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NEW FAMILY OF JACOBI-STIRLING NUMBERS

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The Jacobi-Stirling numbers of the first and second kind were introduced in 2007 by Everitt et al. In this article we find new explicit formulas for Jacobi Stirling numbers. Furthermore, we derive and study new class of the Jacobi Stirling numbers so-called generalized Jacobi-Stirling numbers. Some special cases such as Legendre-Stirling numbers are given. Some interesting combinatorial identities are obtained.

1. INTRODUCTION

The Jacobi Stirling numbers were discovered by Everitt et al. [9]. The Jacobi-Stirling numbers of the second kind $P^{\alpha,\beta}S_n^j$ (for more details, see [1], [5], [12] and [16, 18]), are defined by

$$x^n = \sum_{j=0}^{n} P^{(\alpha,\beta)}S_n^j \langle x\rangle_j^{(\alpha,\beta)} \quad (n \in N_0)$$

where $\langle x\rangle_j^{(\alpha,\beta)}$ is generalized falling factorial defined for $x \in C$, by

$$\langle x\rangle_j^{(\alpha,\beta)} = \begin{cases} 1 & \text{if } j = 0 \\ \prod_{r=0}^{j-1}\big(x - r(r+z)\big) & \text{if } j \in N \end{cases},$$

and $z = \alpha + \beta + 1$.

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The Jacobi-Stirling numbers of the first kind $P^{(\alpha,\beta)}s_n^j$ are defined by, see ([5], [9] and [14])

$$\langle x\rangle_n^{(\alpha,\beta)} = \sum_{j=0}^{n} P^{(\alpha,\beta)}s_n^j x^j.$$

The numbers $P^{(\alpha,\beta)}S_n^j$ have the explicit expression

$$P^{(\alpha,\beta)}S_n^j = \sum_{r=0}^{j} (-1)^{r+j} \frac{\Gamma(z+r)\Gamma(z+2r+1)(r(r+z))^n}{r!(j-r)!\Gamma(z+2r)\Gamma(z+j+r+1)}, \tag{1}$$

for each $n \in N$ and $j \in \{1, 2, \cdots, n\}$, and the generating function

$$f_j^{(\alpha,\beta)}(t) = \sum_{n=0}^{\infty} P^{(\alpha,\beta)}S_n^j t^{n-j} = \prod_{r=1}^{j} \frac{1}{1-r(r+z)t}, \left(|t| < \frac{1}{j(j+z)}\right).$$

The Jacobi-Stirling numbers of the first and second kind satisfy, respectively the recurrence relations

$$P^{(\alpha,\beta)}s_n^j = Ps_{n-1}^{j-1} - (n-1)(n-1+z)P^{(\alpha,\beta)}s_{n-1}^j, \tag{2}$$

$$P^{(\alpha,\beta)}S_n^j = P^{(\alpha,\beta)}S_{n-1}^{j-1} + j(j+z)P^{(\alpha,\beta)}S_{n-1}^j \quad (n, j \in N), \tag{3}$$

where $P^{(\alpha,\beta)}s_n^0 = P^{(\alpha,\beta)}S_n^0 = \delta_{n,0}$ and $P^{(\alpha,\beta)}s_n^j = P^{(\alpha,\beta)}S_n^j = 0$ for $j > n$, $(n, j \in N)$, where $\delta_{n,j}$ is the Kronecker delta.

Note that when $z = \alpha + \beta + 1 = 1$, the Jacobi-Stirling numbers are reduced to the Legendre-Stirling numbers of the first and second kind, see ([1], [8], [10], [13] and [15]). In [1] and [11] we can find the combinatorial interpretation of Legendre-Stirling and Jacobi-Stirling numbers, respectively. We refer to recent studies in [1–7] and [17] for a better details and understanding.
In this article, Section 2, we give new explicit expressions for Jacobi-Stirling numbers of both kinds. Also, in Section 3, we introduce a generalization of these numbers. Moreover, some combinatorial identities are derived. Finally, some interesting special cases are obtained.

2. NEW EXPLICIT FORMULAS FOR JACOBI-STIRLING NUMBERS

In this section we investigate new explicit expressions for the Jacobi-Stirling numbers of both kinds.

Theorem 1. *The Jacobi-Stirling numbers of the second kind have the following explicit formula*

$$P^{(\alpha,\beta)}S_n^k = \sum_{\sigma_{n-1}=n-k, i_j \epsilon\{0,1\}} \binom{1}{i_1}\binom{1+z}{i_1}\binom{2-i_1}{i_2}\binom{2+z-i_1}{i_2}\cdots \binom{n-1-i_1-\cdots-i_{n-2}}{i_{n-1}}\binom{n-1+z-i_1-\cdots-i_{n-2}}{i_{n-1}}, \sigma_n = \sum_{l=1}^{n} i_l. \tag{4}$$

Proof. For $n = k$, we get $P^{(\alpha,\beta)}S_n^n = 1$. Now, if we take $i_{n-1}\epsilon\{0,1\}$, we have

$$P^{(\alpha,\beta)}S_n^k = \sum_{\sigma_{n-2}=(n-1)-(k-1), i_j\epsilon\{0,1\}} \binom{1}{i_1}\binom{1+z}{i_1}\binom{2-i_1}{i_2}\binom{2+z-i_1}{i_2}\cdots \binom{n-2-i_1-\cdots-i_{n-3}}{i_{n-2}}\binom{n-2+z-i_1-\cdots-i_{n-3}}{i_{n-2}}$$
$$+ \sum_{\sigma_{n-2}=(n-1)-k, i_j\epsilon\{0,1\}} (n-1-(n-1-k))(n-1+z-(n-1-k))\binom{1}{i_1}\binom{1+z}{i_1} \binom{2-i_1}{i_2}\binom{2+z-i_1}{i_2}\cdots\binom{n-2-i_1-\cdots-i_{n-3}}{i_{n-2}} \binom{n-2+z-i_1-\cdots-i_{n-3}}{i_{n-2}} = P^{(\alpha,\beta)}S_{n-1}^{k-1} + k(k+z)P^{(\alpha,\beta)}S_{n-1}^k,$$

which is the same recurrence relation (3) of $P^{(\alpha,\beta)}S_n^k$. This completes the proof. Similarly, we can prove the following: □

Theorem 2. *Jacobi-Stirling numbers of the second kind have the following explicit expression*

$$P^{(\alpha,\beta)}S_n^k = \sum_{\sigma_{n-1}=n-k, i_j\epsilon\{0,1\}} \binom{1+z}{i_1}\binom{(1-i_1)(1+z-i_1)}{i_2} \binom{(2-i_1)(2+z-i_1-i_2)}{i_3}\cdots\binom{(n-1-i_1-\cdots-i_{n-2})(n-1+z-i_1-\cdots-i_{n-2})}{i_{n-1}}. \tag{5}$$

Remark 3. *From* (1) *and* (4)*, we have the combinatorial identity*

$$\sum_{r=0}^{k}(-1)^{r+k}\frac{\Gamma(z+r)\Gamma(z+2r+1)(r(r+z))^n}{r!(k-r)!\Gamma(z+2r)\Gamma(z+k+r+1)} = \sum_{\sigma_{n-1}=n-k, i_j\epsilon\{0,1\}} \binom{1}{i_1}\binom{1+z}{i_1}\binom{2-i_1}{i_2}\binom{2+z-i_1}{i_2}\cdots \binom{n-1-i_1-\cdots-i_{n-2}}{i_{n-1}}\binom{n-1+z-i_1-\cdots-i_{n-2}}{i_{n-1}}.$$

Theorem 4. *The Jacobi-Stirling numbers of the second kind $P^{(\alpha,\beta)}S_n^k$ have the operational formula*

$$\prod_{j=0}^{n} x^{-j(j+z)}Dx^{j(j+z)} = \sum_{j=0}^{n} P^{(\alpha,\beta)}S_n^j x^j D^j, \text{ where } D := d/dx. \tag{6}$$

Theorem 5. *The Jacobi-Stirling numbers of the first kind $P^{(\alpha,\beta)}s_n^k$ have the explicit* *expression*

$$P^{(\alpha,\beta)}s_n^k = \sum_{\sigma_n=k, i_j\epsilon\{0,1\}} \binom{i_1}{1-i_1}\binom{i_2-1-z}{1-i_2}\binom{i_3-2(2+z)}{1-i_3}\cdots \binom{i_n-(n-1)(n-1+z)}{1-i_n}.$$

Proof. For $k=0$ we have $P^{(\alpha,\beta)}s_n^0 = 0$ and for $n=k$ we get $P^{(\alpha,\beta)}s_n^n = 1$. Now, if we take $i_n\epsilon\{0,1\}$, we get

$$\begin{aligned} P^{(\alpha,\beta)}s_n^k = &\sum_{\sigma_{n-1}=k, i_j\epsilon\{0,1\}} \binom{i_1}{1-i_1}\binom{i_2-1-z}{1-i_2}\binom{i_3-2(2+z)}{1-i_3}\cdots \\ &\binom{i_{n-1}-(n-2)(n-2+z)}{1-i_{n-2}}(-(n-1)(n-1)) + \sum_{\sigma_{n-1}=k-1, i_j\epsilon\{0,1\}} \binom{i_1}{1-i_1} \\ &\binom{i_2-1-z}{1-i_2}\binom{i_3-2(2+z)}{1-i_3}\cdots\binom{i_{n-1}-(n-2)(n-2+z)}{1-i_{n-2}}(1) \\ &= P^{(\alpha,\beta)}s_{n-1}^{k-1} - (n-1)(n-1+z)P^{(\alpha,\beta)}s_{n-1}^k. \end{aligned}$$

That is the same recurrence relation (2) of $P^{(\alpha,\beta)}s_n^k$. This completes the proof. □

Corollary 6. *The Legendre-Stirling numbers of the second kind PS_n^k have the following new explicit formulas*

$$PS_n^k = \sum_{\sigma_{n-1}=n-k, i_j\epsilon\{0,1\}} \binom{1}{i_1}\binom{2}{i_1}\binom{2-i_1}{i_2}\binom{3-i_1}{i_2}\cdots \binom{n-1-i_1-\cdots-i_{n-2}}{i_{n-1}}\binom{n-i_1-\cdots-i_{n-2}}{i_{n-1}}, \sigma_n = \sum_{l=1}^{n} i_l.$$

$$PS_n^k = \sum_{\sigma_{n-1}=n-k, i_j\epsilon\{0,1\}} \binom{2}{i_1}\binom{(1-i_1)(2-i_1)}{i_2}\binom{(2-i_1)(3-i_1-i_2)}{i_3}\cdots \binom{(n-1-i_1-\cdots-i_{n-2})(n-i_1-\cdots-i_{n-2})}{i_{n-1}}.$$

Proof. The proof follows directly by setting $z=1$ in (4) and (5) respectively. □

Corollary 7. *The Legendre-Stirling numbers of the first kind Ps_n^k have the following new explicit formula*

$$Ps_n^k = \sum_{\sigma_n=k, i_j\epsilon\{0,1\}} \binom{i_1}{1-i_1}\binom{i_2-2}{1-i_2}\binom{i_3-2.3}{1-i_3}\cdots\binom{i_n-(n-1)n}{1-i_n}.$$

Proof. The proof follows directly by setting $z = 1$ in (6) . □

3. GENERALIZED JACOBI-STIRLING NUMBERS

In this section, we derive new family of generalized Jacobi-Stirling numbers to be written as:

Definition 8. *The Generalized Jacobi-Stirling numbers of the second kind denoted by $S_{A,B}(n,k)$, satisfy the recurrence relation*

$$S_{A,B}(n,k) = S_{A,B}(n-1,k-1) + (k+A)(k+B)S_{A,B}(n-1,k), \tag{7}$$

where A and B belongs to R with initial conditions $S_{A,B}(n,0) = \delta_{n,0}$ and $S_{A,B}(n,k) = 0,\ k > n$.

Theorem 9. *The numbers $S_{A,B}(n,k)$ have the explicit formula*

$$S_{A,B}(n,k) = \sum_{\sigma_{n-1}=n-k, i_j \epsilon\{0,1\}} \binom{1+A}{i_1}\binom{1+B}{i_1}\binom{2+A-i_1}{i_2}\binom{2+B-i_1}{i_2}\cdots \binom{n-1+A-i_1-\cdots-i_{n-2}}{i_{n-1}}\binom{n-1+B-i_1-\cdots-i_{n-2}}{i_{n-1}}. \tag{8}$$

Proof. For $n = k$ we get $S_{A,B}(n,n) = 1$. For $i_{n-1}\ \epsilon\{0,1\}$ we get from (8) that

$$S_{A,B}(n,k) = \sum_{\sigma_{n-2}=n-k, i_j \epsilon\{0,1\}} \binom{1+A}{i_1}\binom{1+B}{i_1}\binom{2+A-i_1}{i_2}\binom{2+B-i_1}{i_2}\cdots \binom{n-2+A-i_1-\cdots-i_{n-3}}{i_{n-2}}\binom{n-2+B-i_1-\cdots-i_{n-3}}{i_{n-2}} +$$
$$(n-1+A-n+1+k)(n-1+B-n+1+k). \sum_{\sigma_{n-2}=n-k-1, i_j \epsilon\{0,1\}} \binom{1+A}{i_1}\binom{1+B}{i_1}\binom{2+A-i_1}{i_2}. \binom{2+B-i_1}{i_2}\cdots\binom{n-2+A-i_1-\cdots-i_{n-3}}{i_{n-2}}\binom{n-2+B-i_1-\cdots-i_{n-3}}{i_{n-2}}$$

i.e. relation (7). This completes the proof.

□

Moreover, we define the new family of the generalized Jacobi-Stirling numbers as follows:

Definition 10.

$$\langle x\rangle_j^{\bar{A}} = \begin{cases} 1 & if\, j = 0 \\ \prod_{r=0}^{j-1}\left(x - \prod_{i=0}^{p}(r + A_i)\right) & if\, j \in N \end{cases}$$

where, $A_i, i = 1, 2, ..., p$ are real numbers.

Definition 11. *Let $P^{\bar{A}}S_n^j$, the extended Generalized Jacobi-Stirling numbers of the second kind associated with $\bar{A} = (A_1, A_2, ..., A_p)$, may be called p-multiparameter Jacobi-Stirling numbers of the second kind, be defined by*

$$x^n = \sum_{j=0}^{n} P^{\bar{A}}S_n^j \langle x\rangle_j^{\bar{A}} \quad (n \in N_0)$$

Theorem 12. *The numbers $P^{\bar{A}}S_n^j$ satisfy the recurrence relation*

$$P^{\bar{A}}S_n^j = P^{\bar{A}}S_{n-1}^{j-1} + \prod_{i=1}^{p}(j + A_i)P^{\bar{A}}S_{n-1}^j. \tag{9}$$

Proof. Since $\sum_{j=0}^{n} P^{\bar{A}}S_n^j \langle x\rangle_j^{\bar{A}} = x^n = x^{n-1}\Bigg(x - \prod_{i=1}^{p}(j + A_i) + \prod_{i=1}^{p}(j + A_i)\Bigg)$

$= \sum_{j=0}^{n-1} P^{\bar{A}}S_n^j \big(x - \prod_{i=1}^{p}(j + A_i)\big)\langle x\rangle_j^{\bar{A}} + \sum_{j=0}^{n-1} P^{\bar{A}}S_n^j \big(\prod_{i=1}^{p}(j + A_i)\big)\langle x\rangle_j^{\bar{A}}$

$= \sum_{j=0}^{n-1} P^{\bar{A}}S_n^j \langle x\rangle_{j+1}^{\bar{A}} + \sum_{j=0}^{n-1} P^{\bar{A}}S_n^j \big(\prod_{i=1}^{p}(j + A_i)\big)\langle x\rangle_j^{\bar{A}}$.

Equating the coefficients of $\langle x\rangle_j^{\bar{A}}$ yields (9). □

Similar to **Theorem 12**, we can prove the following theorem.

Theorem 13. *The p-multiparameter Jacobi-Stirling numbers of the second kind, $P^{\bar{A}}S_n^k$ have the explicit formula*

$$P^{\bar{A}}S_n^k = \sum_{\sigma_{n-1}=n-k, i_j \epsilon \{0,1\}} \prod_{j=1}^{p} \binom{1 + A_j}{i_1}\binom{2 + A_j - i_1}{i_2}\cdots \binom{n - 1 + A_j - i_1 - \cdots - i_{n-2}}{i_{n-1}}.$$

Theorem 14. *The numbers $S_{A,B}(n, k)$ are the coefficients of t^{n-j} in the Taylor expansion of*

$$f_j(A, B; t) = \prod_{r=1}^{j} \frac{1}{1 - (r + A)(r + B)t} \qquad \left(|t| < \frac{1}{(j + A)(j + B)}\right).$$

Proof. We have

$$\frac{t^j}{\big(1 - (1 + A)(1 + B)t\big)\big(1 - (2 + A)(2 + B)t\big)\cdots\big(1 - (j + A)(j + B)t\big)}$$
$$= \sum_{m=1}^{j} \frac{A_m}{1 - (m + A)(m + B)t} \qquad \left(|t| < \frac{1}{(j + A)(j + B)}\right),$$

and this implies

$$t^j = A_1\big(1-(2+A)(2+B)t\big)\big(1-(3+A)(3+B)t\big)\cdots\big(1-(j+A)(j+B)t\big)$$
$$+ A_2\big(1-(1+A)(1+B)t\big)\big(1-(3+A)(3+B)t\big)\cdots\big(1-(j+A)(j+B)t\big)$$
$$+\cdots+ A_j\big(1-(1+A)(1+B)t\big)\big(1-(2+A)(2+B)t\big)\cdots$$
$$\big(1-(j-1+A)(j-1+B)t\big).$$

If we let $t=\frac{1}{(1+A)(1+B)}$ we find

$$A_1 = \frac{(-1)^{j+1}}{(j-1)!}\cdot\frac{1}{(A+1)(B+1)}\cdot\frac{\Gamma(A+B+3)}{\Gamma(A+B+j+2)}\ .$$

Similarly, letting $t=\frac{1}{(2+A)(2+B)}$, we have

$$A_2 = \frac{(-1)^{j+2}}{(j-2)!}\cdot\frac{1}{(A+2)(B+2)}\cdot\frac{\Gamma(A+B+3)\Gamma(A+B+5)}{\Gamma(A+B+4)\Gamma(A+B+j+3)}\ .$$

For

$$A_3 = \frac{(-1)^{j+3}}{(j-3)!}\cdot\frac{1}{(A+3)(B+3)}\cdot\frac{\Gamma(A+B+4)\Gamma(A+B+7)}{\Gamma(A+B+6)\Gamma(A+B+j+4)}$$

, and in general

$$A_m = \frac{(-1)^{j+m}}{(j-m)!}\cdot\frac{1}{(A+m)(B+m)}\cdot\frac{\Gamma(A+B+m+1)\Gamma(A+B+2m+1)}{\Gamma(A+B+2m)\Gamma(A+B+j+m+1)}\ .$$

So, we have

$$t^j\cdot\prod_{s=1}^{j}\frac{1}{1-(s+A)(s+B)t} = \prod_{s=1}^{j}\frac{t}{1-(s+A)(s+B)t} = \sum_{s=1}^{j}\frac{A_s}{1-(s+A)(s+B)t}$$
$$= \sum_{s=1}^{j}\sum_{n=0}^{+\infty}A_s\cdot\big((s+A)(s+B)\big)^n t^n \quad \left(|t|<\frac{1}{(j+A)(j+B)}\right)$$
$$= \sum_{n=0}^{+\infty}\left(\sum_{s=1}^{j}A_s\cdot\big((s+A)(s+B)\big)^n\right)t^n,$$

and

$$\prod_{s=1}^{j}\frac{1}{1-(s+A)(s+B)t} = \sum_{n=0}^{+\infty}\left(\sum_{s=1}^{j}A_j\cdot\big((s+A)(s+B)\big)^n\right)t^{n-j}$$
$$\sum_{n=0}^{+\infty}\sum_{s=1}^{j}\frac{(-1)^{j+s}}{(j-s)!}\cdot\frac{1}{(A+s)(B+s)}\cdot\frac{\Gamma(A+B+s+1)\Gamma(A+B+2s+1)}{\Gamma(A+B+2s)\Gamma(A+B+j+s+1)}$$
$$(s+A)(s+B)t^{n-j} = \sum_{n=0}^{+\infty}S_{A,B}(n,j)\cdot t^{n-j},$$

with the explicit formula

$$S_{A,B}(n,k) = \sum_{s=1}^{k} \frac{(-1)^{s+k}}{(k-s)!} \cdot \frac{1}{(A+s)(B+s)}$$
$$\frac{\Gamma(A+B+s+1)\Gamma(A+B+2s+1)}{\Gamma(A+B+2s)\Gamma(A+B+k+s+1)} \big((s+A)(s+B)\big)^n.$$

This competes the proof. □

Remark 15. *For $B=0$, $A=\alpha+\beta+1$ we get the Jacobi-Stirling numbers. For $B=0$ we get the numbers $S_A(k,0)$ studied in [14].*
Similarly, for numbers $P^{\bar{A}}S_n^j$ we have

$$\prod_{s=1}^{j} \frac{1}{1-\prod_{i=1}^{p}(s+A_j)t} = \sum_{n=0}^{+\infty} \Big(\sum_{s=1}^{j} A_j \big(\prod_{i=1}^{p}(s+a_i)\big)^n \Big) t^{n-j},$$

and its coefficients can be obtained in a similar way.

Definition 16. *Let $P^{\bar{A}}s_n^j$, the Generalized Jacobi-Stirling numbers of the first kind associated with $\bar{A}=(A_1,A_2,...,A_p)$, may be called p-multiparameter Jacobi-Stirling numbers of the first kind, be defined by*

$$\langle x\rangle_n^{\bar{A}} = \sum_{j=0}^{n} P^{\bar{A}}s_n^j x^j \quad (n \in N_0). \tag{10}$$

Theorem 17. *The numbers $P^{\bar{A}}s_n^j$ satisfy the recurrence relation*

$$P^{\bar{A}}s_n^j = P^{\bar{A}}s_{n-1}^{j-1} - \prod_{i=1}^{p}(n-1+A_i)P^{\bar{A}}s_{n-1}^j. \tag{11}$$

Proof. Since $\langle x\rangle_n^{\bar{A}} = \langle x\rangle_{n-1}^{\bar{A}}\big(x-\prod_{i=1}^{p}(n-1+A_i)\big)$, hence using (10), we have

$$\sum_{j=0}^{n} P^{\bar{A}}s_n^j x^j = \sum_{j=0}^{n-1} P^{\bar{A}}s_n^j x^j \left(x - \prod_{i=1}^{p}(n-1+A_i) \right)$$
$$= \sum_{j=0}^{n-1} P^{\bar{A}}s_{n-1}^j \left(x^{j+1} - \prod_{i=1}^{p}(n-1+A_i) \right) \sum_{j=0}^{n-1} P^{\bar{A}}s_{n-1}^j x^j.$$

Equating the coefficient of x^j on both sides yields (11). □

Also, we formulate the following theorem:

Theorem 18. *The p-multiparameter Jacobi-Stirling numbers of the first kind, $P^{\bar{A}}s_n^k$ have the explicit formula*

$$P^{\bar{A}}s_n^k = \sum_{\sigma_n=k, i_j\epsilon\{0,1\}} \binom{i_1 - \prod_{j=1}^{p} A_j}{1-i_1} \binom{i_2 - \prod_{j=1}^{p}(1+A_j)}{1-i_2} \cdots \binom{i_n - \prod_{j=1}^{p}(n-1+A_j)}{1-i_n}.$$

Proof. For $n = k$ we get $P^{\bar{A}}s_n^n = 1$. Now, if we take $i_n \;\epsilon\{0,1\}$, we get

$$P^{\bar{A}}s_n^k = \sum_{\sigma_{n-1}=k, i_j\epsilon\{0,1\}} \binom{i_1 - \prod_{j=1}^{p} A_j}{1-i_1} \binom{i_2 - \prod_{j=1}^{p}(1+A_j)}{1-i_2} \binom{i_{n-1} - \prod_{j=1}^{p}(n-2+A_j)}{1-i_{n-1}} \cdots$$

$$\cdot \left(-\prod_{j=1}^{p}(n-1+A_j)\right) + \sum_{\sigma_{n-1}=k-1, i_j\epsilon\{0,1\}} \binom{i_1 - \prod_{j=1}^{p} A_j}{1-i_1} \binom{i_2 - \prod_{j=1}^{p}(1+A_j)}{1-i_2} \cdots$$

$$\binom{i_{n-1} - \prod_{j=1}^{p}(n-2+A_j)}{1-i_{n-1}} = P^{\bar{A}}s_{n-1}^{k-1} - \prod_{j=1}^{p}(n-1+A_j) P^{\bar{A}}s_{n-1}^{k},$$

i.e. relation (11). This completes the proof.

□

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