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WEIGHTED HERMITE-HADAMARD-TYPE INEQUALITIES FOR GENERALIZATIONS OF STEFFENSEN'S INEQUALITY VIA THE EXTENSION OF MONGOMERY IDENTITY

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In this paper, we obtain some new weighted Hermite-Hadamard-type inequalities which involve generalizations of Steffensen's inequality obtained by using the extension of Montgomery identity via Taylor's formula. Further, we show that by using the extension of Montgomery identity via Fink's identity we can obtain some other weighted Hermite-Hadamard-type inequalities.

1. INTRODUCTION

Convexity is one of the most important notions in mathematical analysis. Although it is very simple in nature, it is very powerful. It has many applications in various areas of pure and applied sciences, such as in economics, medicine, optimization theory, etc. A great role in the popularization of the subject of convex functions was played by the famous book "Inequalities" [**6**] which assembled almost all important inequalities. Many inequalities are direct consequences of the applications of the convexity property of functions. One of the most interesting results relating to convexity is Hermite-Hadamard's inequality (see [**5**] and [**7**]). It gives us an estimate of the integral mean value of a continuous convex function. Precisely, if $f : [a, b] \to \mathbb{R}$ is convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

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In this paper we will use the weighted Hermite-Hadamard inequality for convex functions given in the following theorem (see [**10**], [**14**], [**17**]):

Theorem 1. *Let $p : [a, b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function on $[a, b]$, then we have*

$$P(b)f(m) \leq \int_a^b p(x)f(x)dx \leq P(b)\left[\frac{b-m}{b-a}f(a) + \frac{m-a}{b-a}f(b)\right], \tag{1}$$

where

$$P(t) = \int_a^t p(x)\,dx \quad \text{and} \quad m = \frac{1}{P(b)}\int_a^b p(x)x\,dx.$$

In 1918 Steffensen proved the following inequality (see [**16**]):

Theorem 2. *Suppose that f is nonincreasing and g is integrable on $[a, b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Then we have*

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt.$$

The inequalities are reversed for f nondecreasing.

Since its appearance many papers have been devoted to generalizations and refinements of Steffensen's inequality and its connection to other important inequalities such as Gauss-Steffensen's, Hölder's, Jenssen-Steffensen's and other inequalities. A complete overview of the results related to Steffensen's inequality can be found in monographs [**8, 15**].

Now, let us recall the well known Montgomery identity from "Inequalities for Functions and their Integrals and Derivatives" by Mitrinović, Pečarić and Fink (see [**9**]):

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b T_1(x, s)f'(s)ds,$$

where the Peano kernel is

$$T_1(x, s) = \begin{cases} \frac{s-a}{b-a}, & a \leq s \leq x; \\ \frac{s-b}{b-a}, & x < s \leq b. \end{cases}$$

In [**1**], the authors obtained the following extension of Montgomery identity using Taylor's formula:

Theorem 4. *Let $f : I \to \mathbb{R}$ be suct that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. Then the following identity holds*

$$\begin{aligned} f(x) = {} & \frac{1}{b-a}\int_a^b f\left(t\right)dt - \sum_{i=0}^{n-2} f^{(i+1)}\left(x\right)\frac{\left(b-x\right)^{i+2} - \left(a-x\right)^{i+2}}{(i+2)!\,(b-a)} \\ & + \frac{1}{(n-1)!}\int_a^b T_n\left(x, s\right)f^{(n)}\left(s\right)ds, \end{aligned} \tag{2}$$

where

$$T_n(x,s)=\begin{cases}\frac{-1}{n(b-a)}(a-s)^n, & a\le s\le x;\\ \frac{-1}{n(b-a)}(b-s)^n, & x<s\le b.\end{cases}$$

In [**2**], the authors used the identity (2) to generalize Steffensen's inequality for $n-$convex functions. Also, the following identities related to generalizations of Steffensen's inequality for $n-$convex functions are proved in [**2**].

Theorem 5 ([**2**]). *Let $f: I\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\ge 2$, $I\subset\mathbb{R}$ an open interval, $a,b\in I$, $a<b$. Let $g,u:[a,b]\to\mathbb{R}$ be integrable functions such that u is positive and $0\le g\le 1$. Let $\int_a^{a+\lambda}u(t)dt=\int_a^b g(t)u(t)dt$ and let the function G_1 be defined by*

$$G_1(x)=\begin{cases}\int_a^x(1-g(t))u(t)dt, & x\in[a,a+\lambda],\\ \int_x^b g(t)u(t)dt, & x\in[a+\lambda,b].\end{cases} \tag{3}$$

Then

$$\begin{aligned}&\int_a^{a+\lambda}f(t)u(t)dt-\int_a^b f(t)g(t)u(t)dt\\ &+\int_a^b G_1(x)\left(\frac{f(b)-f(a)}{b-a}-\sum_{i=0}^{n-3}f^{(i+2)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}\right)dx\\ &=-\frac{1}{(n-2)!}\int_a^b\left(\int_a^b G_1(x)T_{n-1}(x,s)dx\right)f^{(n)}(s)ds.\end{aligned} \tag{4}$$

Theorem 6 ([**2**]). *Let $f: I\to\mathbb{R}$ be suct that $f^{(n-1)}$ is absolutely continuous for some $n\ge 2$, $I\subset\mathbb{R}$ an open interval, $a,b\in I$, $a<b$. Let $g,u:[a,b]\to\mathbb{R}$ be integrable functions such that u is positive and $0\le g\le 1$. Let $\int_{b-\lambda}^b u(t)dt=\int_a^b g(t)u(t)dt$ and let the function G_2 be defined by*

$$G_2(x)=\begin{cases}\int_a^x g(t)u(t)dt, & x\in[a,b-\lambda],\\ \int_x^b(1-g(t))u(t)dt, & x\in[b-\lambda,b].\end{cases} \tag{5}$$

Then

$$\begin{aligned}&\int_a^b f(t)g(t)u(t)dt-\int_{b-\lambda}^b f(t)u(t)dt\\ &+\int_a^b G_2(x)\left(\frac{f(b)-f(a)}{b-a}-\sum_{i=0}^{n-3}f^{(i+2)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}\right)dx\\ &=-\frac{1}{(n-2)!}\int_a^b\left(\int_a^b G_2(x)T_{n-1}(x,s)dx\right)f^{(n)}(s)ds.\end{aligned} \tag{6}$$

Let us also recall the identity obtained by Fink in 1992 (see [**4**]):

$$\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right) - \frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)dt, \tag{7}$$

where

$$F_k(x)=\frac{n-k}{k!}\cdot\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a}$$

and

$$k(t,x)=\begin{cases} t-a, & a\le t\le x\le b,\\ t-b, & a\le x<t\le b.\end{cases} \tag{8}$$

In [**12**] (see also [**15**]) some generalizations of Steffensen's inequality were obtained using an extension of weighted Montgomery identity via Fink's identity. Using the identity (7) some new generalizations of Steffensen's inequality for $n-$convex functions were obtained in [**13**] using different reasoning from the one used in [**12**].

By $\mathcal{T}_k(x)$ we will denote

$$\mathcal{T}_k(x)=\frac{n-1-k}{k!}\cdot\frac{f^{(k)}(a)(x-a)^k-f^{(k)}(b)(x-b)^k}{b-a}. \tag{9}$$

In [**13**] the authors proved the following identities which were used to obtain generalizations of Steffensen's inequality for $n-$convex functions.

Theorem 7 ([**13**]). *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\ge 2$ and let g,u be integrable functions on $[a,b]$ such that u is positive and $0\le g\le 1$ on $[a,b]$. Let $\int_a^{a+\lambda}u(t)dt=\int_a^b g(t)u(t)dt$ and let the function G_1 be defined by* (3). *Then*

$$\begin{aligned}&\int_a^{a+\lambda} f(t)u(t)dt-\int_a^b f(t)g(t)u(t)dt-\sum_{k=0}^{n-2}\int_a^b \mathcal{T}_k(x)G_1(x)dx\\ &=-\frac{1}{(b-a)(n-2)!}\int_a^b\left(\int_a^b G_1(x)(x-t)^{n-2}k(t,x)dx\right)f^{(n)}(t)dt.\end{aligned} \tag{10}$$

Theorem 8 ([**13**]). *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\ge 2$ and let g,u be integrable functions on $[a,b]$ such that u is positive and $0\le g\le 1$ on $[a,b]$. Let $\int_{b-\lambda}^b u(t)dt=\int_a^b g(t)u(t)dt$ and let the function G_2 be defined by* (5). *Then*

$$\begin{aligned}&\int_a^b f(t)g(t)u(t)dt-\int_{b-\lambda}^b f(t)u(t)dt-\sum_{k=0}^{n-2}\int_a^b \mathcal{T}_k(x)G_2(x)dx\\ &=-\frac{1}{(b-a)(n-2)!}\int_a^b\left(\int_a^b G_2(x)(x-t)^{n-2}k(t,x)dx\right)f^{(n)}(t)dt.\end{aligned} \tag{11}$$

The aim of this paper is to use the identities related to generalizations of Steffensen's inequality for $n-$convex functions by the extension of Montgomery identity via Taylor's formula or via Fink's identity to obtain new weighted Hermite-Hadamard-type inequalities for $(n+2)-$convex functions.

2. MAIN RESULTS

The purpose of this section is to establish some new weighted Hermite-Hadamard-type inequalities for $(n+2)-$convex functions. Motivated by results proved in [**11**] using Theorem 5 we obtain the following result.

Theorem 9. *Let $I \subset \mathbb{R}$ be an open interval and let $a, b \in I$ be such that $a < b$. Let the function $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and f is $(n+2)-$convex on I for $n \geq 2$. Let $g, u : [a, b] \to \mathbb{R}$ be integrable functions such that u is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} u(t)dt = \int_a^b g(t)u(t)dt$, let the function G_1 be defined by* (3) *and let T_{n-1} be defined by*

$$T_{n-1}(x,s) = \begin{cases} \frac{-1}{(n-1)(b-a)}(a-s)^{n-1}, & a \leq s \leq x; \\ \frac{-1}{(n-1)(b-a)}(b-s)^{n-1}, & x < s \leq b. \end{cases} \tag{12}$$

If

$$-\int_a^b G_1(x) T_{n-1}(x,s)dx \geq 0, \quad s \in [a,b] \tag{13}$$

then

$$\begin{aligned} &P_1(b) \cdot f^{(n)}(m_1) \leq \\ &(n-2)! \left[\int_a^{a+\lambda} f(t)u(t)dt - \int_a^b f(t)g(t)u(t)dt \right. \\ &\left. + \int_a^b G_1(x) \left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \right) dx \right] \\ &\leq P_1(b) \cdot \left[\frac{b-m_1}{b-a} f^{(n)}(a) + \frac{m_1-a}{b-a} f^{(n)}(b) \right], \end{aligned} \tag{14}$$

where

$$P_1(b) = \frac{1}{(n-1) \cdot n \cdot (b-a)} \int_a^b G_1(x) \left((b-x)^n - (a-x)^n \right) dx \tag{15}$$

and

$$\begin{aligned} m_1 =& \frac{1}{(n-1) \cdot n \cdot (b-a) \cdot P_1(b)} \\ &\times \int_a^b G_1(x) \left(\frac{(b-x)^{n+1} - (a-x)^{n+1}}{n+1} + x \cdot \left((b-x)^n - (a-x)^n \right) \right) dx. \end{aligned} \tag{16}$$

Proof. The function f satisfies the conditions of Theorem 5, so the identity (4) holds. Now, let us define the function p_1 on $[a, b]$ by

$$p_1(s) = -\int_a^b G_1(x) T_{n-1}(x, s)\, dx, \quad s \in [a, b], \tag{17}$$

where G_1 and T_{n-1} are defined by (3) and (12). Under the assumption (13) it is obvious that p_1 is a non-negative function.

Notice that, for odd number $n \geq 2$, we have $T_{n-1}(x, s) \leq 0$, for $s \in [a, b]$. Therefore, for odd numbers $n \geq 2$ the condition (13) is always satisfied.

Since f is an $(n+2)-$convex function, the function $f^{(n)}$ is convex. Applying Theorem 1 with non-negative function p_1 and convex function $f^{(n)}$ we obtain the following inequality

$$\begin{aligned} & P_1(b) \cdot f^{(n)}(m_1) \leq -\int_a^b \left(\int_a^b G_1(x) T_{n-1}(x, s) dx \right) f^{(n)}(s) ds \\ & \leq P_1(b) \cdot \left[\frac{b - m_1}{b - a} f^{(n)}(a) + \frac{m_1 - a}{b - a} f^{(n)}(b) \right] \end{aligned} \tag{18}$$

where the expressions $P_1(b)$ and m_1 are given by

$$P_1(b) = -\int_a^b \left(\int_a^b G_1(x) T_{n-1}(x, s) dx \right) ds$$

and

$$m_1 = -\frac{1}{P_1(b)} \int_a^b \left(\int_a^b G_1(x) T_{n-1}(x, s) dx \right) s\, ds.$$

By calculating the above expressions we obtain the following:

$$\begin{aligned} P_1(b) &= -\int_a^b \left(\int_a^b G_1(x) T_{n-1}(x, s) dx \right) ds = -\int_a^b G_1(x) \left(\int_a^b T_{n-1}(x, s) ds \right) dx \\ &= \frac{1}{(n-1) \cdot (b-a)} \int_a^b G_1(x) \left(\int_a^x (a-s)^{n-1} ds + \int_x^b (b-s)^{n-1} ds \right) dx \\ &= \frac{1}{(n-1) \cdot n \cdot (b-a)} \int_a^b G_1(x) \left((b-x)^n - (a-x)^n \right) dx \end{aligned}$$

and

$$\begin{aligned}
m_1 &= \frac{-1}{P_1(b)} \int_a^b \left(\int_a^b G_1(x) T_{n-1}(x,s) dx \right) s\, ds \\
&= \frac{-1}{P_1(b)} \int_a^b G_1(x) \left(\int_a^b T_{n-1}(x,s) \cdot s\, ds \right) dx \\
&= \frac{1}{P_1(b) \cdot (n-1) \cdot (b-a)} \int_a^b G_1(x) \left(\int_a^x s(a-s)^{n-1} ds + \int_x^b s(b-s)^{n-1} ds \right) dx \\
&= \frac{1}{(n-1) \cdot n \cdot (b-a) \cdot P_1(b)} \\
&\times \int_a^b G_1(x) \left(\frac{(b-x)^{n+1} - (a-x)^{n+1}}{n+1} + x \cdot ((b-x)^n - (a-x)^n) \right) dx.
\end{aligned}$$

Using the identity (4) for the middle part of the inequality (18), the inequality (18) becomes the inequality (14). This completes the proof. □

Similarly, using Theorem 6 we obtain the following new weighted Hermite-Hadamard-type inequality.

Theorem 10. *Let $I \subset \mathbb{R}$ be an open interval and let $a, b \in I$ be such that $a < b$. Let the function $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and f is $(n+2)-$convex on I for $n \geq 2$. Let $g, u : [a,b] \to \mathbb{R}$ be integrable functions such that u is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b u(t)dt = \int_a^b g(t)u(t)dt$, let the function G_2 be defined by (5) and let T_{n-1} be defined by (12). If*

$$-\int_a^b G_2(x) T_{n-1}(x,s) dx \geq 0, \quad s \in [a,b], \tag{19}$$

then

$$\begin{aligned}
&P_2(b) \cdot f^{(n)}(m_2) \leq \\
&(n-2)! \left[\int_a^b f(t)g(t)u(t)dt - \int_{b-\lambda}^b f(t)u(t)dt \right. \\
&\left. + \int_a^b G_2(x) \left(\frac{f(b)-f(a)}{b-a} - \sum_{i=0}^{n-3} f^{(i+2)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \right) dx \right] \\
&\leq P_2(b) \cdot \left[\frac{b-m_2}{b-a} f^{(n)}(a) + \frac{m_2-a}{b-a} f^{(n)}(b) \right],
\end{aligned} \tag{20}$$

where

$$P_2(b) = \frac{1}{(n-1) \cdot n \cdot (b-a)} \int_a^b G_2(x) \left((b-x)^n - (a-x)^n \right) dx$$

and

$$m_2 = \frac{1}{(n-1)\cdot n\cdot (b-a)\cdot P_2(b)} \times \int_a^b G_2(x)\left(\frac{(b-x)^{n+1}-(a-x)^{n+1}}{(n+1)} + x\cdot((b-x)^n-(a-x)^n)\right)dx.$$

Proof. Similar to the proof of Theorem 9 using the identity (6). □

Remark 1. *If f is an $(n+2)-$concave function, then the inequalities in* (14) *and* (20) *are reversed. This follows from the fact that if f is $(n+2)-$concave function, then $-f^{(n)}$ is convex, so applying the inequality* (1) *on $-f^{(n)}$ we obtain the reversed inequalities in* (14) *and* (20)*.*

Remark 2. *Let us show that Theorems 9 and 10 can also be proved by different method, introduced in [**3**]. Using the idea from the mentioned paper the expressions $P_1(b)$ and m_1 can be calculated as follows.*

The value $P_1(b)$ can be obtained from the identity (4) *taking $f(t)=\frac{t^n}{n!}$. Then $f^{(n)}(t)=1$, and by simple calculation we get*

$$\begin{aligned}
P_1(b) &= (n-2)!\left[\int_a^{a+\lambda}\frac{t^n}{n!}u(t)dt - \int_a^b \frac{t^n}{n!}g(t)u(t)dt\right.\\
&\left. + \int_a^b G_1(x)\left(\frac{b^n-a^n}{n!(b-a)} - \sum_{i=0}^{n-3}\frac{x^{n-i-2}}{(n-i-2)!}\cdot\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}\right)dx\right]\\
&= (n-2)!\left[\int_a^{a+\lambda}\frac{t^n}{n!}u(t)dt - \int_a^b \frac{t^n}{n!}g(t)u(t)dt\right.\\
&\left. + \int_a^b G_1(x)\left(\frac{b^n-a^n}{n!(b-a)} - \sum_{i=2}^{n-1}\frac{x^{n-i}}{(n-i)!}\cdot\frac{(b-x)^{i}-(a-x)^{i}}{i!\,(b-a)}\right)dx\right]\\
&= (n-2)!\left[\int_a^{a+\lambda}\frac{t^n}{n!}u(t)dt - \int_a^b \frac{t^n}{n!}g(t)u(t)dt\right.\\
&\left. + \int_a^b G_1(x)\left(\frac{b^n-a^n}{n!(b-a)} - \left(\frac{b^n-a^n}{n!(b-a)} - \frac{x^{n-1}}{(n-1)!} + \frac{(a-x)^n-(b-x)^n}{n!(b-a)}\right)\right)dx\right]\\
&= \int_a^b G_1(x)\frac{(b-x)^n-(a-x)^n}{(n-1)\cdot n\cdot (b-a)}dx.
\end{aligned}$$

To calculate m_1 we take the function $f(t)=\frac{t^{n+1}}{(n+1)!}$ since its $n-th$ derivative *is* $f^{(n)}(t) = 1$ *and from the identity* (4) *we get*

$$\begin{aligned}
m_1 &= -\frac{1}{P_1(b)} \int_a^b \left(\int_a^b G_1(x) T_{n-1}(x,s) dx \right) s ds \\
&= \frac{(n-2)!}{P_1(b)} \left[\int_a^{a+\lambda} \frac{t^{n+1}}{(n+1)!} u(t) dt - \int_a^b \frac{t^{n+1}}{(n+1)!} g(t) u(t) dt \right. \\
&+ \left. \int_a^b G_1(x) \left(\frac{b^{n+1} - a^{n+1}}{(n+1)!(b-a)} - \sum_{i=0}^{n-3} \frac{x^{n-i-1}}{(n-i-1)!} \cdot \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \right) dx \right] \\
&= \frac{(n-2)!}{P_1(b)} \left[\int_a^{a+\lambda} \frac{t^{n+1}}{(n+1)!} u(t) dt - \int_a^b \frac{t^{n+1}}{(n+1)!} g(t) u(t) dt \right. \\
&+ \left. \int_a^b G_1(x) \left(\frac{b^{n+1} - a^{n+1}}{(n+1)!(b-a)} - \sum_{i=1}^{n-2} \frac{x^{n-i}}{(n-i)!} \cdot \frac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!\,(b-a)} \right) dx \right] \\
&= \frac{(n-2)!}{P_1(b)} \left[\int_a^{a+\lambda} \frac{t^{n+1}}{(n+1)!} u(t) dt - \int_a^b \frac{t^{n+1}}{(n+1)!} g(t) u(t) dt \right. \\
&+ \int_a^b G_1(x) \left(\frac{b^{n+1} - a^{n+1}}{(n+1)!(b-a)} - \left(\frac{b^{n+1} - a^{n+1}}{(n+1)!(b-a)} - \frac{x^n}{n!} + x \cdot \frac{(a-x)^n - (b-x)^n}{n!(b-a)} \right. \right. \\
&+ \left. \left. \left. \frac{(a-x)^{n+1} - (b-x)^{n+1}}{(n+1)!(b-a)} \right) \right) dx \right] \\
&= \frac{1}{P_1(b)} \left[\int_a^b x \cdot G_1(x) \frac{(b-x)^n - (a-x)^n}{(n-1) \cdot n \cdot (b-a)} dx \right. \\
&+ \left. \int_a^b G_1(x) \frac{(b-x)^{n+1} - (a-x)^{n+1}}{(n-1) \cdot n \cdot (n+1) \cdot (b-a)} dx \right].
\end{aligned}$$

Now using Theorem 7 we obtain the following new weighted Hermite-Hadamard-type inequalities for $(n+2)-$convex functions related to generalization of Steffensen's inequality by the extension of Montgomery identity via Fink's identity.

Theorem 11. *Let the function* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous and* f *is* $(n+2)-$*convex on* $[a,b]$ *for* $n \geq 2$. *Let* $g, u : [a,b] \to \mathbb{R}$ *be integrable functions such that* u *is positive and* $0 \leq g \leq 1$. *Let* $\int_a^{a+\lambda} u(t)dt = \int_a^b g(t)u(t)dt$, *let the function* G_1 *be defined by* (3) *and let* $k(t,x)$ *be defined by* (8). *If*

$$-\int_a^b G_1(x)(x-t)^{n-2} k(t,x) dx \geq 0, \quad t \in [a,b] \tag{21}$$

then

(22)
$$P_3(b)\cdot f^{(n)}(m_3)\le (n-2)!(b-a)\left[\int_a^{a+\lambda} f(t)u(t)dt-\int_a^b f(t)g(t)u(t)dt-\sum_{k=0}^{n-2}\int_a^b \mathcal{T}_k(x)G_1(x)dx\right] \le P_3(b)\cdot\left[\frac{b-m_3}{b-a}f^{(n)}(a)+\frac{m_3-a}{b-a}f^{(n)}(b)\right],$$

where

$$P_3(b)=\frac{1}{(n-1)\cdot n}\int_a^b G_1(x)\,\left((x-b)^n-(x-a)^n\right)dx, \tag{23}$$

(24)
$$m_3=\frac{1}{(n-1)\cdot n\cdot P_3(b)}\times\int_a^b G_1(x)\left(2\cdot\frac{(x-b)^{n+1}-(x-a)^{n+1}}{n+1}+\left(b(x-b)^n-a(x-a)^n\right)\right)dx$$

and $\mathcal{T}_k$ *is defined by* (9).

Proof. Let us define the function $p_3:[a,b]\to\mathbb{R}$ by

$$p_3(t)=-\int_a^b G_1(x)(x-t)^{n-2}k(t,x)dx. \tag{25}$$

From the condition (21) we have that the function p_3 is non-negative.

Further, for $(n+2)-$convex function function f we have that $f^{(n)}$ is convex. Hence, we can apply Theorem 1 on functions $f^{(n)}$ and p_3 to obtan

$$\begin{aligned}P_3(b)\cdot f^{(n)}(m_3)&\le-\int_a^b\left(\int_a^b G_1(x)(x-t)^{n-2}k(t,x)dx\right)f^{(n)}(t)dt\\&\le P_3(b)\cdot\left[\frac{b-m_3}{b-a}f^{(n)}(a)+\frac{m_3-a}{b-a}f^{(n)}(b)\right].\end{aligned}\tag{26}$$

From Theorem 1 we see that $P_3(b)$ and m_3 can be calculated as follows

$$\begin{aligned}P_3(b)&=\int_a^b p_3(t)dt=-\int_a^b\left(\int_a^b G_1(x)(x-t)^{n-2}k(t,x)dx\right)dt\\&=-\int_a^b G_1(x)\left(\int_a^b (x-t)^{n-2}k(t,x)dt\right)dx\\&=-\int_a^b G_1(x)\left(\int_a^x (x-t)^{n-2}(t-a)dt+\int_x^b (x-t)^{n-2}(t-b)dt\right)dx\\&=\frac{1}{(n-1)\cdot n}\int_a^b G_1(x)\,\left((x-b)^n-(x-a)^n\right)dx\end{aligned}$$

and

$$\begin{aligned}
m_3 &= \frac{1}{P_3(b)}\int_a^b p_3(t)t\,dt = \frac{-1}{P_3(b)}\int_a^b\left(\int_a^b G_1(x)(x-t)^{n-2}k(t,x)dx\right)t\,dt\\
&= \frac{-1}{P_3(b)}\int_a^b G_1(x)\left(\int_a^b (x-t)^{n-2}k(t,x)\cdot t\,dt\right)dx\\
&= \frac{-1}{P_3(b)}\int_a^b G_1(x)\left(\int_a^x t(x-t)^{n-2}(t-a)dt+\int_x^b t(x-t)^{n-2}(t-b)dt\right)dx\\
&= \frac{1}{P_3(b)\cdot(n-1)\cdot n}\\
&\times\int_a^b G_1(x)\left(2\cdot\frac{(x-b)^{n+1}-(x-a)^{n+1}}{n+1}+(b(x-b)^n-a(x-a)^n)\right)dx.
\end{aligned}$$

Since the function f satisties the conditions of Theorem 7 we can apply the identity (10) for the middle part in (26), i.e. we have

(27)

$$\begin{aligned}
&-\int_a^b\left(\int_a^b G_1(x)(x-t)^{n-2}k(t,x)dx\right)f^{(n)}(t)dt\\
&=(n-2)!\cdot(b-a)\left(\int_a^{a+\lambda}f(t)u(t)dt-\int_a^b f(t)g(t)u(t)dt-\sum_{k=0}^{n-2}\int_a^b \mathcal{T}_k(x)G_1(x)dx\right).
\end{aligned}$$

Thus we have proved the desired assertion (22). □

Similar using Theorem 8 we obtain the following weighted Hermite-Hadamard-type inequalities .

Theorem 12. *Let the function $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous and f is $(n+2)-$convex on $[a,b]$ for $n\geq 2$. Let $g,u:[a,b]\to\mathbb{R}$ be integrable functions such that u is positive and $0\leq g\leq 1$. Let $\int_{b-\lambda}^b u(t)dt=\int_a^b g(t)u(t)dt$ and let the function G_2 be defined by* (5) *and let $k(t,x)$ be defined by* (8)*. If*

$$-\int_a^b G_2(x)(x-t)^{n-2}k(t,x)dx\geq 0,\quad t\in[a,b] \tag{28}$$

then

(29)

$$\begin{aligned}
&P_4(b)\cdot f^{(n)}(m_4)\leq\\
&(n-2)!(b-a)\left[\int_a^b f(t)g(t)u(t)dt-\int_{b-\lambda}^b f(t)u(t)dt-\sum_{k=0}^{n-2}\int_a^b \mathcal{T}_k(x)G_2(x)dx\right]\\
&\leq P_4(b)\cdot\left[\frac{b-m_4}{b-a}f^{(n)}(a)+\frac{m_4-a}{b-a}f^{(n)}(b)\right],
\end{aligned}$$

where

$$P_4(b) = \frac{1}{(n-1)\cdot n} \int_a^b G_2(x)\ ((x-b)^n - (x-a)^n)\, dx, \tag{30}$$

(31)

$$\begin{aligned} m_4 =& \frac{1}{(n-1)\cdot n \cdot P_4(b)} \\ &\times \int_a^b G_2(x) \left(2 \cdot \frac{(x-b)^{n+1} - (x-a)^{n+1}}{n+1} + (b(x-b)^n - a(x-a)^n) \right) dx \end{aligned}$$

and $\mathcal{T}_k$ *is defined by* (9).

Proof. Similar to the proof of Theorem 11. □

Remark 3. *As in Remark 1 we have the following:*
If f *is an* $(n+2)-$*concave function, then the inequalities in* (22) *and* (29) *are reversed.*

Remark 4. *Theorems 11 and 12 can also be proved by different method, introduced in [**3**] as showed in Remark 2.*

3. CONCLUSION

The Hermite-Hadamard inequality is very important in mathematics since it can be used in many different studies in pure and applied mathematics. As a result, the aim of this article is to use weighted Hermite-Hadamard inequality on some generalizations of Steffensen's inequality which were obtained by using the extension of Montgomery identity via Taylor's formula in [**2**] or by using the extension of Montgomery identity via Fink's identity in [**13**]. In this way we obtain new weighted Hermite-Hadamard inequalities for $(n+2)-$convex functions which have identities related to generalizations of Steffensen's inequality as a middle part of the inequality.

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