Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **17** (2023), 525–537.
https://doi.org/10.2298/AADM220810024H

ANALYTICAL AND ASYMPTOTIC REPRESENTATIONS FOR TWO SEQUENCE RELATED TO GAUSS' LEMNISCATE FUNCTIONS

Xue-Feng Han, Chao-Ping Chen and H. M. Srivastava*

Let the sequences G_n and g_n be defined by

$$G_n := \int_0^1 \frac{\mathrm{d}t}{(1-t^{2n})^{1/n}} \quad (n \geqq 2) \qquad \text{and} \qquad g_n := \int_0^\infty \frac{\mathrm{d}t}{(1+t^{2n})^{1/n}} \quad (n \geqq 1).$$

In this paper, we first derive analytical representations for these two sequences G_n and g_n in terms of the gamma function. By using the obtained analytical representations, we then deduce asymptotic expansions for G_n and g_n.

1. INTRODUCTION AND MOTIVATION

The lemniscate, also called the lemniscate of Bernoulli (see, for example, [**20**]), is the locus of points (x, y) in the plane satisfying the following equation:

$$(x^2 + y^2)^2 = x^2 - y^2.$$

In the polar coordinates (r, θ), the equation becomes $r^2 = \cos(2\theta)$ and its arc length is given by the function

$$\text{(1)} \qquad \operatorname{arcsl} x = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^4}} \qquad (|x| \leqq 1),$$

*Corresponding author. Chao-Ping Chen
2020 Mathematics Subject Classification. Primary 41A60; Secondary 11M35, 40A05
Keywords and Phrases. Gamma and Beta functions, Lemniscate functions, Asymptotic expansions, Zeta functions, Bell polynomials.

where $\operatorname{arcsl} x$ is called the arc lemniscate sine function studied by Carl Friedrich Gauss (1777–1855) during the period 1797–1798. Another lemniscate function, investigated by Gauss, is the hyperbolic arc lemniscate sine function, defined as follows:

$$\operatorname{arcslh} x = \int_0^x \frac{\mathrm{d}t}{\sqrt{1+t^4}} \qquad (x \in \mathbb{R}). \tag{2}$$

The functions (1) and (2) can be found to be investigated in several recent works (see, for example, [**2, 3, 13, 14, 15**], [**4**, p. 259], [**6**, (2.5)–(2.6)], [**16**, Ch. 1] and [**17**, p. 286]). In particular, Neuman [**13**] introduced Gauss' arc lemniscate tangent and the hyperbolic arc lemniscate tangent functions.

Gauss' constant G is given by

$$G = \frac{1}{\operatorname{agm}(1, \sqrt{2})} = \frac{2}{\pi} \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}} = 0.8346268\cdots,$$

where $\operatorname{agm}(a, b)$ is the arithmetic-geometric mean, so that

$$G = \frac{1}{2\pi} B\left(\frac{1}{4}, \frac{1}{2}\right)$$

where $B(x, y)$ denotes the beta function. Gauss' constant G is used in the definition of the lemniscate constant L given by

$$L = \pi G.$$

For a very informative history of the lemniscate integral and its importance in the early developments of the elliptic integrals and the elliptic functions, see Ayoub's historical survey article [**2**]. Siegel [**16**] considered the lemniscate integral to be so important that he began his development of the theory of elliptic functions with a thorough discussion of the lemniscate integral.

The integrals in (1) and (2) are closely related. Indeed, if we set

$$v = \frac{\sqrt{2}x}{\sqrt{1+x^4}},$$

then an easy elementary calculation reveals that

$$\int_0^v \frac{\mathrm{d}t}{\sqrt{1-t^4}} = \sqrt{2} \int_0^x \frac{\mathrm{d}t}{\sqrt{1+t^4}}. \tag{3}$$

The relation (3) is very important because it represents the key intermediary step in the famous problem of doubling the arc length of the lemniscate. For discussions of this historically important problem, see the aforementioned paper of Ayoub [**2**] and Siegel's textbook [**16**].

Klamkin [**11**] proposed the following elegant problem:

Without performing any integration, determine the following ratio:

$$\left(\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^4}}\right) : \left(\int_0^1 \frac{\mathrm{d}t}{\sqrt{1+t^4}}\right).$$

Subsequently, Farnell [**10**] proved that the desired ratio is $\sqrt{2}$. Raynor and the proposer (Klamkin) obtained the following more general result (see [**10**, Editorial Note]):

$$\text{(4)}\qquad \left(\int_0^1 \frac{\mathrm{d}t}{(1-t^{2n})^{1/n}}\right) : \left(\int_0^1 \frac{\mathrm{d}t}{(1+t^{2n})^{1/n}}\right) = \sec\left(\frac{\pi}{2n}\right) \qquad (n = 2, 3, 4, \cdots).$$

Motivated by (4), and observing that

$$\int_0^1 \frac{\mathrm{d}t}{(1+t^{2n})^{1/n}} = \int_1^\infty \frac{\mathrm{d}u}{(1+u^{2n})^{1/n}},$$

we define the sequences G_n and g_n by

$$\text{(5)}\qquad G_n = \int_0^1 \frac{\mathrm{d}t}{(1-t^{2n})^{1/n}} \qquad (n \geqq 2)$$

and

$$\text{(6)}\qquad g_n = \int_0^\infty \frac{\mathrm{d}t}{(1+t^{2n})^{1/n}} = 2\int_0^1 \frac{\mathrm{d}t}{(1+t^{2n})^{1/n}} \qquad (n \geqq 1).$$

Thus, clearly, the formula (4) can be written as follows:

$$\text{(7)}\qquad \frac{G_n}{\frac{1}{2}g_n} = \sec\left(\frac{\pi}{2n}\right) \qquad (n = 2, 3, 4, \cdots).$$

In this paper, we first give analytical representations for the above-defined sequences G_n and g_n in terms of the gamma function. By using the obtained analytical representations, we then present asymptotic expansions for the sequences G_n and g_n.

2. ANALYTICAL REPRESENTATIONS FOR G_n AND g_n

The proposed analytical representations for the sequences G_n and g_n are given by Theorem 1 below.

Theorem 1. *The sequences G_n and g_n have the following analytical representations*:

$$\text{(8)}\qquad G_n = \frac{\Gamma(1+\frac{1}{2n})\Gamma(\frac{1}{2}-\frac{1}{2n})}{2^{\frac{1}{n}}\sqrt{\pi}} \qquad (n \geqq 2)$$

and

$$\text{(9)}\qquad g_n = \frac{2^{1-\frac{1}{n}}\sqrt{\pi}\,\Gamma\left(1+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)} \qquad (n \geqq 1).$$

Proof. By an elementary change of variables given by

$$u = 1 - t^{2n} \qquad (0 \leqq t \leqq 1),$$

we obtain

$$\begin{aligned} G_n &= \frac{1}{2n}\int_0^1 u^{(1-\frac{1}{n})-1}(1-u)^{\frac{1}{2n}-1}\mathrm{d}u = \frac{1}{2n}B\left(1-\frac{1}{n},\frac{1}{2n}\right) \\ &= \frac{\Gamma(1-\frac{1}{n})\Gamma(\frac{1}{2n})}{2n\Gamma(1-\frac{1}{2n})} = \frac{\Gamma(1-\frac{1}{n})\Gamma(1+\frac{1}{2n})}{\Gamma(1-\frac{1}{2n})}. \end{aligned} \tag{10}$$

by using the recurrence formula

$$\Gamma(z+1) = z\Gamma(z). \tag{11}$$

The gamma function satisfies the following duplication formula [**1**, p. 256]:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{12}$$

The choice $z = 1 - \frac{1}{2n}$ in (12) yields

$$\frac{\Gamma(1-\frac{1}{n})}{\Gamma(1-\frac{1}{2n})} = \frac{\Gamma(\frac{1}{2}-\frac{1}{2n})}{2^{\frac{1}{n}}\sqrt{\pi}}. \tag{13}$$

Substitution of the expression (13) into (10) leads us to the desired result (8).

Next, by an elementary change of variables given by

$$u = 1 + t^{2n} \qquad (t \geqq 0),$$

we obtain

$$\begin{aligned} g_n &= \int_0^\infty \frac{\mathrm{d}t}{(1+t^{2n})^{1/n}} = \frac{1}{2n}\int_1^\infty u^{-\frac{1}{n}}(u-1)^{\frac{1}{2n}-1}\mathrm{d}u \\ &= \frac{1}{2n}\int_0^1 v^{\frac{1}{2n}-1}(1-v)^{\frac{1}{2n}-1}\mathrm{d}v = \frac{1}{2n}B\left(\frac{1}{2n},\frac{1}{2n}\right) \qquad (\textit{where} \quad u = 1/v) \\ &= \frac{1}{2n}\frac{\Gamma(\frac{1}{2n})\Gamma(\frac{1}{2n})}{\Gamma\big(2\left(\frac{1}{2n}\right)\big)} = \frac{2^{1-\frac{1}{n}}\sqrt{\pi}\Gamma\left(1+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right)}, \end{aligned}$$

where we have made use of (11) and (12). The proof of Theorem 1 is thus completed. □

Remark 2. By the following reflection formula for the gamma function $[1, p.256]$:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad (z \notin \mathbb{Z} := \{0, \pm1, \pm2, \cdots\}),$$

we have

$$\Gamma\left(\frac{1}{2}-\frac{1}{2n}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2n}\right) = \pi\sec\left(\frac{\pi}{2n}\right). \tag{14}$$

Thus, from (8) and (9) we retrieve (7) by means of (14).

3. ASYMPTOTIC EXPANSIONS FOR G_n AND g_n

In this section, we establish the asymptotic expansions for G_n and g_n, which are based upon the Bell polynomials. The Bell polynomials, named in honor of Eric Temple Bell (1883–1960), are a triangular array of polynomials given by (see, for example, Comtet [**8**, pp. 133–134], Cvijović [**9**] and Masjed-Jamei et al. [**12**])

$$B_{n,k}(x_1, x_2, \cdots, x_{n-k+1}) = \sum \frac{n!}{j_1!\, j_2! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where the sum is taken over all non-negative integers $j_1, j_2, j_3, \cdots, j_{n-k+1}$ such that

$$j_1 + j_2 + \cdots + j_{n-k+1} = k \qquad \text{and} \qquad j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n.$$

The following sum:

$$B_n(x_1, x_2, x_3, \cdots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, x_2, x_3, \cdots, x_{n-k+1})$$

is sometimes called the nth complete Bell polynomial. These complete Bell polynomials satisfy the following identity:

$$B_n(x_1, x_2, x_3, \cdots, x_n) = \begin{vmatrix} x_1 & \binom{n-1}{1}x_2 & \binom{n-1}{2}x_3 & \binom{n-1}{3}x_4 & \binom{n-1}{4}x_5 & \cdots & \cdots & x_n \\ -1 & x_1 & \binom{n-2}{1}x_2 & \binom{n-2}{2}x_3 & \binom{n-2}{3}x_4 & \cdots & \cdots & x_{n-1} \\ 0 & -1 & x_1 & \binom{n-3}{1}x_2 & \binom{n-3}{2}x_3 & \cdots & \cdots & x_{n-2} \\ 0 & 0 & -1 & x_1 & \binom{n-4}{1}x_2 & \cdots & \cdots & x_{n-3} \\ 0 & 0 & 0 & -1 & x_1 & \cdots & \cdots & x_{n-4} \\ 0 & 0 & 0 & 0 & -1 & \cdots & \cdots & x_{n-5} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & x_1 \end{vmatrix}. \tag{15}$$

In order to contrast them with complete Bell polynomials, the polynomials $B_{n,k}$ defined above are sometimes called partial Bell polynomials. The complete Bell polynomials appear in the exponential of a formal power series:

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n!} u^n\right) = \sum_{n=0}^{\infty} \frac{B_n(x_1, \cdots, x_n)}{n!}\, u^n. \tag{16}$$

The Bell polynomials are quite general polynomials and they have been found in many applications in combinatorics. In his monograph, Comtet [**8**] devoted much to a thorough presentation of the Bell polynomials in the chapter on identities and expansions. For more results, see the works by Charalambides [**7**, Chapter 11] and Riordan [**18**, Chapter 5].

We now state and prove the asymptotic expansion of the sequence G_n defined by (5)

Theorem 3. *The sequence G_n, defined in (5), has the following asymptotic expansion:*

$$\begin{aligned} G_n = \sum_{j=0}^{\infty} \frac{b_j}{n^j} = 1 + \frac{\pi^2}{12n^2} + \frac{\zeta(3)}{4n^3} + \frac{\pi^4}{160n^4} + \left(\frac{3\zeta(5)}{16} + \frac{\pi^2\zeta(3)}{48}\right)\frac{1}{n^5} \\ + \left(\frac{61\pi^6}{120960} + \frac{\zeta(3)^2}{32}\right)\frac{1}{n^6} + \cdots \qquad (n \to \infty), \end{aligned} \tag{17}$$

with the coefficients b_j given by the recursive formula:

$$b_0 = 1, \quad b_1 = 0, \quad b_j = \sum_{\ell=0}^{j-1} \left(1 - \frac{\ell}{j}\right) \frac{J^{(j-\ell)}(0)}{(j-\ell)!} b_\ell \qquad (j \geqq 2), \tag{18}$$

where

$$J(0) = 0, \quad J'(0) = 0, \quad J^{(k)}(0) = \frac{1}{2^k}\Big[(-1)^k + (2^k - 1)\Big](k-1)!\,\zeta(k) \qquad (k \geqq 2)$$

*and $\zeta(s)$ denotes the Riemann zeta function given by (see, for example, [**19**])*

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big) \end{cases}$$

Proof. First of all, we define the functions $I(x)$ and $J(x)$ by

$$I(x) = \frac{\Gamma(1+\frac{x}{2})\Gamma(\frac{1}{2}-\frac{x}{2})}{2^x\sqrt{\pi}} \qquad \text{and} \qquad J(x) = \ln I(x)$$

for $0 < x < 1$. We thus find that

$$J(x) = \ln\Gamma\left(1+\frac{x}{2}\right) + \ln\Gamma\left(\frac{1}{2}-\frac{x}{2}\right) - x\ln 2 - \ln(\sqrt{\pi}).$$

Elementary calculations would show that

$$J'(x) = \frac{1}{2}\left[\psi\left(1+\frac{x}{2}\right) - \psi\left(\frac{1}{2}-\frac{x}{2}\right)\right] - \ln 2,$$

$$J^{(j)}(x) = \frac{1}{2^j}\left[\psi^{(j-1)}\left(1+\frac{x}{2}\right) - (-1)^{j-1}\psi^{(j-1)}\left(\frac{1}{2}-\frac{x}{2}\right)\right] \qquad (j \geqq 2),$$

where the Psi (or the Digamma) function $\psi(x)$ is defined by

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x)$$

and $\psi^{(j)}(x)$ $(j \geqq 1)$ are called the Polygamma functions. We then obtain

$$J(0) = 0, \quad J'(0) = \frac{1}{2}\left[\psi\left(1\right) - \psi\left(\frac{1}{2}\right)\right] - \ln 2 = 0,$$

$$J^{(j)}(0) = \frac{1}{2^j}\left[\psi^{(j-1)}\left(1\right) - (-1)^{j-1}\psi^{(j-1)}\left(\frac{1}{2}\right)\right] \qquad (j \geqq 2).$$

Noting that (see [**21**, p. 34])

$$\psi^{(j)}\left(1\right) = (-1)^{j+1}\, j!\, \zeta(j+1) \qquad \text{and} \qquad \psi^{(j)}\left(\frac{1}{2}\right) = (-1)^{j+1}\, j!\, (2^{j+1}-1)\,\zeta(j+1),$$

we get

$$J(0) = 0, \quad J'(0) = 0, \quad J^{(j)}(0) = \frac{1}{2^j}\Big[(-1)^j + 2^j - 1\Big]\,(j-1)!\,\zeta(j) \qquad (j \geqq 2).$$

We are thus led to the following power series:

$$J(x) = \sum_{j=2}^{\infty}\frac{J^{(j)}(0)}{j!}\,x^j$$

$$= \frac{\pi^2}{12}x^2 + \frac{\zeta(3)}{4}\,x^3 + \frac{\pi^4}{360}x^4 + \frac{3\zeta(5)}{16}x^5 + \frac{\pi^6}{5670}x^6 + \frac{9\zeta(7)}{64}x^7 + \frac{\pi^8}{75600}x^8 + \cdots . \tag{19}$$

Also, in linght of the following limit formula:

$$\lim_{j\to\infty}\left|\frac{\frac{J^{(j+1)}(0)}{(j+1)!}x^{j+1}}{\frac{J^{(j)}(0)}{j!}x^j}\right| = \lim_{j\to\infty}\frac{2^{j+1}-1+(-1)^{j+1}}{2^{j+1}-2+2(-1)^j}\,\frac{j}{j+1}\,\frac{\zeta(j+1)}{\zeta(j)}|x| = |x|,$$

we see that the power series (19) converges absolutely on $(-1,1)$.

We now let

$$a_0 = J(0) = 0, \quad a_1 = J'(0) = 0, \quad a_j = \frac{J^{(j)}(0)}{j!} \qquad (j \geqq 2).$$

The equation (19) can then be written as follows:

$$J(x) = \sum_{j=1}^{\infty} a_j x^j.$$

By using (16), we find that

$$I(x) = \exp(J(x)) = \exp\left(\sum_{j=1}^{\infty} \frac{j!\, a_j}{j!}\, x^j\right) = \sum_{j=0}^{\infty} b_j x^j,$$

where

$$b_j = \frac{B_j\Big(1!\, a_1, 2!\, a_2, \cdots, j!\, a_j\Big)}{j!}. \tag{20}$$

Bulò *et al.* [**5**, Theorem 1] proved that the complete Bell polynomials can be expressed by using the following recursive relation:

$$B_n(x_1, x_2, \cdots, x_n) = \begin{cases} \sum\limits_{\ell=0}^{n-1} \binom{n-1}{\ell} x_{n-\ell} B_\ell(x_1, x_2, \cdots, x_\ell) & (n > 0) \\ 1 & (\text{otherwise}). \end{cases} \tag{21}$$

Therefore, by employing (21), the formula (20) can be rewritten as follows:

$$\begin{aligned} b_0 &= 1 \quad \text{and} \\ b_j &= \frac{1}{j!} \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} (j-\ell)!\, a_{j-\ell} B_\ell\Big(1!\, a_1, 2!\, a_2 \cdots, \ell!\, a_\ell\Big) \\ &= \frac{1}{j!} \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} (j-\ell)!\, a_{j-\ell} \ell! b_\ell \\ &= \sum_{\ell=0}^{j-1} \frac{j-\ell}{j}\, a_{j-\ell} b_\ell = \sum_{\ell=0}^{j-1} \left(1 - \frac{\ell}{j}\right) \frac{J^{(j-\ell)}(0)}{(j-\ell)!} b_\ell \qquad (j \in \mathbb{N}). \end{aligned}$$

We then obtain the following asymptotic expansion:

$$\begin{aligned} G_n = I\left(\frac{1}{n}\right) = \sum_{j=0}^{\infty} \frac{b_j}{n^j} &= 1 + \frac{\pi^2}{12n^2} + \frac{\zeta(3)}{4n^3} + \frac{\pi^4}{160n^4} + \left(\frac{3\zeta(5)}{16} + \frac{\pi^2\zeta(3)}{48}\right)\frac{1}{n^5} \\ &\quad + \left(\frac{61\pi^6}{120960} + \frac{\zeta(3)^2}{32}\right)\frac{1}{n^6} + \cdots \qquad (n \to \infty). \end{aligned}$$

This completes the proof of Theorem 3. □

Remark 4. We can calculate the coefficients b_j in (17) by using the formulas (20) and (15). We thus find that

(22)

$$b_n = \frac{1}{n!}\begin{vmatrix} 1!\,a_1 & \binom{n-1}{1}2!\,a_2 & \binom{n-1}{2}3!\,a_3 & \binom{n-1}{3}4!\,a_4 & \cdots & \cdots & n!\,a_n \\ -1 & 1!\,a_1 & \binom{n-2}{1}2!\,a_2 & \binom{n-2}{2}3!\,a_3 & \cdots & \cdots & (n-1)!\,a_{n-1} \\ 0 & -1 & 1!\,a_1 & \binom{n-3}{1}2!\,a_2 & \cdots & \cdots & (n-2)!\,a_{n-2} \\ 0 & 0 & -1 & 1!\,a_1 & \cdots & \cdots & (n-3)!\,a_{n-3} \\ 0 & 0 & 0 & -1 & \cdots & \cdots & (n-4)!\,a_{n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1!\,a_1 \end{vmatrix}.$$

The representation using a recursive algorithm for the coefficients b_j in (18) is more practical for numerical evaluation than the expression in (22).

We next state and prove the asymptotic expansion of the sequence g_n defined by (6)

Theorem 5. *The sequence g_n, defined in* (6), *has the following asymptotic expansion*:

$$g_n = \sum_{j=0}^{\infty} \frac{\beta_j}{n^j} = 2\left\{1 - \frac{\pi^2}{24n^2} + \frac{\zeta(3)}{4n^3} - \frac{\pi^4}{640n^4} + \left(\frac{3\zeta(5)}{16} - \frac{\pi^2\zeta(3)}{96}\right)\frac{1}{n^5} + \left(-\frac{79\pi^6}{967680} + \frac{\zeta(3)^2}{32}\right)\frac{1}{n^6} + \cdots\right\} \qquad (n \to \infty),$$

with the coefficients β_j given by the following recursive formula:

$$\beta_0 = 1, \quad \beta_1 = 0, \quad \beta_j = \sum_{\ell=0}^{j-1}\left(1 - \frac{\ell}{j}\right)\frac{V^{(j-\ell)}(0)}{(j-\ell)!}\,\beta_\ell \qquad (j \geqq 2),$$

where

$$V(0) = \ln 2, \quad V'(0) = 0, \quad V^{(k)}(0) = \frac{(-1)^{k-1}(2^{k-1}-1)\,(k-1)!\,\zeta(k)}{2^{k-1}} \qquad (k \geqq 2).$$

Proof. We begin by defining the functions $U(x)$ and $V(x)$ by

$$U(x) = \frac{2^{1-x}\sqrt{\pi}\,\Gamma\left(1+\frac{x}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{x}{2}\right)} \qquad \text{and} \qquad V(x) = \ln U(x).$$

We thus find that

$$V(x) = \ln\Gamma\left(1+\frac{x}{2}\right) - \ln\Gamma\left(\frac{1}{2}+\frac{x}{2}\right) + (1-x)\ln 2 + \ln(\sqrt{\pi}).$$

Elementary calculations would show that

$$V'(x) = \frac{1}{2}\left[\psi\left(1+\frac{x}{2}\right) - \psi\left(\frac{1}{2}+\frac{x}{2}\right)\right] - \ln 2,$$

$$V^{(j)}(x) = \frac{1}{2^j}\left[\psi^{(j-1)}\left(1+\frac{x}{2}\right) - \psi^{(j-1)}\left(\frac{1}{2}+\frac{x}{2}\right)\right] \qquad (j \geqq 2).$$

We then find that

$$V(0) = \ln 2, \quad V'(0) = \frac{1}{2}\left[\psi(1) - \psi\left(\frac{1}{2}\right)\right] - \ln 2 = 0,$$

$$V^{(j)}(0) = \frac{1}{2^j}\left[\psi^{(j-1)}(1) - \psi^{(j-1)}\left(\frac{1}{2}\right)\right] = \frac{(-1)^{j-1}(2^{j-1}-1)\,(j-1)!\,\zeta(j)}{2^{j-1}} \quad (j \geqq 2).$$

We are thus led to the following power series:

$$\begin{aligned} V(x) &= \ln 2 + \sum_{j=2}^{\infty} \frac{V^{(j)}(0)}{j!}\, x^j \\ &= \ln 2 - \frac{\pi^2}{24}x^2 + \frac{\zeta(3)}{4}x^3 - \frac{7\pi^4}{2880}x^4 + \frac{3\zeta(5)}{16}x^5 \\ &\quad - \frac{31\pi^6}{181440}x^6 + \frac{9\zeta(7)}{64}x^7 - \frac{127\pi^8}{9676800}x^8 + \cdots . \end{aligned} \tag{23}$$

Also, by noting that

$$\lim_{j\to\infty}\left|\frac{\frac{V^{(j+1)}(0)}{(j+1)!}\, x^{j+1}}{\frac{V^{(j)}(0)}{j!}\, x^j}\right| = \lim_{j\to\infty}\frac{2^j-1}{2^j-2}\,\frac{j}{j+1}\,\frac{\zeta(j+1)}{\zeta(j)}|x| = |x|,$$

we see that the power series (23) converges absolutely on $(-1,1)$.

We now let

$$\alpha_0 = V(0) = \ln 2, \quad \alpha_1 = V'(0) = 0, \quad \alpha_j = \frac{V^{(j)}(0)}{j!} \qquad (j \geqq 2).$$

The equation (23) can then be written as follows:

$$V(x) = \alpha_0 + \sum_{j=1}^{\infty} \alpha_j x^j.$$

Furthermore, by using (16), we find that

$$U(x) = \exp(V(x)) = 2\exp\left(\sum_{j=1}^{\infty}\frac{j!\,\alpha_j}{j!}\, x^j\right) = 2\sum_{j=0}^{\infty}\,\beta_j x^j,$$

where

$$\beta_j = \frac{B_j\Big(1!\,\alpha_1, 2!\,\alpha_2, \cdots, j!\,\alpha_j\Big)}{j!}. \tag{24}$$

By means of (21), the formula (24) can be rewritten as follows:

$$\begin{aligned}
\beta_0 &= 1 \quad \text{and} \\
\beta_j &= \frac{1}{j!}\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}(j-\ell)!\,\alpha_{j-\ell}B_\ell\Big(1!\alpha_1, 2!\alpha_2\cdots, \ell!\alpha_\ell\Big) \\
&= \frac{1}{j!}\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}(j-\ell)!\,\alpha_{j-\ell}\ell!\,\beta_\ell \\
&= \sum_{\ell=0}^{j-1}\frac{j-\ell}{j}\alpha_{j-\ell}\beta_\ell = \sum_{\ell=0}^{j-1}\left(1-\frac{\ell}{j}\right)\,\frac{V^{(j-\ell)}(0)}{(j-\ell)!}\beta_\ell \qquad (j\in\mathbb{N}).
\end{aligned}$$

Thus, finally, we obtain

$$\begin{aligned}
g_n = U\left(\frac{1}{n}\right) = 2\sum_{j=0}^{\infty}\frac{\beta_j}{n^j} = 2\Bigg\{1 - \frac{\pi^2}{24n^2} + \frac{\zeta(3)}{4n^3} - \frac{\pi^4}{640n^4} + \left(\frac{3\zeta(5)}{16} - \frac{\pi^2\zeta(3)}{96}\right)\frac{1}{n^5} \\
+\left(-\frac{79\pi^6}{967680} + \frac{\zeta(3)^2}{32}\right)\frac{1}{n^6} + \cdots\Bigg\}. \qquad (n\to\infty).
\end{aligned}$$

The proof of Theorem 5 is thus completed. □

Acknowledgements. The authors thank the referee for helpful comments. This work was supported by the Key Science Research Project in the Universities of Henan Province (20B110007) and the Fundamental Research Funds for the Universities of the Henan Province (NSFRF210446).

REFERENCES

1. M. Abramowitz and I. A. Stegun (Editors): *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Applied Mathematics Series **55**, Ninth printing, National Bureau of Standards, Washington, D.C., 1972.
2. R. Ayoub: *The lemniscate and Fagnano's contributions to elliptic integrals*. Arch. History Exact Sci. **29** (1984), 131–149.
3. B. C. Berndt, S. Bhargava: *Ramanujan's inversion formulas for the lemniscate and allied functions*. J. Math. Anal. Appl. **160** (1991), 504–524
4. J. M. Borwein, P. B. Borwein: *Pi and the AGM: A Study in the Analytic Number Theory and Computational Complexity*. John Wiley and Sons, New York, 1987.

5. S. R. Bulò, E. R. Hancock, F. Aziz, M. Pelillo: *Efficient computation of Ihara coefficients using the Bell polynomial recursion*. Linear Algebra Appl. **436** (2012), 1436–1441.

6. B. C. Carlson: *Algorithms involving arithmetic and geometric means*. Amer. Math. Monthly, **78** (1971), 496–505.

7. C. A. Charalambides: *Enumerative Combinatorics*. CRC Press Series on Discrete Mathematics and Its Applications, Chapman & Hall (CRC Press), Boca Raton, Florida, 2002.

8. L. Comtet: *Advanced Combinatorics*. D. Reidel Publishing Company, Dordrecht, 1974.

9. D. Cvijović: *New identities for the partial Bell polynomials*. Appl. Math. Lett. **24** (2011), 1544–1547.

10. A. B. Farnell: *Problem E4848 (Solution)*. Amer. Math. Monthly, **67** (1960), 300–300.

11. M. S. Klamkin: *Problem E4848*. Amer. Math. Monthly, **66** (1959), 427–427.

12. M. Masjed-Jamei, Z. Moalemi, W. Koepf, H. M. Srivastava: *An extension of the Taylor series expansion by using the Bell polynomials*. Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) **113** (2019), 1445–1461.

13. E. Neuman: *On Gauss lemniscate functions and lemniscatic mean*. Math. Pannon. **18** (2007), 77–94.

14. E. Neuman: *Two-sided inequalities for the lemniscate functions*. J. Inequal. Spec. Funct. **1** (2010), 1–7.

15. E. Neuman: *On lemniscate functions*. Integral Transforms Spec. Funct. **24** (2013), 164–171.

16. C. L. Siegel: *Topics in Complex Function Theory*. Vol. 1, John Wiley and Sons, New York, 1969.

17. S. Ramanujan: *Notebooks (2 volumes)*. Tata Institute of Fundamental Research, Bombay, 1957.

18. J. Riordan: *Combinatorial Identities (Reprint of the 1968 original)*. Robert E. Krieger Publishing Company, Huntington, New York, 1979.

19. H. M. Srivastava: *The zeta and related functions: Recent developments*. J. Adv. Engrg. Comput. **3** (2019), 329–354.

20. H. M. Srivastava, Q. Z. Ahmad, M. Darus, N. Khan, B. Khan, N. Zaman, H. H. Shah: *Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli*. Mathematics **7** (2019), Article ID 848, 1–10.

21. H. M. Srivastava:, *Zeta and q-Zeta Functions and Associated Series and Integrals*. Elsevier Science Publishers, Amsterdam, London and New York, 2012.

(Received 10. 08. 2022.)
(Revised 15. 04. 2023.)

Xue-Feng Han
School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City 454000, Henan Province,
People's Republic of China.
E-Mail: *hanxuefeng8110@sohu.com*

Chao-Ping Chen
School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City 454000, Henan Province,
People's Republic of China.
E-Mail: *chenchaoping@sohu.com*

H. M. Srivastava
Department of Mathematics and Statistics,
University of Victoria,
Victoria, British Columbia V8W 3R4,
Canada.

Department of Medical Research,
China Medical University Hospital,
China Medical University, Taichung 40402, Taiwan,
Republic of China.

Department of Mathematics and Informatics,
Azerbaijan University,
71 Jeyhun Hajibeyli Street, AZ1007 Baku,
Azerbaijan.

Section of Mathematics, International Telematic
University Uninettuno, I-00186 Rome,
Italy.
E-Mail: *harimsri@math.uvic.ca*