4n-2}, [2]^{2n-2}, 2n, 2n+2\right\}.
\end{aligned}$$

And if n is odd, we have

$$\begin{aligned}
Spec(X(\mathrm{SD}_{8n}, S)) &= \{2n+2, -2n, -2n+2, 2n, 0, -2, 0, -2, [2]^{2n-2}, [0]^{2n-2},\\
&\qquad [-2]^{2n-2}, [0]^{2n-2}\}\\
&= \left\{-2n, -2n+2, [-2]^{2n}, [0]^{4n-2}, [2]^{2n-2}, 2n, 2n+2\right\}.
\end{aligned}$$

This completes the proof. $\square$

The Corollary 27 implies

Corollary 28. *For any natural number* $n \geq 2$*, there is at least a connected* $(2n+2)$*-regular integral graph with* $8n$ *vertices.*

5. TWO SPECIAL SEMI–DIHEDRAL GROUPS

In this section, we will consider two special semi-dihedral groups.

5.1 Integral Cayley Graphs over SD_{2^n}

At first, we study the integral Cayley graph over semi-dihedral group SD_{2^n} for $n \geq 4$, where

$$\mathrm{SD}_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, bab = a^{2^{n-2}-1} \rangle.$$

In Bertram Huppert's book *Endliche Gruppen*, this group is called a *Quasidiedergruppe*. In Dummit and Foote's book *Abstract Algebra*, it is called the *quasi-dihedral group*. In Daniel Gorenstein's book *Finite Groups*, this group is called the *semi-dihedral group* [**31**], we adopt this name in this section.

For cyclic group $\langle a \rangle$ of order 2^{n-1}, the set of atoms of the Boolean algebra $B(\langle a \rangle)$ is $[B\langle a \rangle] = \left\{[a], [a^2], [a^4], \ldots, [a^{2^{n-2}}]\right\}$. It is easy to know that the set of atoms of the Boolean algebra $B(\langle a^2 \rangle)$ is $[B\langle a^2 \rangle] = \left\{[a^2], [a^4], \ldots, [a^{2^{n-2}}]\right\}$ and thus $[B(\langle a \rangle)] \backslash [B(\langle a^2 \rangle)] = \{[a]\}$. Under these symbols, we have

Theorem 29. *Let $\mathrm{SD}_{2^n} (n \geq 4)$ be the semi-dihedral group, and let $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{2^n} \backslash \{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a \rangle$ and $S_2 \subseteq b\langle a \rangle$. If $S_1 \in B(\langle a \rangle)$ and $2\varsigma_j(S_2^2)$, $2\psi_l(S_2^2)$ are square numbers for all $j \in C_{even}^{\dagger}$, $l \in C_{odd}^{\dagger}$, then $X(\mathrm{SD}_{2^n}, S)$ is integral.*

Proof. Let $S_1 = H_{even} \cup H_{odd}$, where $H_{even} \subseteq \langle a^2 \rangle$ and $H_{odd} \subseteq a\langle a^2 \rangle$. Since $S_1 \in B(\langle a \rangle)$, we have $H_{even} \in B(\langle a^2 \rangle)$ and $H_{odd} \in [B(\langle a \rangle)] \backslash [B(\langle a^2 \rangle)]$. Thus, $H_{odd} = \emptyset$ or $H_{odd} = [a]$.

If $H_{odd} = \emptyset$, then obviously $\psi_l(H_{odd}^2) = 0$.

If $H_{odd} = [a]$, then $H_{odd}^2 = 2^{n-2} * \langle a^2 \rangle$, so, for each $l \in C_{odd}^{\dagger}$, we have

$$\psi_l(H_{odd}^2) = 2^{n-2} \sum_{k=1}^{2^{n-2}} (\omega^{2kl} + \omega^{-2kl}) = 2^{n-2} \cdot 0 = 0.$$

Therefore,

$$2\left(\psi_l(H_{odd}^2) + \psi_l(S_2^2)\right) = 2(0 + \psi_l(S_2^2)) = 2\psi_l(S_2^2).$$

By Theorem 21, $X(\mathrm{SD}_{2^n}, S)$ is integral. $\square$

According to Theorem 29, we can derive infinite classes of integral Cayley graphs over SD_{2^n}.

Corollary 30. *Let $\mathrm{SD}_{2^n} (n \geq 4)$ be the semi-dihedral group, and let $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{2^n} \backslash \{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a \rangle$ and $S_2 \subseteq b\langle a \rangle$. If $S_1, bS_2 \in B(\langle a \rangle)$, then $X(\mathrm{SD}_{2^n}, S)$ is integral.*

Proof. By Theorem 29, it suffices to show that $2\varsigma_j(S_2^2)$, $2\psi_l(S_2^2)$ are square numbers for all $j \in C^{\dagger}_{even}$, $l \in C^{\dagger}_{odd}$. Let $bS_2 = H_{even} \cup H_{odd}$, where $H_{even} \subseteq \langle a^2 \rangle$ and $H_{odd} \subseteq a\langle a^2 \rangle$. Since $bS_2 \in B(\langle a \rangle)$, we have $H_{even} \in B(\langle a^2 \rangle)$, and $H_{odd} = \emptyset$ or $[a]$.

Case 1: If $H_{odd} = \emptyset$, then $bS_2 = H_{even} \in B(\langle a^2 \rangle)$. By Lemma 4, $bS_2 = (bS_2)^{-1}$, without loss of generality, suppose that $bS_2 = \{a^{2k} | k \in \Phi\}$, where Φ is a set of integers satisfying $\Phi = -\Phi$, then $S_2 = \{ba^{2k} \mid k \in \Phi\}$. Therefore, by Lemma 7, we have

$$(bS_2)^2 = \{a^{2(k_1+k_2)} \mid k_1, k_2 \in \Phi\} \qquad \text{and} \qquad S_2^2 = \{a^{2(t_1-t_2)} \mid t_1, t_2 \in \Phi\}.$$

Note that $\Phi = -\Phi$, we have $(bS_2)^2 = S_2^2$. Since $bS_2 \in B(\langle a^2 \rangle)$, by Lemmas 18 and 19,

$$2\varsigma_j(S_2^2) = 2\varsigma_j((bS_2)^2) = [\varsigma_j(bS_2)]^2 \quad \text{and} \quad 2\psi_l(S_2^2) = 2\psi_l((bS_2)^2) = [\psi_l(bS_2)]^2$$

are square numbers.

Case 2: If $H_{odd} = [a]$, then $bS_2 = H_{even} \cup [a]$ and $S_2 = bH_{even} \cup b[a]$, where $H_{even} \in B(\langle a^2 \rangle)$ and $[a] = \{a^{2k-1} \mid k = 1, 2, \ldots, 2^{n-2}\}$. By Lemma 4, $H_{even} = H_{even}^{-1}$. Suppose that $H_{even} = \{a^{2l} | l \in \Omega\}$, where Ω is a set of integers satisfying $\Omega = -\Omega$, then $bH_{even} = \{ba^{2l} \mid l \in \Omega\}$. Thus, we have

$$(bS_2)^2 = H_{even}^2 \cup H_{even}[a] \cup [a]H_{even} \cup [a]^2,$$

and

$$S_2^2 = (bH_{even})^2 \cup (bH_{even})(b[a]) \cup (b[a])(bH_{even}) \cup (b[a])^2.$$

Next we will prove $(bS_2)^2 = S_2^2$. By Lemma 7, we have

$$(bH_{even})(b[a]) = (bH_{even}b)[a] = H_{even}^{-1}[a] = H_{even}[a].$$

Similarly,

$$(b[a])(bH_{even}) = [a]H_{even} \quad \text{and} \quad (b[a])^2 = [a]^2.$$

Further, by the same method as in Case 1, we have $(bH_{even})^2 = H_{even}^2$. Therefore we conclude $(bS_2)^2 = S_2^2$.

Since $bS_2 \in B(\langle a \rangle)$, by Lemma 18, for each $j \in C^{\dagger}_{even}$,

$$2\varsigma_j(S_2^2) = 2\varsigma_j((bS_2)^2) = [\varsigma_j(bS_2)]^2$$

is a square number. Note that

$$H_{even}[a] = [a]H_{even} = |\Omega| * [a] = |\Omega| * \{a^{2k-1} \mid k = 1, 2, \ldots, 2^{n-2}\},$$

and

$$[a]^2 = 2^{n-2} * \langle a^2 \rangle.$$

We have

$$\begin{aligned}\psi_l(H_{even}[a]) &= \psi_l([a]H_{even}) = |\Omega|\sum_{k=1}^{2^{n-2}}\psi_l(a^{2k-1})\\ &= |\Omega|\sum_{k=1}^{2^{n-2}}(\omega^{l(2k-1)}-\omega^{-l(2k-1)})\\ &= |\Omega|(0-0)=0,\end{aligned}$$

and

$$\psi_l([a]^2)=2^{n-2}\sum_{k=1}^{2^{n-2}}(\omega^{2kl}+\omega^{-2kl})=2^{n-2}\cdot 0=0.$$

Therefore, combining the above expressions, we obtain

$$\begin{aligned}\psi_l(S_2^2) &= \psi_l((bS_2)^2)=\psi_l(H_{even}^2)+\psi_l(H_{even}[a])+\psi_l([a]H_{even})+\psi_l([a]^2)\\ &= \psi_l(H_{even}^2)+0+0+0=\psi_l(H_{even}^2).\end{aligned}$$

Moreover, by Lemma 19, we have

$$2\psi_l(S_2^2)=2\psi_l((bS_2)^2)=2\psi_l(H_{even}^2)=[\psi_l(H_{even})]^2$$

is a square number too, for each $l\in C_{odd}^{\dagger}$. □

Remark. We would like to point out that the condition in Corollary 30 is not necessary. Let $S_1=a\langle a^2\rangle$ and $S_2=\{ba^2\}$, by Corollary 16, $X(\mathrm{SD}_{2^n},S)$ is connected, integral and bipartite, but $bS_2=\{a^2\}\notin B(\langle a\rangle)$.

5.2 Integral Cayley Graphs over SD_{8p}

For an odd prime p, let $\mathrm{SD}_{8p}=\langle a,b\mid a^{4p}=b^2=1, bab=a^{2p-1}\rangle$ be the semi-dihedral group of order $8p$.

Firstly, for cyclic group $\langle a\rangle$ of order $4p$, the set of atoms of the Boolean algebra $B(\langle a\rangle)$ is $[B(\langle a\rangle)]=\big\{[a],[a^2],[a^4],[a^p],[a^{2p}]\big\}$. It is easy to find the set of atoms of the Boolean algebra $B(\langle a^2\rangle)$ is $[B(\langle a^2\rangle)]=\{[a^2],[a^4],[a^{2p}]\}$ and $[B(\langle a\rangle)]\backslash[B(\langle a^2\rangle)]=\{\emptyset,[a],[a^p],a\langle a^2\rangle\}$, where, by easy calculation from the lines before Lemma 4, $[a^p]=\{a^p,a^{3p}\}$ and $[a]=\{a^{2k+1}\mid k=0,1,2,\ldots,2p-1\}\setminus\{a^p,a^{3p}\}$. Thus, we have

Theorem 31. *For an odd prime p, let $\mathrm{SD}_{8p}=\langle a,b\mid a^{4p}=b^2=1,bab=a^{2p-1}\rangle$, and let $S=S_1\cup S_2\subseteq \mathrm{SD}_{8p}\backslash\{1\}$ be such that $S=S^{-1}$, where $S_1\in\langle a\rangle$ and $S_2\in b\langle a\rangle$. If $S_1\in B(\langle a\rangle)$ and $2\varsigma_j(S_2^2)$, $2\psi_l(S_2^2)$ are square numbers for all $j\in C_{even}^{\dagger}$, $l\in C_{2,3}^{odd}\backslash\{p,3p\}$, then $X(\mathrm{SD}_{8p},S)$ is integral.*

Proof. Let $S_1=H_{even}\cup H_{odd}$, where $H_{even}\subseteq\langle a^2\rangle$ and $H_{odd}\subseteq a\langle a^2\rangle$. Since $S_1\in B(\langle a\rangle)$, we have $H_{even}\in B(\langle a^2\rangle)$ and $H_{odd}\in[B(\langle a\rangle)]\backslash[B(\langle a^2\rangle)]=\{\emptyset,[a],[a^p],a\langle a^2\rangle\}$. Thus we distinguish into the following four cases.

Case 1: If $H_{odd} = \emptyset$, then obviously $\psi_l(H_{odd}^2) = 0$.
Case 2: If $H_{odd} = [a] = a\langle a^2\rangle\backslash\{a^p, a^{3p}\}$, then

$$\begin{aligned} H_{odd}^2 &= \big((a\langle a^2\rangle \cdot a\langle a^2\rangle)\backslash 2*(a\langle a^2\rangle \cdot \{a^p, a^{3p}\})\big) \cup \big(\{a^p, a^{3p}\} \cdot \{a^p, a^{3p}\}\big) \\ &= 2p * \langle a^2\rangle\backslash 4 * \langle a^2\rangle \cup 2 * \{1, a^{2p}\} \\ &= (2p-4) * \langle a^2\rangle \cup 2 * \{1, a^{2p}\}. \end{aligned}$$

Thus, by Lemma 10, we have

$$\begin{aligned} \psi_l(H_{odd}^2) &= (2p-4)\sum_{k=0}^{2p-1} \psi_l(a^{2k}) + 2(\psi_l(1) + \psi_l(a^{2p})) \\ &= (2p-4)\sum_{k=0}^{2p-1} (\omega^{2kl} + \omega^{-2kl}) + 2\cdot(2-2) \\ &= (2p-4)\cdot(0+0) + 0 \\ &= 0. \end{aligned}$$

Case 3: If $H_{odd} = [a^p] = \{a^p, a^{3p}\}$, then $H_{odd}^2 = 2 * \{1, a^{2p}\}$. So we have

$$\psi_l(H_{odd}^2) = 2(\psi_l(1) + \psi_l(a^{2p})) = 2\cdot(2-2) = 0.$$

Case 4: If $H_{odd} = a\langle a^2\rangle$, then $H_{odd}^2 = 2p * \langle a^2\rangle$. So we have

$$\psi_l(H_{odd}^2) = 2p\sum_{k=0}^{2p-1} \psi_l(a^{2k}) = 2p\sum_{k=0}^{2p-1} (\omega^{2kl} + \omega^{-2kl}) = 2p\cdot(0+0) = 0.$$

Therefore, for each $l \in C_{2,3}^{odd}\backslash\{p, 3p\}$,

$$2(\psi_l(H_{odd}^2) + \psi_l(S_2^2)) = 2(0 + \psi_l(S_2^2)) = 2\psi_l(S_2^2).$$

By Theorem 21, $X(\mathrm{SD}_{8p}, S)$ is integral. □

Theorem 31 implies

Corollary 32. *For an odd prime p, let $\mathrm{SD}_{8p} = \langle a, b \mid a^{4p} = b^2 = 1, bab = a^{2p-1}\rangle$, and let $S = S_1 \cup S_2 \subseteq \mathrm{SD}_{8p}\backslash\{1\}$ be such that $S = S^{-1}$, where $S_1 \subseteq \langle a\rangle$ and $S_2 \subseteq b\langle a\rangle$. If $S_1, bS_2 \in B(\langle a\rangle)$, then $X(\mathrm{SD}_{8p}, S)$ is integral.*

Proof. The proof is similar to that of Corollary 30. □

Remark. Let $\mathrm{SD}_{8n} (n \geq 4)$ be the semi-dihedral group, where $n = 2^{t-3}$ $(t \geq 4)$ or $n = p$ for odd prime p. For $S_1 \in B(\langle a\rangle)$, by Theorems 23, 29 and 31, if we want to keep the integrality of $X(\mathrm{SD}_{8n}, S)$, we just need to make sure that $S_2 = S_2^{-1}$ and $2\varsigma_j(S_2^2)$, $2\psi_l(S_2^2)$ are square numbers.

If $|S_2| = 1$, since $S_2 = S_2^{-1}$, then we must have $S_2 = \{ba^{2k}\}$ for some $k \in \{0, 1, 2, \ldots, 2n-1\}$. Thus we have $S_2^2 = \{1\}$. Therefore $2\varsigma_j(S_2^2) = 2\psi_l(S_2^2) = 4$ is a square number.

If $|S_2| = 2$, since $S_2 = S_2^{-1}$, then we must have $S_2 = \{ba^{2k+1}, (ba^{2k+1})^{-1}\}$ for some $k \in \{0, 1, 2, \ldots, 2n-1\}$. Thus we have $S_2^2 = 2*[a^{2n}] = 2*\{1, a^{2n}\}$. Therefore, for each j, l, since j is even and l is odd, we have

$$2\varsigma_j(S_2^2) = 4\left(\varsigma_j(1) + \varsigma_j(a^{2n})\right) = 4\left(2 + (\omega^{2nj} + \omega^{-2nj})\right) = 4(2+2) = 16$$

and

$$2\psi_l(S_2^2) = 4\left(\psi_l(1) + \psi(a^{2n})\right) = 4\left(2 + (\omega^{2nl} + \omega^{-2nl})\right) = 4(2-2) = 0$$

are square numbers.

Thus, for these two special semi-dihedral groups in this section, if $S_1 \in B(\langle a \rangle)$ and $|S_2| < 3$, then $X(\mathrm{SD}_{8n}, S)$ is integral if and only if $S_2 \in \left\{\emptyset, \{ba^{2k}\}, \{ba^{2k+1}, (ba^{2k+1})^{-1}\}\right\}$ for some $k \in \{0, 1, 2, \ldots, 2n-1\}$.

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