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A NEW CLASS OF GENERALIZED FUBINI POLYNOMIALS AND THEIR COMPUTATIONAL ALGORITHMS

Neslihan Kilar

The aim of this paper is to give many new and elegant formulas for a new class of generalized Fubini polynomials with the aid of generating functions and their functional equations. By using these formulas, some computational algorithms involving a new class of generalized Fubini polynomials and special polynomials and numbers are constructed. Using these algorithms, some values of these numbers and polynomials are computed. Finally, some remarks and observations on the results of this paper are presented.

1. INTRODUCTION

Recently, many researchers have studied on special numbers and polynomials involving the Bernoulli type numbers and polynomials, Fubini type numbers and polynomials, the Stirling numbers, and the combinatorial numbers and sums. Especially, the Fubini type numbers and polynomials have been studied by many researchers in different methods. The Fubini type numbers are related to the binomial coefficients, special numbers and polynomials, such as the Apostol-Bernoulli numbers and polynomials of higher order, the Apostol-Euler numbers and polynomials of higher order, the Apostol-Genocchi numbers and polynomials of higher order, the Stirling type numbers and polynomials, the Apostol type Frobenius–Euler polynomials of higher order (see, for detail, [**4**], [**6**]-[**10**], [**12**], [**13**], [**15**], [**16**], [**35**], [**37**], [**50**]).

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In order to give main results of this paper, we can use generating functions and their functional equations methods. By applying functional equations of the genearting functions for the certain families of the special numbers and polynomials, we derive many new formulas and relations including the Fubini type numbers and polynomials, the Apostol-Bernoulli numbers and polynomials of higher order, the Apostol-Euler numbers and polynomials of higher order, the Apostol-Genocchi numbers of higher order, the Stirling numbers, the Apostol type Frobenius–Euler numbers of higher order, the array polynomials, the combinatorial numbers, and other special numbers and polynomials.

Let us briefly give the notations and definitions to be used throughout this paper as follows:

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, the set of integers, the set of real numbers, and the set of complex numbers, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Besides

$$0^n = \begin{cases} 1, & (n = 0) \\ 0, & (n \in \mathbb{N}) \end{cases}$$

and

$$\binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{n} = \frac{(\lambda)_n}{n!} \quad (n \in \mathbb{N};\ \lambda \in \mathbb{C}),$$

where $(\lambda)_n$ is the falling factorial defined by

$$(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \lambda (\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1).$$

We assuming that $\ln z$ denotes the principal branch of the many-valued function $\ln z$ with the imaginary part $\operatorname{Im}(\ln z)$ constrained by

$$-\pi < \operatorname{Im}(\ln z) \leq \pi$$

(*cf.* [**1**]-[**52**]).

We also need the following generating functions for the special numbers and polynomials.

The Stirling numbers of the first kind, $s(n, m)$, are defined by

$$F_{S_1}(t, m) = \frac{(\ln(1+t))^m}{m!} = \sum_{n=0}^{\infty} s(n, m) \frac{t^n}{n!}, \tag{1}$$

where $m \in \mathbb{N}_0$ and

$$(x)_n = \sum_{v=0}^{n} s(n, v) x^v, \tag{2}$$

with $s(0, 0) = 1$ and for $v > n$, $s(n, v) = 0$ (*cf.* [**3**], [**30**], [**34**], [**40**], [**41**], [**51**]; see also the references cited therein).

The Stirling numbers of the second kind, $S(n, m)$, are defined by

$$F_{S_2}(t, m) = \frac{(e^t - 1)^m}{m!} = \sum_{n=0}^{\infty} S(n, m) \frac{t^n}{n!} \tag{3}$$

and

$$x^n = \sum_{v=0}^{n} S(n,v)(x)_v\,,$$

(*cf.* [**3**], [**30**], [**34**], [**40**], [**41**], [**51**]; see also the references cited therein).

By using (3), the following formula for the numbers $S(n,m)$ is given:

$$S(n,m) = \frac{1}{m!}\sum_{v=0}^{m}(-1)^{m-v}\binom{m}{v}v^n, \tag{4}$$

where $m,n \in \mathbb{N}_0$ and $S(0,0)=1$. For $m>n$ (or $m<0$), we have $\binom{n}{m}=0$ and

$$S(n,m)=0$$

(*cf.* [**3**], [**30**], [**34**], [**40**], [**41**], [**51**]).

Let $\alpha \in \mathbb{R}$ (or $\mathbb{C}$). The Apostol-Bernoulli numbers and polynomials of order α are defined by means of the following generating functions:

$$F_{AB}(t,\alpha;\lambda) = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} = \sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(\lambda)\frac{t^n}{n!} \tag{5}$$

and

$$G_{AB}(t,x,\alpha;\lambda) = F_{AB}(t,\alpha;\lambda)e^{xt} = \sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{6}$$

where $|t|<2\pi$ when $\lambda=1$; $|t|<|\ln(\lambda)|$ when $\lambda\neq 1$; $1^\alpha=1$ (*cf.* [**32**]-[**34**], [**45**], [**51**]). One can observe that

$$\mathcal{B}_n^{(\alpha)}(\lambda) = \mathcal{B}_n^{(\alpha)}(0;\lambda).$$

Substituting $\lambda=1$ into (5) and (6), we have

$$B_n^{(\alpha)} = \mathcal{B}_n^{(\alpha)}(1) \quad\text{and}\quad B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x;1),$$

where $B_n^{(\alpha)}$ and $B_n^{(\alpha)}(x)$ denotes the Bernoulli numbers and polynomials of order α (*cf.* [**32**]-[**34**], [**51**]).

When $\alpha=1$ in (5) and (6), we get

$$\mathcal{B}_n(\lambda) = \mathcal{B}_n^{(1)}(\lambda) \quad\text{and}\quad \mathcal{B}_n(x;\lambda) = \mathcal{B}_n^{(1)}(x;\lambda).$$

Substituting $\lambda=1$ and $\alpha=-k$ ($k\in\mathbb{N}$) into (5), and using (3), we have the following well-known relation (*cf.* [**51**, Eq. (7.17)]):

$$B_n^{(-k)} = \binom{n+k}{k}^{-1} S(n+k,k)\,.$$

Let $\alpha \in \mathbb{R}$ (or $\mathbb{C}$). The Apostol-Euler numbers and polynomials of order α are defined by means of the following generating functions:

$$F_{AE}(t,\alpha;\lambda) = \left(\frac{2}{\lambda e^t + 1}\right)^\alpha = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(\lambda)\frac{t^n}{n!} \tag{7}$$

and

$$G_{AE}(t,x,\alpha;\lambda) = F_{AE}(t,\alpha;\lambda)\, e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{8}$$

where $|t| < \pi$ when $\lambda = 1$; $|t| < |\ln(-\lambda)|$ when $\lambda \neq 1$; $1^\alpha = 1$ (*cf.* [**32**]-[**34**], [**45**], [**51**]). One can observe that

$$\mathcal{E}_n^{(\alpha)}(\lambda) = \mathcal{E}_n^{(\alpha)}(0;\lambda).$$

When $\alpha = 1$ in (7) and (8), we get

$$\mathcal{E}_n(\lambda) = \mathcal{E}_n^{(1)}(\lambda) \quad \text{and} \quad \mathcal{E}_n(x;\lambda) = \mathcal{E}_n^{(1)}(x;\lambda).$$

Substituting $\lambda = 1$ into (7) and (8), we have

$$E_n^{(\alpha)} = \mathcal{E}_n^{(\alpha)}(1) \quad \text{and} \quad E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x;1),$$

where $E_n^{(\alpha)}$ and $E_n^{(\alpha)}(x)$ denotes the Euler numbers and polynomials of order α (*cf.* [**32**]-[**34**], [**45**], [**51**]).

By using (8) and (3), we have

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = 2^\alpha \sum_{r=0}^{n} \binom{n}{r} x^{n-r} \sum_{j=0}^{r} \binom{\alpha+j-1}{j} \frac{j!\,(-\lambda)^j}{(\lambda+1)^{j+\alpha}} S(r,j), \tag{9}$$

(*cf.* [**32**, Eq. (20)], [**34**], [**51**]).

Let $\alpha \in \mathbb{R}$ (or $\mathbb{C}$). The Apostol-Genocchi numbers and polynomials of order α are defined by means of the following generating functions:

$$F_{AG}(t,\alpha;\lambda) = \left(\frac{2t}{\lambda e^t + 1}\right)^\alpha = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(\lambda)\frac{t^n}{n!} \tag{10}$$

and

$$G_{AG}(t,x,\alpha;\lambda) = F_{AG}(t,\alpha;\lambda)\, e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}, \tag{11}$$

where $|t| < \pi$ when $\lambda = 1$; $|t| < |\ln(-\lambda)|$ when $\lambda \neq 1$; $1^\alpha = 1$ (*cf.* [**34**], [**51**], [**52**]). By using (11), we get

$$\mathcal{G}_n^{(\alpha)}(\lambda) = \mathcal{G}_n^{(\alpha)}(0;\lambda).$$

Setting $\alpha = 1$ in (10) and (11), we have

$$\mathcal{G}_n(\lambda) = \mathcal{G}_n^{(1)}(\lambda) \text{ and } \mathcal{G}_n(x;\lambda) = \mathcal{G}_n^{(1)}(x;\lambda).$$

Substituting $\lambda = 1$ into (10) and (11), we have

$$G_n^{(\alpha)} = \mathcal{G}_n^{(\alpha)}(1) \quad \text{and} \quad G_n^{(\alpha)}(x) = \mathcal{G}_n^{(\alpha)}(x;1),$$

where $G_n^{(\alpha)}$ and $G_n^{(\alpha)}(x)$ denotes the Genocchi numbers and polynomials of order α (*cf.* [**34**], [**51**], [**52**]).

Remark 1. *The Apostol-Bernoulli polynomials of order α, $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ defined by* (6) *when $\lambda \neq 1$ and the Apostol-Genocchi polynomials of order α, $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ defined by* (11) *when $\lambda \neq -1$ should be restricted correctly to nonnegative integer values in which cases each of these two polynomial families in* (6) *and* (11) *has been commonly used in the literature when*

$$\lambda \neq 1 \qquad \text{and} \qquad \lambda \neq -1$$

respectively. Similarly, this constraint on the order α is tacitly assumed to be satisfied in all these and other analogous situations in this paper (cf. [**51**], [**52**]*).*

In view of (6), (8) and (11), we see that

$$\mathcal{G}_{n+m}^{(m)}(x;\lambda) = (n+m)_m \, \mathcal{E}_n^{(m)}(x;\lambda) = (-2)^m \, \mathcal{B}_{n+m}^{(m)}(x;-\lambda), \tag{12}$$

where $n, m \in \mathbb{N}_0$ (*cf.* [**34**, Lemma 2-3], [**51**]).

The Apostol type Frobenius–Euler numbers and polynomials of order m are defined by means of the following generating functions:

$$F_{AH}(t, m; \lambda, u) = \left(\frac{1-u}{\lambda e^t - u}\right)^m = \sum_{n=0}^{\infty} \mathcal{H}_n^{(m)}(\lambda, u) \frac{t^n}{n!} \tag{13}$$

and

$$G_{AH}(t, x, m; \lambda, u) = F_{AH}(t, m; \lambda, u) e^{xt} = \sum_{n=0}^{\infty} \mathcal{H}_n^{(m)}(x; \lambda, u) \frac{t^n}{n!}, \tag{14}$$

where $m \in \mathbb{N}_0$, $u, \lambda \in \mathbb{C}$ with $u \neq \lambda$ and $u \neq 1$ (*cf.* [**2**], [**21**], [**41**], [**49**]).

Substituting $u = -1$ into (13) and (14), we get

$$\mathcal{H}_n^{(m)}(\lambda, -1) = \mathcal{E}_n^{(m)}(\lambda) \quad \text{and} \quad \mathcal{H}_n^{(m)}(x; \lambda, -1) = \mathcal{E}_n^{(m)}(x;\lambda)$$

(*cf.* [**2**], [**21**], [**41**]).

The λ-Stirling numbers of the second kind, $S_2(n, m; \lambda)$, are defined by

$$\frac{(\lambda e^t - 1)^m}{m!} = \sum_{n=0}^{\infty} S_2(n, m; \lambda) \frac{t^n}{n!}, \tag{15}$$

where $m \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [**34**], [**41**], [**51**]). Substituting $\lambda = 1$ into (15), we have

$$S_2\left(n, m; 1\right) = S\left(n, m\right).$$

Cakic and Milovanovic [**3**] gave many applications of the array polynomials, which are defined by the following generating function:

$$\frac{\left(e^t - 1\right)^m}{m!} e^{xt} = \sum_{n=0}^{\infty} S_m^n\left(x\right) \frac{t^n}{n!}, \tag{16}$$

where $m \in \mathbb{N}_0$ (*cf.* [**3**], [**41**], [**43**]).

By using (16), the following formula for the polynomials $S_m^n\left(x\right)$ is given:

$$S_m^n\left(x\right) = \frac{1}{m!} \sum_{v=0}^{m} \left(-1\right)^{m-v} \binom{m}{v} \left(x+v\right)^n$$

(*cf.* [**3**], [**5**], [**41**], [**43**]).

Combining the above relation with (4), we have the following well-known formula:

$$S_m^n\left(x\right) = \sum_{j=0}^{n} \binom{n}{j} S(j, m) x^{n-j}.$$

Since $S(j, m) = 0$ for $m > j$, we have

$$S_0^0\left(x\right) = S_n^n\left(x\right) = 1, \quad S_0^n\left(x\right) = x^n$$

and if $m > n$, then we see that

$$S_m^n\left(x\right) = 0$$

(*cf.* [**3**], [**5**], [**41**], [**43**]).

The numbers $y_1(n, m; \lambda)$ are defined by means of the following generating function

$$F_{y_1}\left(t, m; \lambda\right) = \frac{\left(\lambda e^t + 1\right)^m}{m!} = \sum_{n=0}^{\infty} y_1\left(n, m; \lambda\right) \frac{t^n}{n!}, \tag{17}$$

where $m \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [**43**]; see also [**20**]).

By using (17), the following formula for the numbers $y_1(n, k; \lambda)$ is given (*cf.* [**43**, Eq. (9)]):

$$y_1\left(n, m; \lambda\right) = \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} j^n \lambda^j.$$

The numbers $y_2(n, m; \lambda)$ are defined by means of the following generating function

$$\frac{\left(\lambda e^t + \lambda^{-1} e^{-t} + 2\right)^m}{(2m)!} = \sum_{n=0}^{\infty} y_2\left(n, m; \lambda\right) \frac{t^n}{n!}, \tag{18}$$

where $m \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [**43**]). The numbers $y_1(n,m;\lambda)$ and the numbers $y_2(n,m;\lambda)$ are also so-called the combinatorial numbers.

By using (17) and (18), we have

$$y_1\left(n,2m;\lambda\right) = \lambda^m \sum_{j=0}^{n} \binom{n}{j} m^{n-j} y_2\left(j,m;\lambda\right) \tag{19}$$

(*cf.* [**43**]).

The Peters polynomials are defined by means of the following generating function:

$$\frac{(1+t)^x}{\left(1+(1+t)^\lambda\right)^\mu} = \sum_{n=0}^{\infty} s_n\left(x;\lambda,\mu\right)\frac{t^n}{n!} \tag{20}$$

(*cf.* [**11**], [**29**], [**39**], [**47**]).

When $x=0$ in (20), the Peters polynomials are reduced to the Peters numbers:

$$s_n\left(0;\lambda,\mu\right) = s_n\left(\lambda,\mu\right).$$

When $\mu = 1$ and $x = 0$ in (20), we have the Boole polynomials and numbers, respectively:

$$s_n\left(x;\lambda,1\right) = \xi\left(x;\lambda\right)$$

and

$$s_n\left(0;\lambda,1\right) = \xi\left(\lambda\right)$$

(*cf.* [**11**]) and also when $\lambda = 1$ in the above equation, we have

$$\xi\left(1\right) = s_n\left(0;1,1\right) = \frac{1}{2}Ch_n = \frac{(-1)^n n!}{2^{n+1}}, \tag{21}$$

where Ch_n denote so-called the Changhee numbers (*cf.* [**11**], [**17**], [**24**], [**45**], [**47**]).

By using (20), we obtain the following relation for the Peters polynomials:

$$(x)_n = \sum_{v=0}^{n}\sum_{d=0}^{\mu}\binom{\mu}{d}\binom{n}{v}(\lambda d)_v\, s_{n-v}\left(x;\lambda,\mu\right), \tag{22}$$

where $\mu \in \mathbb{N}$ (*cf.* [**47**, Eq. (28)]).

The polynomials $Y_{n,2}(x;\lambda)$ are defined by means of the following generating function:

$$\frac{2\left(1+\lambda t\right)^x}{\lambda^2 t + 2(\lambda-1)} = \sum_{n=0}^{\infty} Y_{n,2}\left(x;\lambda\right)\frac{t^n}{n!} \tag{23}$$

(*cf.* [**47**]). Substituting $x=0$ into (23), we have

$$Y_{n,2}\left(\lambda\right) = Y_{n,2}\left(0;\lambda\right)$$

(*cf.* [**47**]).

With the aid of (3), (6) and (23), we obtain

$$\mathcal{B}_n\left(x;\frac{\lambda}{2-\lambda}\right)=\frac{(2-\lambda)\,n}{2}\sum_{j=0}^{n-1}\lambda^{-j}S(n-1,j)Y_{j,2}\left(x;\lambda\right), \tag{24}$$

where $n\in\mathbb{N}$ (*cf.* [**47**, Eq. (25)]).

In [**24**] Kucukoglu and Simsek defined the numbers $\beta_n(k)$ by means of the following generating function:

$$\left(1-\frac{z}{2}\right)^k=\sum_{n=0}^{\infty}\beta_n\left(k\right)\frac{z^n}{n!}, \tag{25}$$

where $k\in\mathbb{N}_0$, $z\in\mathbb{C}$ with $|z|<2$.

By using (25), we have

$$\beta_n\left(k\right)=\frac{(-1)^n n!}{2^n}\binom{k}{n}=\binom{k}{n}Ch_n, \tag{26}$$

hence

$$Ch_n=\frac{(-1)^n n!}{2^n},$$

where $n,k\in\mathbb{N}_0$ (*cf.* [**24**, Eq. (4.9)]).

1.1 Generating functions for Fubini type numbers and polynomials

The Fubini numbers, which are denoted by $w_g(n)$, count the number of weak orderings on a set of n elements (*cf.* [**8**]). Here weak ordering is a mathematical formalization of the intuitive notion of a ranking of a set, some of whose members may be tied with each other. That is, weak orders are also a generalization of totally ordered sets (see, for detail, [**38**]).

The numbers $w_g(n)$ are defined by the following generating function:

$$\frac{1}{2-e^t}=\sum_{n=0}^{\infty}w_g(n)\frac{t^n}{n!}, \tag{27}$$

where $w_g\left(0\right)=1$ (*cf.* [**9**]; see also [**4**], [**6**]-[**8**], [**10**], [**13**], [**35**], [**36**], [**50**]).

By using (27), Koninck [**8**] defined by following combinatorial sum:

$$w_g(k)=\sum_{v=1}^{k}v^k\left(\sum_{j=0}^{k-v}(-1)^j\binom{j+v}{j}\right),$$

and for $k \geq 1$, few values of the numbers $w_g(k)$ are given as follows:

$$w_g(1) = 1, \qquad w_g(2) = 3, \qquad w_g(3) = 13, \qquad w_g(4) = 75, \qquad w_g(5) = 541$$

(*cf.* [**4**], [**8**], [**10**], [**37**]; see also the references cited therein).

Apart from the above notation, it is known that the Fubini numbers are also denoted by $\phi(n)$, $a(n)$ and H_n by some sources.

The Fubini type polynomials of order m are defined by means of the following generating function:

$$F_a(t, x, m) = \frac{2^m}{(2 - e^t)^{2m}} e^{xt} = \sum_{n=0}^{\infty} a_n^{(m)}(x) \frac{t^n}{n!}, \tag{28}$$

where $m \in \mathbb{N}_0$ and $|t| < \ln 2$ (*cf.* [**13**]; see also [**14**], [**15**]).

Substituting $x = 0$ into (28), we get

$$a_n^{(m)}(0) = a_n^{(m)},$$

where $a_n^{(m)}$ denote the Fubini type numbers of order m (*cf.* [**13**]).

Substituting $x = 0$ and $m = 1$ into (28), and using (27), we have

$$a_n^{(1)} = 2 \sum_{j=0}^{n} \binom{n}{j} w_g(j) w_g(n-j)$$

(*cf.* [**13**, Theorem 4.7]).

The generalized Fubini numbers of order m are defined by means of the following generating function

$$F_g(t, m; k) = \left(\frac{e^t - 1}{k + 1 - ke^t} \right)^m = \sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{t^n}{n!}, \tag{29}$$

where $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$ (*cf.* [**13**]).

Substituting $m = 1$ into (29), we have

$$f_{n,k}^{(1)} = f_{n,k}$$

(*cf.* [**36**, p. 398, Eq. (10.24)]).

Putting $k = 0$ in (29), we get

$$f_{n,0}^{(m)} = m! S(n, m).$$

By using (29), the following formula for the numbers $f_{n,k}^{(m)}$ is given:

$$f_{n,k}^{(m)} = \sum_{j=1}^{n-1} \binom{n}{j} f_{j,k}^{(l)} f_{n-j,k}^{(m-l)},$$

where $n > 1$ (*cf.* [**13**]).

Substituting $k = m = 1$ into (29), for $n > 0$, we get

$$f_{n,1} = w_M(n),$$

where the numbers $w_M(n)$ are defined by means of the following generating function

$$\frac{e^t - 1}{2 - e^t} = \sum_{n=0}^{\infty} w_M(n) \frac{t^n}{n!} \tag{30}$$

with $w_M(0) = 0$ (*cf.* [**36**, p. 397]).

By using (30) and (27), we have

$$w_M(n) = \sum_{j=0}^{n-1} \binom{n}{j} w_g(j),$$

where $n \in \mathbb{N}$ (*cf.* [**13**, Eq. (16)]).

The two variable Fubini polynomials, $F_n^{(m)}(x;y)$, are defined by

$$\frac{e^{xt}}{(1 - y(e^t - 1))^m} = \sum_{n=0}^{\infty} F_n^{(m)}(x;y) \frac{t^n}{n!}, \tag{31}$$

where $m \in \mathbb{N}$ (*cf.* [**16**, Eq. (2.1)]).

Setting $x = 0$ in (31), we get

$$F_n^{(m)}(y) = F_n^{(m)}(0;y)$$

(*cf.* [**16**]).

The results of this paper is briefly summarized as below:

In Section 2, we define new classes of special polynomials. By using generating functions with their functional equations methods, we give not only some fundamental properties of these polynomials, but also we derive some new formulas, identities and relations associated with these polynomials and other special numbers and polynomials.

In Section 3, by using generating functions of special numbers and polynomials and their functional equations, we give some identities and formulas involving the Apostol type numbers and polynomials of higher order, the generalized Fubini numbers of higher order, the Stirling type numbers, the Changhee numbers, the numbers $y_1(n,m;\lambda)$, the numbers $y_2(n,m;\lambda)$, and the numbers $\beta_n(k)$.

In Section 4, we give some algorithms for the generalized Fubini numbers and polynomials of higher order, the Apostol-Euler polynomials of higher order, and the Stirling numbers of the second kind. By using these algorithms, we calculate the numerical values of the generalized Fubini numbers and polynomials of higher order. Additionally, we present some plots of the generalized Fubini polynomials of higher order for some of their special cases.

In Section 5, we give further remarks and observations on the special polynomials with conclusion.

2. GENERATING FUNCTION FOR TWO NEW CLASSES OF SPECIAL POLYNOMIALS

In this section, we define generating functions for two new classes of special polynomials. By using these generating functions, we both investigate some properties of these polynomials, and give some identities and relations related to the Apostol-Bernoulli numbers and polynomials of higher order, the Apostol-Euler numbers and polynomials of higher order, the Peters polynomials, the Stirling numbers, the Fubini type numbers.

We known define two new classes of special polynomials. By Appell polynomials method, the first new class polynomials are defined by the following generating function:

$$\left(\frac{e^t-1}{k+1-ke^t}\right)^m e^{xt}=\sum_{n=0}^{\infty}P_n\left(x;k,m\right)\frac{t^n}{n!}. \tag{32}$$

By using (32), we have the following properties for the polynomials $P_n\left(x;k,m\right)$:

$$\frac{d}{dx}P_n\left(x;k,m\right)=nP_{n-1}\left(x;k,m\right)$$

and

$$P_n\left(x;k,m\right)=\sum_{j=0}^{n-1}\binom{n}{j+1}x^{n-1-j}f_{j+1,k}^{(m)}.$$

In order to give the following generating function for the new second class of polynomials, we use similar method in the works of Simsek [**46**, Eq. (19)] and [**48**, Eq. (3)]:

$$H(t,x,m;k)=\left(\frac{e^t-1}{k+1-ke^t}\right)^m\left(1+t\right)^x=\sum_{n=0}^{\infty}Q_n\left(x;k,m\right)\frac{t^n}{n!}, \tag{33}$$

where $m,k\in\mathbb{N}$ and $|t|<\left|\ln\left(\frac{k}{k+1}\right)\right|$.

When $x=0$ in (33), we have

$$\sum_{n=1}^{\infty}f_{n,k}^{(m)}\frac{t^n}{n!}=\sum_{n=0}^{\infty}Q_n\left(0;k,m\right)\frac{t^n}{n!}.$$

After some elementary calculations, then comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get

$$Q_{n+1}\left(0;k,m\right)=f_{n+1,k}^{(m)}.$$

For $m = 0$ in (33), we obtain

$$Q_n\,(x;k,0) = (x)_n.$$

When $k = 0$ in (33) and using (3), we have

$$Q_n\,(x;0,m) = m! \sum_{j=0}^{n} \binom{n}{j} S(j,m)\,(x)_{n-j}.$$

Setting $x = k = 0$ in (33), we get

$$Q_n\,(0;0,m) = f_{n,0}^{(m)}.$$

When $k = m = 1$ and $x = 0$ in (33), then using (30), we have

$$Q_n\,(0;1,1) = w_M\,(n)\,.$$

Theorem 2. *Let $n \in \mathbb{N}$. Then we have*

$$Q_n\,(x;k,m) = \sum_{j=0}^{n-1} \binom{n}{j+1} (x)_{n-1-j}\, f_{j+1,k}^{(m)}. \tag{34}$$

Proof. Combining (33) with (29), we obtain

$$H(t,x,m;k) = (1+t)^x\, F_g(t,m;k).$$

From the above functional equation, we get

$$\sum_{n=0}^{\infty} Q_n\,(x;k,m)\,\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n-1} \binom{n}{j+1} (x)_{n-1-j}\, f_{j+1,k}^{(m)} \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Combining (34) with (22), we have the following relation including the Peters polynomials and the polynomials $Q_n\,(x;k,m)$.

Theorem 3. *Let $n, \mu \in \mathbb{N}$. Then we have*

$$Q_n\,(x;k,m) = \sum_{j=0}^{n-1}\sum_{v=0}^{n-1-j}\sum_{d=0}^{\mu} \binom{n}{j+1}\binom{\mu}{d}\binom{n-1-j}{v} (\lambda d)_v\, s_{n-1-j-v}\,(x;\lambda,\mu)\, f_{j+1,k}^{(m)}.$$

Theorem 4. *Let $n \in \mathbb{N}_0$. Then we have*

$$Q_n\,(x;1,2m) = \frac{(2m)!}{2^m} \sum_{r=0}^{n}\sum_{j=0}^{r} \binom{n}{r}\binom{r}{j} (x)_{r-j}\, S\,(n-r,2m)\, a_j^{(m)}.$$

Proof. Substituting $k=1$ into (33), using (3) and (28), we get

$$H(t,x,2m;1)=\frac{(2m)!}{2^m}F_{S_2}\left(t,2m\right)F_a\left(t,0,m\right)\left(1+t\right)^x.$$

From the above functional equation, we have

$$\sum_{n=0}^{\infty}Q_n\left(x;1,2m\right)\frac{t^n}{n!}=\frac{(2m)!}{2^m}\sum_{n=0}^{\infty}S\left(n,2m\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}a_n^{(m)}\frac{t^n}{n!}\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}.$$

Thus,

$$\sum_{n=0}^{\infty}Q_n\left(x;1,2m\right)\frac{t^n}{n!}=\frac{(2m)!}{2^m}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{j=0}^{r}\binom{n}{r}\binom{r}{j}\left(x\right)_{r-j}S\left(n-r,2m\right)a_j^{(m)}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. □

Theorem 5. *Let $v\in\mathbb{N}$. Then we have*

$$Q_v\left(x;k,m\right)=\sum_{j=0}^{v-1}\binom{v}{j+1}f_{j+1,k}^{(m)}\sum_{n=0}^{v-1-j}x^n s\left(v-1-j,n\right). \tag{35}$$

Proof. By using (33), we have

$$\sum_{v=0}^{\infty}Q_v\left(x;k,m\right)\frac{t^v}{v!}=\sum_{v=1}^{\infty}f_{v,k}^{(m)}\frac{t^v}{v!}\sum_{n=0}^{\infty}x^n\frac{\left(\ln\left(1+t\right)\right)^n}{n!}.$$

Combining the above equation with (1), we obtain

$$\sum_{v=0}^{\infty}Q_v\left(x;k,m\right)\frac{t^v}{v!}=\sum_{v=1}^{\infty}f_{v,k}^{(m)}\frac{t^v}{v!}\sum_{v=0}^{\infty}\sum_{n=0}^{v}x^n s\left(v,n\right)\frac{t^v}{v!}.$$

Therefore

$$\sum_{v=0}^{\infty}Q_v\left(x;k,m\right)\frac{t^v}{v!}=\sum_{v=0}^{\infty}\sum_{j=0}^{v-1}\binom{v}{j+1}f_{j+1,k}^{(m)}\sum_{n=0}^{v-1-j}x^n s\left(v-1-j,n\right)\frac{t^v}{v!}.$$

Comparing the coefficients of $\frac{t^v}{v!}$ on both sides of the above equation, we arrive at the desired result. □

Remark 6. *By combining* (35) *with* (2)*, we see that* (35) *is reduced to* (34).

Theorem 7. *Let $n\in\mathbb{N}_0$. Then we have*

$$\begin{aligned}Q_n\left(x;k,m\right) &= \frac{m!}{2^m\left(k+1\right)^m}\sum_{v=0}^{n}\binom{n}{v}\sum_{j=0}^{v}\binom{v}{j}\\ &\quad\times\mathcal{E}_j^{(m)}\left(-\frac{k}{k+1}\right)S\left(v-j,m\right)\left(x\right)_{n-v}.\end{aligned} \tag{36}$$

Proof. By using (3), (7) and (33), we have the following functional equation:

$$H(t,x,m;k) = \frac{m!}{2^m (k+1)^m} F_{S_2}(t,m) F_{AE}\left(t,m;-\frac{k}{k+1}\right)(1+t)^x.$$

From the above equation, we get

$$\sum_{n=0}^{\infty} Q_n(x;k,m)\frac{t^n}{n!} = \frac{m!}{2^m (k+1)^m}\sum_{n=0}^{\infty} S(n,m)\frac{t^n}{n!} \times \sum_{n=0}^{\infty} \mathcal{E}_n^{(m)}\left(-\frac{k}{k+1}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Q_n(x;k,m)\frac{t^n}{n!} = \frac{m!}{2^m (k+1)^m}\sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v} \times \sum_{j=0}^{v}\binom{v}{j} S(v-j,m)\mathcal{E}_j^{(m)}\left(-\frac{k}{k+1}\right)(x)_{n-v}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Combining (36) with (2), we have the following theorem:

Theorem 8. *Let $n \in \mathbb{N}_0$. Then we have*

$$Q_n(x;k,m) = \frac{m!}{2^m (k+1)^m}\sum_{v=0}^{n}\binom{n}{v}\sum_{j=0}^{v}\sum_{r=0}^{n-v}\binom{v}{j} \times \mathcal{E}_j^{(m)}\left(-\frac{k}{k+1}\right) S(v-j,m)\, s(n-v,r)\, x^r.$$

Theorem 9. *Let $n \in \mathbb{N}_0$. Then we have*

$$Q_n(x;k,m) = \frac{(-1)^m}{(2k+2)^m}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\sum_{v=0}^{n}\binom{n}{v}\mathcal{E}_v^{(m)}\left(j;-\frac{k}{k+1}\right)(x)_{n-v}. \tag{37}$$

Proof. By using (33), we have

$$\sum_{n=0}^{\infty} Q_n(x;k,m)\frac{t^n}{n!} = (1+t)^x\sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{m-j}}{(2k+2)^m}\left(\frac{2}{1-\frac{k}{k+1}e^t}\right)^m e^{jt}.$$

Combining the above equation with (8), and by applying the Cauchy product formula to the final equation, we obtain

$$\sum_{n=0}^{\infty} Q_n(x;k,m)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{m}\frac{(-1)^{m-j}}{(2k+2)^m}\binom{m}{j}\sum_{v=0}^{n}\binom{n}{v}\mathcal{E}_v^{(m)}\left(j;-\frac{k}{k+1}\right)(x)_{n-v}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Combining (37) with (12), the following relation including the Apostol-Bernoulli polynomials and the generalized Fubini polynomials is given:

Corollary 10. *Let $n \in \mathbb{N}_0$. Then we have*

$$Q_n\left(x;k,m\right)=\frac{1}{(k+1)^m}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\sum_{v=0}^{n}\binom{n}{v}\frac{(x)_{n-v}}{(v+m)_m}\mathcal{B}_{v+m}^{(m)}\left(j;\frac{k}{k+1}\right).$$

Theorem 11. *Let $n \in \mathbb{N}_0$. Then we have*

$$Q_n\left(x;k,m\right)=m!\sum_{j=0}^{n}\sum_{v=0}^{j}\binom{n}{j}\binom{j}{v}S\left(j-v,m\right)(x)_{n-j}\,F_v^{(m)}\left(k\right).$$

Proof. By using (3), (33) and (31), we obtain

$$\sum_{n=0}^{\infty}Q_n\left(x;k,m\right)\frac{t^n}{n!}=m!\sum_{n=0}^{\infty}S\left(n,m\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}F_n^{(m)}\left(k\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}.$$

Thus,

$$\sum_{n=0}^{\infty}Q_n\left(x;k,m\right)\frac{t^n}{n!}=m!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{v=0}^{j}\binom{n}{j}\binom{j}{v}S\left(j-v,m\right)(x)_{n-j}\,F_v^{(m)}\left(k\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result. □

3. RELATIONS AMONG GENERALIZED FUBINI NUMBERS, APOSTOL TYPE NUMBERS AND STIRLING TYPE NUMBERS

In this section, using the properties of generating functions and their functional equations for many special numbers and polynomials, we give some identities and formulas including the Apostol-Bernoulli numbers of higher order, the Apostol-Euler numbers and polynomials of higher order, the Apostol-Genocchi numbers of higher order, the Apostol type Frobenius–Euler numbers of higher order, the generalized Fubini numbers of higher order, the Stirling numbers, the λ-Stirling numbers, the array polynomials, the numbers $y_1(n,m;\lambda)$, the numbers $y_2\left(n,m;\lambda\right)$, the numbers $Y_{n,2}(\lambda)$, and the numbers $\beta_n\left(k\right)$.

Theorem 12. *Let $n, m \in \mathbb{N}$. Then we have*

$$f_{n,k}^{(m)} = \frac{m!}{2^m (k+1)^m} \sum_{j=0}^{n} \binom{n}{j} S(n-j, m) \mathcal{E}_j^{(m)} \left(-\frac{k}{k+1}\right). \tag{38}$$

Proof. By using (3), (7) and (29), we have the following functional equation:

$$F_g(t, m; k) = \frac{m!}{2^m (k+1)^m} F_{S_2}(t, m) F_{AE}\left(t, m; -\frac{k}{k+1}\right).$$

From the above equation, we get

$$\sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{t^n}{n!} = \frac{m!}{2^m (k+1)^m} \sum_{n=0}^{\infty} S(n, m) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{E}_n^{(m)} \left(-\frac{k}{k+1}\right) \frac{t^n}{n!}.$$

Since $S(0, m) = 0$ for $m \neq 0$, we have

$$\sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{t^n}{n!} = \frac{m!}{2^m (k+1)^m} \sum_{n=1}^{\infty} \sum_{j=0}^{n} \binom{n}{j} S(n-j, m) \mathcal{E}_j^{(m)} \left(-\frac{k}{k+1}\right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Combining (38) with (12), we obtain the following corollaries:

Corollary 13. *Let $n, m \in \mathbb{N}$. Then we have*

$$f_{n,k}^{(m)} = \frac{(-1)^m}{(k+1)^m} \sum_{j=0}^{n} \binom{n}{j} \binom{j+m}{m}^{-1} S(n-j, m) \mathcal{B}_{j+m}^{(m)} \left(\frac{k}{k+1}\right). \tag{39}$$

Corollary 14. *Let $n, m \in \mathbb{N}$. Then we have*

$$f_{n,k}^{(m)} = \frac{1}{2^m (k+1)^m} \sum_{j=0}^{n} \binom{n}{j} \binom{j+m}{m}^{-1} S(n-j, m) \mathcal{G}_{j+m}^{(m)} \left(-\frac{k}{k+1}\right).$$

Substituting $m = 1$ into (39) and using (24), we obtain the following result.

Corollary 15. *Let $n, m \in \mathbb{N}$. Then we have*

$$f_{n,k} = -\frac{1}{2k+1} \sum_{j=0}^{n-1} \binom{n}{j} \sum_{v=0}^{j} \left(\frac{2k+1}{2k}\right)^v S(j, v) Y_{v,2} \left(\frac{2k}{2k+1}\right).$$

Combining (38) with (21) and (26), we arrive at the following theorem:

Theorem 16. *Let $n, m \in \mathbb{N}$. Then we have*

$$f_{n,k}^{(m)} = \frac{(-1)^m Ch_m}{(k+1)^m} \sum_{j=0}^{n} \frac{\beta_j(n)}{Ch_j} S(n-j, m) \mathcal{E}_j^{(m)} \left(-\frac{k}{k+1}\right).$$

Theorem 17. *Let $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then we have*

$$f_{n+1,k}^{(m)} = \frac{(-1)^m}{2^m (k+1)^m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \mathcal{E}_{n+1}^{(m)} \left(j; -\frac{k}{k+1} \right). \tag{40}$$

Proof. By using (29), we have

$$\sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{t^n}{n!} = \frac{1}{(k+1)^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \frac{e^{tj}}{\left(1 - \frac{k}{k+1} e^t\right)^m}.$$

Combining above equation with (8), we get

$$\sum_{n=0}^{\infty} \frac{1}{n+1} f_{n+1,k}^{(m)} \frac{t^n}{n!} = \frac{(-1)^m}{2^m (k+1)^m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{n=0}^{\infty} \frac{1}{n+1} \mathcal{E}_{n+1}^{(m)} \left(j; -\frac{k}{k+1} \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Taking $x = 0$ in (37), we also arrive at the equation (40).

Theorem 18. *Let $n, m \in \mathbb{N}$. Then we have*

$$f_{n,k}^{(m)} = \frac{m!}{k^m} \sum_{j=0}^{n} \binom{n}{j} S(j,m) \mathcal{H}_{n-j}^{(m)} (k, k+1).$$

Proof. By using (3), (14) and (29), we have

$$F_g(t, m; k) = \frac{m!}{k^m} F_{S_2}(t, m) F_{AH}(t, m; k, k+1).$$

From the above functional equation, we get

$$\sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{t^n}{n!} = \frac{m!}{k^m} \sum_{n=0}^{\infty} S(n,m) \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathcal{H}_n^{(m)} (k, k+1) \frac{t^n}{n!}.$$

Since $S(0,m) = 0$ for $m \neq 0$, we obtain

$$\sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{t^n}{n!} = \frac{m!}{k^m} \sum_{n=1}^{\infty} \sum_{j=0}^{n} \binom{n}{j} S(j,m) \mathcal{H}_{n-j}^{(m)} (k, k+1) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 19. *Let $n, m \in \mathbb{N}$. Then we have*

$$S(n,m) = (k+1)^m \sum_{j=0}^{n-1} \binom{n}{j+1} y_1 \left(n-1-j, m; -\frac{k}{k+1} \right) f_{j+1,k}^{(m)}. \tag{41}$$

Proof. By using (3), (17) and (29), we have the following functional equation:

$$F_{S_2}(t,m) = (k+1)^m F_g(t,m;k) F_{y_1}\left(t,m;-\frac{k}{k+1}\right).$$

From the above functional equation, we get

$$\sum_{n=0}^{\infty} S(n,m)\frac{t^n}{n!} = (k+1)^m \sum_{n=0}^{\infty} f_{n+1,k}^{(m)} \frac{t^{n+1}}{(n+1)!} \sum_{n=0}^{\infty} y_1\left(n,m;-\frac{k}{k+1}\right)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} S(n,m)\frac{t^n}{n!} = (k+1)^m \sum_{n=0}^{\infty}\sum_{j=0}^{n-1}\binom{n}{j+1} y_1\left(n-1-j,m;-\frac{k}{k+1}\right) f_{j+1,k}^{(m)}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Combining (41) with (19), we arrive at the following theorem:

Theorem 20. *Let $n,m \in \mathbb{N}$. Then we have*

$$\begin{aligned} S(n,2m) &= (k+1)^m (-k)^m \sum_{j=0}^{n-1}\binom{n}{j+1} f_{j+1,k}^{(2m)} \\ &\quad \times \sum_{v=0}^{n-1-j}\binom{n-1-j}{v} m^{n-1-j-v} y_2\left(v,m;-\frac{k}{k+1}\right). \end{aligned}$$

Theorem 21. *Let $n,m \in \mathbb{N}$. Then we have*

$$S(n,m) = (-1)^m (k+1)^m \sum_{j=0}^{n-1}\binom{n}{j+1} S_2\left(n-1-j,m;\frac{k}{k+1}\right) f_{j+1,k}^{(m)}.$$

Proof. By using (3), (15) and (29), we have

$$\sum_{n=0}^{\infty} S(n,m)\frac{t^n}{n!} = (-1)^m (k+1)^m \sum_{n=1}^{\infty} f_{n,k}^{(m)}\frac{t^n}{n!} \sum_{n=0}^{\infty} S_2\left(n,m;\frac{k}{k+1}\right)\frac{t^n}{n!}.$$

From the above equation, it is easily to find that

$$\begin{aligned} \sum_{n=0}^{\infty} S(n,m)\frac{t^n}{n!} &= (-1)^m (k+1)^m \\ &\quad \times \sum_{n=0}^{\infty} n \sum_{j=0}^{n-1}\binom{n-1}{j}\frac{f_{j+1,k}^{(m)}}{j+1} S_2\left(n-1-j,m;\frac{k}{k+1}\right)\frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 22. *Let $n, m \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} B_{n-j}^{(-m)} \mathcal{B}_{j}^{(m)} \left(\frac{k}{k+1} \right) = (-1)^m (k+1)^m f_{n,k}^{(m)}.$$

Proof. By using (5) and (29), we obtain

$$\frac{(-1)^m}{(k+1)^m} F_{AB}(t,-m;1) F_{AB}\left(t,m;\frac{k}{k+1}\right) = \sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{t^n}{n!}.$$

Thus,

$$\frac{(-1)^m}{(k+1)^m} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} B_{n-j}^{(-m)} \mathcal{B}_{j}^{(m)} \left(\frac{k}{k+1} \right) \frac{t^n}{n!} = \sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{t^n}{n!}.$$

Since $\mathcal{B}_{0}^{(m)} \left(\frac{k}{k+1} \right) = 0$, we get

$$\frac{(-1)^m}{(k+1)^m} \sum_{n=1}^{\infty} \sum_{j=0}^{n} \binom{n}{j} B_{n-j}^{(-m)} \mathcal{B}_{j}^{(m)} \left(\frac{k}{k+1} \right) \frac{t^n}{n!} = \sum_{n=1}^{\infty} f_{n,k}^{(m)} \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 23. *Let $n, m \in \mathbb{N}$. Then we have*

$$S(n,m) = \frac{(k+1)^m 2^m}{m!} \sum_{j=0}^{n-1} \binom{n}{j+1} \mathcal{E}_{n-1-j}^{(-m)} \left(-\frac{k}{k+1} \right) f_{j+1,k}^{(m)}. \tag{42}$$

Proof. By using (3), (7) and (29), we obtain

$$F_{S_2}(t,m) = \frac{(k+1)^m 2^m}{m!} F_{AE}\left(t,-m;-\frac{k}{k+1}\right) F_g(t,m;k).$$

From the above equation, we get

$$\sum_{n=0}^{\infty} S(n,m) \frac{t^n}{n!} = \frac{(k+1)^m 2^m}{m!} t \sum_{n=0}^{\infty} \mathcal{E}_{n}^{(-m)} \left(-\frac{k}{k+1} \right) \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{f_{n+1,k}^{(m)}}{n+1} \frac{t^n}{n!}.$$

Hence,

$$\sum_{n=0}^{\infty} S(n,m) \frac{t^n}{n!} = \frac{(k+1)^m 2^m}{m!} \sum_{n=0}^{\infty} n \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{f_{j+1,k}^{(m)}}{j+1} \mathcal{E}_{n-1-j}^{(-m)} \left(-\frac{k}{k+1} \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Remark 24. *By using* (41) *and* (42)*, we obtain the following well-known result:*

$$\mathcal{E}_n^{(-m)}\left(-\frac{k}{k+1}\right) = m!2^{-m}y_1\left(n,m;-\frac{k}{k+1}\right)$$

(cf. [**43**, Eq. (28)]*).*

4. COMPUTATIONAL ALGORITHMS FOR THE GENERALIZED FUBINI NUMBERS AND POLYNOMIALS

Algorithms are used in many branches of science such as computer science, applied mathematics and communications systems. Due to their importance, in this section, by using (4), (9), (34) and (38), we give algorithms for the generalized Fubini numbers and polynomials of higher order, the Apostol-Euler polynomials of higher order, and the Stirling numbers of the second kind. Here note that, we do not study on the computational complexity of algorithms. We only use these algorithms to calculate the numerical values of the generalized Fubini numbers and polynomials of higher order. That is, with the aid of these algorithms, some numerical values of these numbers and polynomials are given by the tables. Moreover, using Mathematica version 12.0 with the Wolfram language, we illustrate some plots of the generalized Fubini polynomials of higher for some of their randomly chosen cases.

By using similar method in Kucukoglu and Simsek [**22**] and equation (4), we give the following algorithms:

Algorithm 1 For $n, m \in \mathbb{N}_0$, this algorithm will return values of the numbers $S(n,m)$.

```
procedure STIRLING_SEC_NUM(n: nonnegative integer,
m: nonnegative integer)
    Global variable S ← 0
    Local variable v: nonnegative integer
    if n = 0 ∧ m = 0  then
        return 1
    else
        if m > 0 ∨ n > 0 ∨ m > n  then
            return 0
        else
            for v = 0; v ≤ m; v = v + 1  do
                    S ← S + Power(−1, m − v)
                        ↪ ∗Binomial_Coef(m, v) ∗ Power(v, n)
            end for
            return (1/Factorial(m)) ∗ S
        end if
    end if
end procedure
```

By using (9), we give an algorithm for the Apostol-Euler polynomials of higher order as follows:

Algorithm 2 For $n \in \mathbb{N}_0$ and $\lambda, \alpha \in \mathbb{C}$, this algorithm will return values of the polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ by the aid of Algorithm 1.

procedure APOST_EULER_POLY_HIG(n: nonnegative integer,
x: parameter, λ: complex number, α: complex number)
  Global variable E $\leftarrow 0$
  Local variable r, j: nonnegative integer
  for $r = 0$; $r \leq n$; $r = r + 1$ **do**
    for $j = 0$; $j \leq r$; $j = j + 1$ **do**
      E $\leftarrow$ E $+ Binomial\_Coef(n, r) * Power(x, n - r)$
        $\hookrightarrow *Binomial\_Coef(\alpha + j - 1, j)$
        $\hookrightarrow * (Factorial(j) * Power(-\lambda, j)/Power(\lambda + 1, j + \alpha))$
        $\hookrightarrow *$ STIRLING_SEC_NUM(r, j)
    end for
  end for
  return $Power(2, \alpha) *$ E
end procedure

In the literature, there are some computational algorithms for special numbers and polynomials (see, for detail, [**18**]-[**20**], [**22**], [**23**], [**25**]-[**28**], [**42**], [**44**]). It should be also noted that there are many different programming languages for the Algorithm 1 and Algorithm 2. For example, the Stirling numbers of the second kind are computed the command StirlingS2[n,m] in Mathematica with the Wolfram language, see [**53**]. Moreover, Kucukoglu and Simsek also gave an algorithm for the generalized Stirling numbers (see [**25**, Algorithm 1 and Eq. (14)]). Here, these algorithms have been given in order to run the Algorithm 3. Moreover, many algorithms can be given to calculate numerical values of the generalized Fubini numbers and polynomials of higher order. We only use the equation (38). By using this equation, we give an algorithm for the generalized Fubini numbers of higher order as follows:

Algorithm 3 For $n, m \in \mathbb{N}$ and $k \in \mathbb{N}_0$, this algorithm will return values of the numbers $f_{n,k}^{(m)}$ by the aid of Algorithm 1 and Algorithm 2.

procedure GENERALIZED_FUBINI_NUM_HIG(n: positive number, k: nonnegative number, m: positive number)
  Global variable F $\leftarrow 0$
  Local variable j: nonnegative integer
  for $j = 0$; $j \leq n$; $j = j + 1$ **do**
    F $\leftarrow$ F + $Binomial\_Coef(n, j)$
      $\hookrightarrow *$ STIRLING_SEC_NUM($n - j, m$)
      $\hookrightarrow *$APOST_EULER_POLY_HIG($j, 0, -k/(k + 1), m$)
  end for
  return $(Factorial(m)/\,(Power(k + 1, m) * Power(2, m))) *$ F
end procedure

By using (34), we give an algorithm for the polynomials $Q_n(x; k, m)$ as follows:

Algorithm 4 For $n, m, k \in \mathbb{N}$, this algorithm will return values of the polynomials $Q_n(x; k, m)$ by the aid of Algorithm 3.

procedure *Q_POLY* (n: positive number, x: parameter, k: positive number, m: positive number)
  Global variable Q $\leftarrow 0$
  Local variable j: nonnegative integer
  for $j = 0$; $j \leq n - 1$; $j = j + 1$ **do**
    Q $\leftarrow$ Q + $Binomial\_Coef(n, j + 1) * FallingFact(x, n - 1 - j)$
      $\hookrightarrow *$GENERALIZED_FUBINI_NUM_HIG($j + 1, k, m$)
  end for
  return Q
end procedure

By using Algorithm 3, we compute a few values of the numbers $f_{n,k}^{(m)}$ for $m = 1$ as follows:

Table 1: A few values of the numbers $f_{n,k}^{(1)}$ (see also [**36**]).

n/k	1	2	3	4	5
1	1	1	1	1	1
2	3	5	7	9	11
3	13	37	73	121	181
4	75	365	1015	2169	3971
5	541	4501	17641	48601	108901

By using Algorithm 3, we compute a few values of the numbers $f_{n,k}^{(m)}$ for $m = 2$ as follows:

Table 2: A few values of the numbers $f_{n,k}^{(2)}$.

n/k	1	2	3	4	5
1	0	0	0	0	0
2	2	2	2	2	2
3	18	30	42	54	66
4	158	446	878	1454	2174
5	1530	7350	20370	43470	79530

By using Algorithm 3, we compute a few values of the numbers $f_{n,k}^{(m)}$ for $m = 3$ as follows:

Table 3: A few values of the numbers $f_{n,k}^{(3)}$.

n/k	1	2	3	4	5
1	0	0	0	0	0
2	0	0	0	0	0
3	6	6	6	6	6
4	108	180	252	324	396
5	1590	4470	8790	14550	21750

Note that, using (38) and the properties of the Stirling numbers of the second kind for $m > n$, $S(n, m) = 0$, some values of the numbers $f_{n,k}^{(m)}$ are equal to zero, given in Table 2 and Table 3.

By using Algorithm 4, we compute a few values of the polynomials $Q_n(x; k, m)$ for $m = 1$ as follows:

$$
\begin{aligned}
Q_0(x;k,1) &= 0, \\
Q_1(x;k,1) &= 1, \\
Q_2(x;k,1) &= 1 + 2k + 2x, \\
Q_3(x;k,1) &= 1 + 6k^2 + 3x^2 + 6k(1+x), \\
Q_4(x;k,1) &= 1 + 24k^3 + 6x - 6x^2 + 4x^3 + 12k^2(3+2x) + 2k(7 + 6x + 6x^2), \\
Q_5(x;k,1) &= 1 + 120k^4 - 15x + 35x^2 - 20x^3 + 5x^4 + 120k^3(2+x) \\
&\quad +30k^2(5 + 4x + 2x^2) + 10k(3 + 5x + 2x^3).
\end{aligned}
$$

By using Algorithm 4, we compute a few values of the polynomials $Q_n(x; k, m)$

for $m = 2$ as follows:

$$\begin{aligned}
Q_0(x;k,2) &= 0,\\
Q_1(x;k,2) &= 0,\\
Q_2(x;k,2) &= 2,\\
Q_3(x;k,2) &= 6(1+2k+x),\\
Q_4(x;k,2) &= 2(7+36k^2+6x+6x^2+12k(3+2x)),\\
Q_5(x;k,2) &= 10(3+48k^3+5x+2x^3+36k^2(2+x))+60k(5+4x+2x^2).
\end{aligned}$$

By using Algorithm 4, we calculate a few values of the polynomials $Q_n(x;k,m)$ for $m = 3$ as follows:

$$\begin{aligned}
Q_0(x;k,3) &= 0,\\
Q_1(x;k,3) &= 0,\\
Q_2(x;k,3) &= 0,\\
Q_3(x;k,3) &= 6,\\
Q_4(x;k,3) &= 12(3+6k+2x),\\
Q_5(x;k,3) &= 30(5+24k^2+4x+2x^2+12k(2+x)).
\end{aligned}$$

It is time to give plots of the polynomials $Q_n(x;k,m)$. Using Mathematica [**53**] via Wolfram language, we show some plots of the polynomials $Q_n(x;k,m)$ for some of their randomly chosen special cases.

Figure 1 is obtained by $k = m = 1$, and $n \in \{0,1,2,3,4,5,6,7,8\}$ using (34) for $x \in [-60, 60]$.

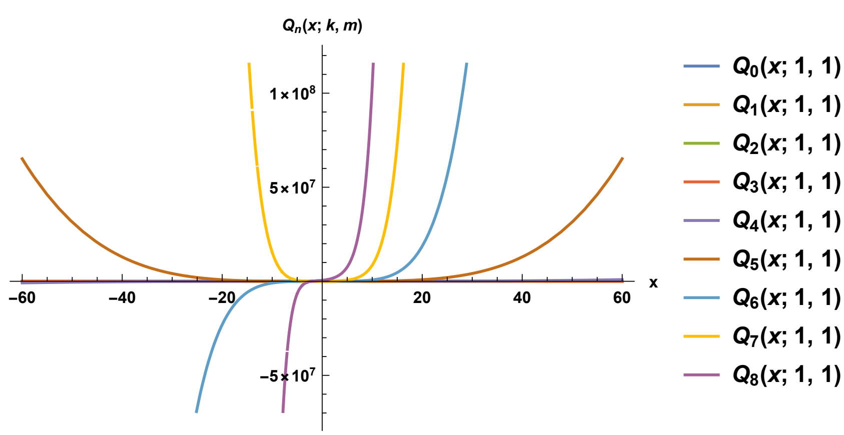

Figure 1: Plots of the polynomials $Q_n(x;1,1)$ in state that $n \in \{0,1,2,3,4,5,6,7,8\}$ and $x \in [-60, 60]$.

Figure 2 is obtained by $k = 1$, $m = 2$ and $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ using (34) for $x \in [-60, 60]$.

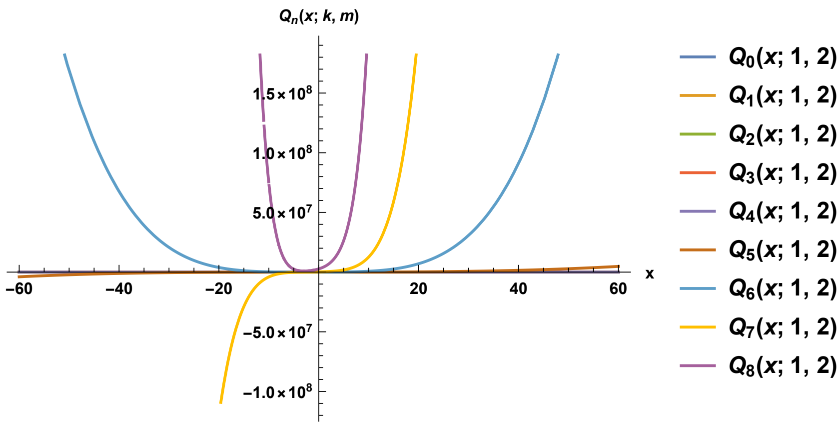

Figure 2: Plots of the polynomials $Q_n(x; 1, 2)$ in state that $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $x \in [-60, 60]$.

Figure 3 is obtained by $k = 1$, $m = 3$ and $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ using (34) for $x \in [-60, 60]$.

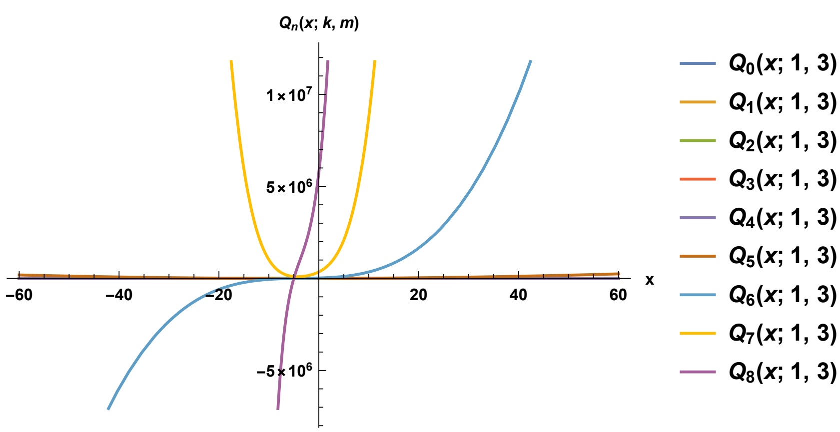

Figure 3: Plots of the polynomials $Q_n(x; 1, 3)$ in state that $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $x \in [-60, 60]$.

Notes that the polynomials $Q_0(x; 1, 1)$, $Q_0(x; 1, 2)$, $Q_1(x; 1, 2)$, $Q_0(x; 1, 3)$, $Q_1(x; 1, 3)$ and $Q_2(x; 1, 3)$ are equal to zero. On the other hand, it can be observed that the values of the polynomials $Q_1(x;1,1)$, $Q_2(x;1,1)$, $Q_3(x;1,1)$, $Q_2(x;1,2)$, $Q_3(x;1,2)$, $Q_3(x;1,3)$ and $Q_4(x;1,3)$ are quite close to the Ox-axis.

5. CONCLUSIONS

In this paper, we constructed new families of special polynomials with their generating functions. By using method of generating functions and their functional equations, we investigated some properties of these new polynomials. We also derived many interesting identities, relations and formulas related to the Fubini type numbers, the Apostol type numbers and polynomials of higher order, the Stirling type numbers, the Peters polynomials, the array polynomials, the combinatorial numbers, and other special numbers and polynomials. Furthermore, we gave not only algorithms for the calculation of the generalized Fubini numbers and polynomials of higher order, the Apostol-Euler polynomials of higher order, and the Stirling numbers of the second kind, but also illustrated some plots of the generalized Fubini polynomials of higher order for some of their randomly selected special cases. Therefore, the results of this paper may be used especially in mathematics, mathematical physics, computer engineering, communications systems, and other branches of engineering and related areas.

It is among my future plans to investigate the relationships of this new family of polynomials given in this paper in depth with other fields and to study mathematical modeling of real world problems with the help of these polynomials.

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Neslihan Kilar
Department of Computer Technologies,
Niğde Ömer Halisdemir University,
TR-51700 Niğde, Turkey
E-mail: *neslihankilar@ohu.edu.tr; neslihankilar@gmail.com*

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