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SHARP INEQUALITIES FOR THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS

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By studying the monotonicity properties of $\mathcal{K}(r), \mathcal{E}(r)$ and some combinations of elementary functions and special functions, some new inequalities for the complete elliptic integrals of the first and second kinds are established. where $\mathcal{K}(r) = \int_0^{\pi/2}(1 - r^2 \sin^2 \theta)^{-1/2} d\theta, \mathcal{E}(r) = \int_0^{\pi/2}(1 - r^2 \sin^2 \theta)^{1/2} d\theta$ denote the complete elliptic integrals of the first and second kind, $r \in (0, 1)$.

1. INTRODUCTION

For $0 < r < 1$ and $r' = \sqrt{1 - r^2}$. Legendre's complete elliptic integrals of the first and second kinds [**14, 15**] are defined by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2}(1 - r^2 \sin^2 \theta)^{-1/2} d\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \pi/2, \qquad \mathcal{K}(1) = \infty \end{cases}$$

and

$$\begin{cases} \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2}(1 - r^2 \sin^2 \theta)^{1/2} d\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \pi/2, \qquad \mathcal{E}(1) = 1, \end{cases}$$

respectively.

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It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory, quasiconformal analysis, theory of mean values, number theory and other related fields [**4, 5, 6, 9, 14, 15, 28, 24, 30, 32, 25, 23, 35**].

Recently, the complete elliptic integrals have attracted the attention of numerous mathematicians. In particular, many remarkable properties and inequalities for the complete elliptic integrals can be found in the literature [**1, 2, 3, 7, 8, 10, 11, 17, 18, 19, 20, 21, 36, 41, 46, 22, 47, 37, 38, 39, 42, 27, 40, 43, 44, 45, 33, 29, 48**]

In 1992, Anderson et al.[**8**] discovered that $\mathcal{K}$ can be approximated by the inverse hyperbolic tangent function, and proved that

$$\frac{\pi}{2}\left(\frac{\mathrm{arthr}}{r}\right)^{1/2} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\mathrm{arthr}}{r}\right) \tag{1}$$

for $r \in (0,1)$.

It is also worth mentioning that the left hand side of inequality (1) was improved by Alzer and Qiu [**2**, Thoerem 18]. They proved that the double inequality

$$\frac{\pi}{2}\left(\frac{\mathrm{arthr}}{r}\right)^{\alpha} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\mathrm{arthr}}{r}\right)^{\beta} \tag{2}$$

holds for all $r \in (0,1)$ with the best possible constants $\alpha = 3/4$ and $\beta = 1$ and proposed an open problem as follows.

Open problem: The double inequality

$$\frac{\pi}{2}\left(\frac{\mathrm{arthr}}{r}\right)^{3/4+\alpha r} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\mathrm{arthr}}{r}\right)^{3/4+\beta r} \tag{3}$$

holds for all $r \in (0,1)$ with the best possible constants $\alpha = 0$ and $\beta = 1/4$. This problem has been proved in [**18**].

Alzer and Qiu [**2**, Theorem 20] proved that the following double inequality

$$a + \alpha_3(1-r') + \log(1+1/r') < \mathcal{K}(r) < a + \beta_3(1-r') + \log(1+1/r') \tag{4}$$

hold For $a = \pi/2 - \log 2 = 0.2853...$ and all real numbers $r \in (0,1)$ with the best possible constants

$$\alpha_3 = \frac{\pi}{4} - \frac{1}{2} = 0.2853... \text{ and } \beta_3 = 3\log 2 - \frac{\pi}{2} = 0.5086....$$

M. Vuorinen [**31**] conjectured that inequality

$$\mathcal{E}(r) > \frac{\pi}{2}\left(\frac{1+r'^{3/2}}{2}\right)^{2/3} \tag{5}$$

holds for all $r \in (0,1)$. This conjecture was proved by R. W. Barnard et al. in [**12**]. Later, they also provided an upper bound for $\mathcal{E}(r)$ [**13**]

$$\mathcal{E}(r) < \frac{\pi}{2}\left(\frac{1+r'^2}{2}\right)^{1/2},\ 0 < r < 1. \tag{6}$$

Recently, some bounds for $\mathcal{E}(r)$ were discovered in the paper [**34**]. For example, Theorem 1.2 in [**34**] states that, for $r \in (0,1)$, the following double inequalities hold.

$$\frac{\pi}{2} - \frac{3\pi}{16}\frac{r - r'^2\mathrm{arthr}}{r} < \mathcal{E}(r) < \frac{\pi}{2} - \left(\frac{\pi}{2} - 1\right)\frac{r - r'^2\mathrm{arthr}}{r} \tag{7}$$

$$\frac{\pi}{2} - \log r' - r\mathrm{arthr} < \mathcal{E}(\mathrm{r}) < \frac{\pi}{2} - \left(2 - \frac{\pi}{4}\right)\log \mathrm{r}' - \mathrm{rarthr} \tag{8}$$

In this paper, inspired by the double inequalities (7) and (8), we obtain several optimal upper and lower bounds for complete elliptic integrals of the first and second kind, by studying the monotonicity properties of functions, which are defined in terms of $\mathcal{K}$ and $\mathcal{E}$.

2. PRELIMINARIES AND LEMMAS

In order to establish our main results we need several formulas and Lemmas, which we present in this section.

For real numbers a, b and c with $c \neq 0, -1, -2, ...$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) =_2 F_1(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}, for|x| < 1. \tag{9}$$

Here, $(a,0) = 1$ for $a \neq 0$ and (a,b) denotes the shifted factorial function

$$(a,n) = a(a+1)(a+2)(a+3)\cdots(a+n-1)$$

for $n = 1, 2\cdots$.

For $0 < r < 1$, the following formulas were presented in [**9**, 1.20 Exercises] and [**9**, (3.13)]:

$$\begin{aligned}\arcsin r &= rF(\frac{1}{2},\frac{1}{2};\frac{3}{2};r^2)\\ \text{arthr} &= rF(\frac{1}{2},1;\frac{3}{2};r^2)\\ \mathcal{K}(r) &= \frac{\pi}{2}F(\frac{1}{2},\frac{1}{2};1;r^2)\\ \mathcal{E}(r) &= \frac{\pi}{2}F(\frac{1}{2},-\frac{1}{2};1;r^2).\end{aligned}$$

Lemma 1. *[16, 24] If $n \geq 1$, then*

$$n+\frac{1}{4} < \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 < n+\frac{4}{\pi}-1. \tag{10}$$

Lemma 2. *[2, 26] Let a_n and $b_n (n = 0,1,2,\cdots)$ be real numbers, and let the power series $A(t)=\sum_{n=0}^{\infty} a_n t^n$ and $B(t)=\sum_{n=0}^{\infty} b_n t^n$ be convergent for $|t|<R$. If $b_n > 0$ for $n = 0,1,2,\cdots$, and if $\frac{a_n}{b_n}$ is strictly increasing (or decreasing) for $n = 0,1,2,\cdots$, then the function $\frac{A(t)}{B(t)}$ is strictly increasing (or decreasing) on $(0,R)$.*

3. MAIN RESULTS

Theorem 1. *The function $f_1(r) = r[\pi/2-\mathcal{E}(r)]/(r-r'^2\sinh r)$ in strictly increasing from $(0,1)$ onto $(3\pi/20, \pi/2-1)$. Moreover, the double inequality*

$$\frac{\pi}{2}-\left(\frac{\pi}{2}-1\right)\frac{r-r'^2\sinh r}{r} < \mathcal{E}(r) < \frac{\pi}{2}-\frac{3\pi}{20}\frac{r-r'^2\sinh r}{r} \tag{11}$$

holds for all $r\in(0,1)$.

Proof. Using series expansion

$$\sinh x = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!},$$

and

$$\mathcal{E}(r) = \frac{\pi}{2}F(\frac{1}{2},-\frac{1}{2};1;r^2),$$

we have

$$f_1(r) = \frac{\frac{\pi}{2}-\mathcal{E}(r)}{1-(1-r^2)\frac{\sinh r}{r}} = \frac{\pi}{2}\frac{\sum_{n=0}^{\infty} R_n r^{2n}}{\sum_{n=0}^{\infty} S_n r^{2n}} \tag{12}$$

where $R_n = \frac{(\frac{1}{2},n)(\frac{1}{2},n+1)}{2[(n+1)!]^2}$ and $S_n = \frac{(2n+3)(2n+2)-1}{(2n+3)!}$.

Let $T_n = R_n/S_n$. Then

$$\frac{T_{n+1}}{T_n} = \frac{(2n+5)(2n+3)(2n+1)(4n^2+10n+5)}{2(4n^2+18n+19)(n+2)}. \tag{13}$$

simple computations lead to

$$\begin{aligned}&(2n+5)(2n+3)(2n+1)(4n^2+10n+5) - 2(4n^2+18n+19)(n+2)\\ &= 224n^4 + 576n^3 + 648n^2 + 32n^5 + 270n - 1 > 0\end{aligned}$$

for all $n \geq 1$. This implies that

$$\frac{T_{n+1}}{T_n} > 1. \tag{14}$$

Inequality (14) implies that T_n is strictly increasing for $n = 1, 2, \cdots$, therefore, from (12) and Lemma 2 we clearly see that $f_1(r)$ is strictly increasing in $(0,1)$. Moreover, making use of l'Hôpital's rule we have $f_1(0^+) = 3\pi/20$ and $f_1(1^-) = \pi/2 - 1$. □

Theorem 2. *The function* $f_2(r) = [\frac{\pi}{2}\mathrm{arthr} - \mathrm{r}\mathcal{E}(\mathrm{r})]/(\mathrm{r}^2\mathrm{arthr})$ *in strictly increasing from* $(0,1)$ *onto* $(7\pi/24, \pi/2)$. *Moreover, the double inequality*

$$\frac{\pi}{2}\frac{\mathrm{arthr}}{r}(1-r^2) < \mathcal{E}(r) < \frac{\pi}{2}\frac{\mathrm{arthr}}{r}\left(1 - \frac{7}{12}r^2\right) \tag{15}$$

holds for all $r \in (0,1)$.

Proof. Using series expansion we have

$$f_2(r) = \frac{\frac{\pi}{2}\mathrm{arthr} - \mathrm{r}\mathcal{E}(\mathrm{r})}{r^2\mathrm{arthr}} = \frac{\pi}{2}\frac{\sum_{n=1}^{\infty} R_n r^{2n}}{\sum_{n=1}^{\infty} S_n r^{2n}}, \tag{16}$$

where $R_n = \frac{1}{2n+1} + \frac{(1/2,n)^2}{(2n-1)(n!)^2}$ and $S_n = \frac{1}{2n-1}$.

Let $T_n = \frac{R_n}{S_n} = \frac{2n-1}{2n+1} + \frac{1}{\pi}\left(\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right)^2$. Then

$$T_{n+1} - T_n = \frac{4}{(2n+1)(2n+3)} - \frac{4n+3}{4\pi(n+1)^2}\left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right)^2. \tag{17}$$

We only need to prove $T_{n+1} - T_n > 0$,which is equivalent to

$$\frac{16\pi(n+1)^2}{(2n+1)(2n+3)(4n+3)} > \left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right)^2. \tag{18}$$

By Lemma 1, we only need to prove

$$\frac{16\pi(n+1)^2}{(2n+1)(2n+3)(4n+3)} > \frac{4}{4n+1} \tag{19}$$

Since $\pi > 3$, we only to prove

$$\frac{12(n+1)^2}{(2n+1)(2n+3)(4n+3)} > \frac{1}{4n+1} \tag{20}$$

easy calculation gives

$$\begin{aligned} &12(n+1)^2(4n+1) - (2n+1)(2n+3)(4n+3) \\ &= 32n^3 + 64n^2 + 36n + 3 > 0 \end{aligned}$$

for all $n \geq 1, n \subseteq N$.

Thus $T_{n+1} > T_n$, which implies that T_n is strictly increasing for $n = 1, 2, \cdots$, therefore, from (16) and Lemma 2 we clearly see that $f_2(r)$ is strictly increasing in $(0,1)$. Moreover, using l'Hôpital's rule we have $f_2(0^+) = 7\pi/24$ and $f_2(1^-) = \pi/2$. $\square$

Theorem 3. *The function* $f_3(r) = [\frac{\pi}{2}\arcsin r - r\mathcal{E}(r)]/(r^2 \arcsin r)$ *in strictly increasing from* $(0,1)$ *onto* $(5\pi/24, 1/2(\pi - 4/\pi))$. *Moreover, the double inequality*

$$\frac{\pi}{2}\frac{\arcsin r}{r}\left[1 - (1 - \frac{4}{\pi^2})r^2\right] < \mathcal{E}(r) < \frac{\pi}{2}\frac{\arcsin r}{r}\left(1 - \frac{5}{12}r^2\right) \tag{21}$$

holds for all $r \in (0,1)$.

Proof. Using series expansion we have

$$f_3(r) = \frac{\frac{\pi}{2}\arcsin r - r\mathcal{E}(r)}{r^2 \arcsin r} = \frac{\pi}{2}\frac{\sum_{n=1}^{\infty} R_n r^{2n}}{\sum_{n=1}^{\infty} S_n r^{2n}} \tag{22}$$

where

$$R_n = \frac{(\frac{1}{2}, n)(\frac{1}{2}, n)}{n!}\left[\frac{1}{(1+2n)(\frac{1}{2}, n)} + \frac{1}{(2n-1)n!}\right]$$

and

$$S_n = \frac{(\frac{1}{2}, n)(\frac{1}{2}, n)}{n!}\frac{2n}{(\frac{1}{2}, n)(2n-1)^2},$$

Let $T_n = \frac{R_n}{S_n} = \frac{2n-1}{2n}\left[\frac{2n-1}{2n+1} + \frac{(\frac{1}{2}, n)}{n!}\right] = \frac{2n-1}{2n}\left[\frac{2n-1}{2n+1} + \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}\right]$. Then

$$T_{n+1} - T_n = \frac{12n^2 + 8n - 3}{2n(2n+3)(2n+1)(n+1)} + \frac{(2n+3)\Gamma(n+1/2)}{4\sqrt{\pi}n(n+1)\Gamma(n+1)} > 0. \tag{23}$$

Inequalities (22) and (23) together with Lemma 2 lead to the conclusion that $f_3(r)$ is strictly increasing in $(0,1)$. Moreover, making use of l'Hôpital's rule we have $f_3(0^+) = 5\pi/24$ and $f_3(1^-) = \frac{\pi}{2}\left(1 - \frac{4}{\pi^2}\right)$. □

Theorem 4. *The function* $f_4(r) = [\frac{\pi}{2}\mathrm{arthr} - \mathrm{r}\mathcal{K}(\mathrm{r})]/(\mathrm{r}^2\mathrm{arthr})$ *in strictly increasing from* $(0,1)$ *onto* $(\pi/24, \pi/2 - 1)$*. Moreover, the double inequality*

$$\frac{\pi}{2}\frac{\mathrm{arthr}}{r}\left[1 - (1 - \frac{2}{\pi})r^2\right] < \mathcal{K}(r) < \frac{\pi}{2}\frac{\mathrm{arthr}}{r}\left(1 - \frac{1}{12}r^2\right) \tag{24}$$

holds for all $r \in (0,1)$.

Proof. Using series expansion we have

$$f_4(r) = \frac{\frac{\pi}{2}\mathrm{arthr} - \mathrm{r}\mathcal{K}(\mathrm{r})}{r^2\mathrm{arthr}} = \frac{\pi}{2}\frac{\sum_{n=1}^{\infty} R_n r^{2n}}{\sum_{n=1}^{\infty} S_n r^{2n}} \tag{25}$$

where $R_n = \frac{1}{2n+1} - \frac{(1/2,n)^2}{(n!)^2}$ and $S_n = \frac{1}{2n-1}$.

Let $T_n = \frac{R_n}{S_n} = \frac{2n-1}{2n+1} - \frac{2n-1}{\pi}\left(\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right)^2$. Then

$$T_{n+1} - T_n = \frac{4}{(2n+1)(2n+3)} - \frac{6n+5}{4\pi(n+1)^2}\left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right)^2 \tag{26}$$

We only need to prove $T_{n+1} - T_n > 0$,which is equivalent to

$$\frac{16\pi(n+1)^2}{(2n+1)(2n+3)(6n+5)} > \left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right)^2 \tag{27}$$

By Lemma 1, we only need to prove

$$\frac{16\pi(n+1)^2}{(2n+1)(2n+3)(6n+5)} > \frac{4}{4n+1} \tag{28}$$

Since $\pi > 3$, we only to prove

$$\frac{12(n+1)^2}{(2n+1)(2n+3)(6n+5)} > \frac{1}{4n+1} \tag{29}$$

easy calculation gives

$$\begin{aligned} &12(n+1)^2(4n+1) - (2n+1)(2n+3)(6n+5) \\ &= 24n^3 + 40n^2 + 14n - 3 > 0 \end{aligned}$$

for all $n \geq 1, n \subseteq N$.

This means that the sequence T_n is strictly increasing for $n = 1, 2, \cdots$, therefore, from (25) and Lemma 2 we clearly see that $f_4(r)$ is strictly increasing in $(0,1)$. Moreover, using l'Hôpital's rule we have $f_4(0^+) = \pi/24$ and $f_4(1^-) = \pi/2 - 1$. □

Theorem 5. *The function* $f_5(r) = [\frac{\pi}{2} - \mathcal{K}(r) - r\mathrm{arthr}]/(\log(\mathrm{r}'))$ *in strictly decreasing from* $(0,1)$ *onto* $(2, \pi/4+2)$. *Moreover, the double inequality*

$$\frac{\pi}{2} - (2+\frac{\pi}{4})\log r' - r\mathrm{arthr} < \mathcal{K}(\mathrm{r}) < \frac{\pi}{2} - 2\log \mathrm{r}' - \mathrm{rarthr}. \tag{30}$$

holds for all $r \in (0,1)$.

Proof. Making Use of series expansion one has

$$f_5(r) = \frac{\frac{\pi}{2} - r\mathrm{arthr} - \mathcal{K}(\mathrm{r})}{\frac{1}{2}\log(1-r^2)} = \frac{\sum_{n=1}^{\infty} R_n r^{2n}}{\sum_{n=1}^{\infty} S_n r^{2n}} \tag{31}$$

where $R_n = \pi\frac{(1/2,n)^2}{(n!)^2} + \frac{2}{2n-1}$ and $S_n = \frac{1}{n}$.

Let $T_n = \frac{R_n}{S_n} = n\left(\pi\frac{(1/2,n)^2}{(n!)^2} + \frac{2}{2n-1}\right) = n\left[\left(\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right)^2 + \frac{2}{2n-1}\right]$. Then

$$T_{n+1} - T_n = \frac{1}{4(n+1)^2}\left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right)^2 - \frac{2}{4n^2-1} \tag{32}$$

We only need to prove $T_{n+1} - T_n < 0$,which is equivalent to

$$\left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right)^2 > \frac{4n^2-1}{8(n+1)^2} \tag{33}$$

By Lemma 1, we only need to prove

$$\frac{4n+1}{4} > \frac{4n^2-1}{8(n+1)^2} \tag{34}$$

Simple calculation gives

$$\frac{4n+1}{4} - \frac{4n^2-1}{8(n+1)^2} = \frac{8n^3+14n^2+12n+3}{8(n+1)^2} > 0$$

for all $n \geq 1, n \subseteq N$.

Hence, the sequence T_n is strictly decreasing for $n = 1, 2, \cdots$, therefore, from (31) and Lemma 2 we clearly see that $f_5(r)$ is strictly increasing in $(0,1)$. Moreover, using l'Hôpital's rule we have $f_5(0^+) = 2 + \frac{\pi}{4}$ and $f_5(1^-) = 2$ □

4. CONCLUDING REMARKS

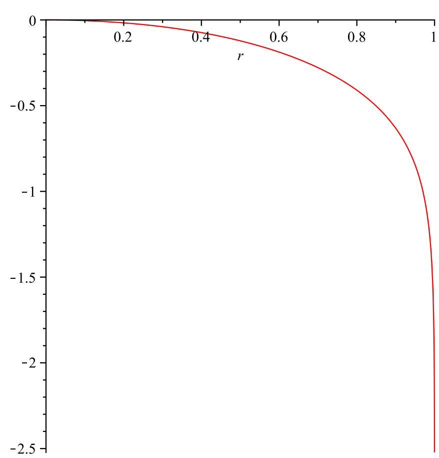
Remark 1. *The upper bound for $\mathcal{K}(r)$ in Theorem 4 is sharper than the corresponding one in* (1)*.*

Remark 2. *Moreover, based on numerical experiments, we note that our upper and lower bounds from Theorem 5 are better than the corresponding upper and lower bounds from* (3)*. Indeed, we consider the functions $u, l : (0,1) \to \mathbb{R}$ defined by*

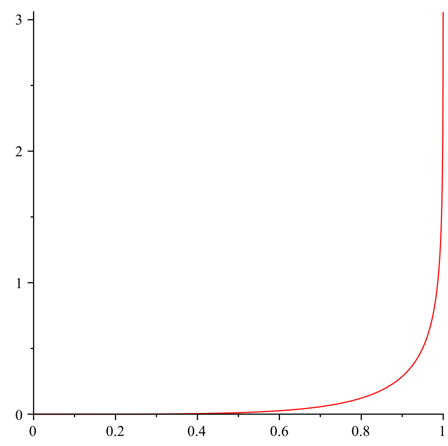
$$u(r) = \frac{\pi}{2} - 2\log r' - r\mathrm{arthr} - \frac{\pi}{2}\left(\frac{\mathrm{arthr}}{\mathrm{r}}\right)^{3/4+\mathrm{r}/4},$$

$$l(r) = \frac{\pi}{2} - (2+\frac{\pi}{4})\log r' - r\mathrm{arthr} - \frac{\pi}{2}\left(\frac{\mathrm{arthr}}{\mathrm{r}}\right)^{3/4}.$$

Then Figure 1 shows that the upper and lower bounds in (30) *for $\mathcal{K}(r)$ are better than the upper and lower bounds in* (3)*.*



(a) The graph of the functions $u(r)$



(b) The graph of the functions $l(r)$

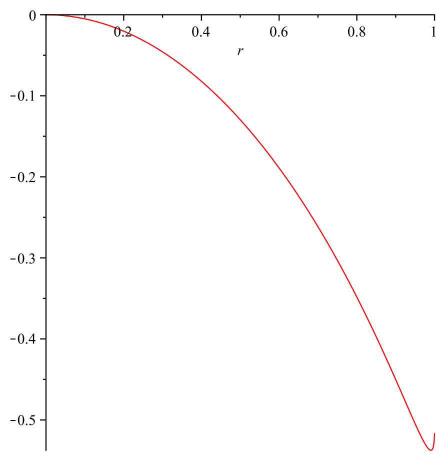
Figure 1: The graph of the functions $u(r)$ and $l(r)$

Remark 3. *Consider the functions $zu, zl : (0,1) \to \mathbb{R}$ defined by*

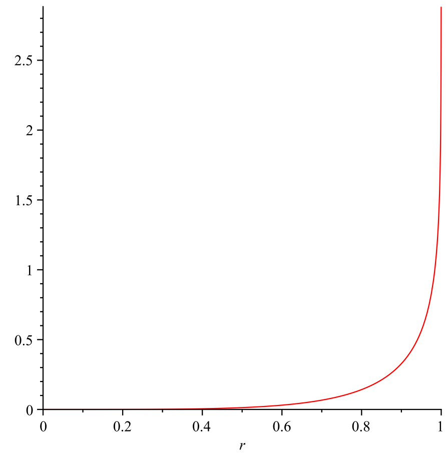
$$zu(r) = \frac{\pi}{2} - 2\log r' - r\mathrm{arthr} - \left[\frac{\pi}{2} - \log 2 + (3\log 2 - \frac{\pi}{2})(1-\mathrm{r}') + \log(1+1/\mathrm{r}')\right],$$

$$zl(r) = \frac{\pi}{2} - (2+\frac{\pi}{4})\log r' - r\mathrm{arthr} - \left[\frac{\pi}{2} - \log 2 + (\frac{\pi}{4} - \frac{1}{2})(1-\mathrm{r}') + \log(1+1/\mathrm{r}')\right].$$

The plots presented in Figure 2 demonstrates that the upper and lower bounds in (30) *for $\mathcal{K}(r)$ are better than the upper and lower bounds in* (4)*.*



(a) The graph of the functions $zu(r)$



(b) The graph of the functions $zl(r)$

Figure 2: The graph of the functions $zu(r)$ and $zl(r)$

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