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FAMILIES OF LOG LEGENDRE CHI FUNCTION INTEGRALS

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In this paper we investigate the representation of integrals involving the product of the Legendre Chi function, polylogarithm function and log function. We will show that in many cases these integrals take an explicit form involving the Riemann zeta function, the Dirichlet Eta function, Dirichlet lambda function and many other special functions. Some examples illustrating the theorems will be detailed.

1. INTRODUCTION PRELIMINARIES AND NOTATION

In this paper we investigate the representations of integrals of the type

$$\int_0^1 x^a \chi_p(x) \operatorname{Li}_q(\delta x^b) \ln^m(x)\ dx, \tag{1}$$

in terms of special functions such as zeta functions, Dirichlet eta functions, polylogarithmic functions, beta functions and others. We note that $\chi_p(x)$ is the Legendre-Chi function (LCF), $\operatorname{Li}_q(\delta x^b)$ is the polylogarithmic function, $a \in \mathbb{R}$, $a \geq -2$, $b \in \mathbb{R}^+$, $p \in \mathbb{N}$, $q \in \mathbb{N}$, $\delta \in [-1,1] \setminus \{0\}$ and $m \in \mathbb{N}$ for the set of complex numbers $\mathbb{C}$, natural numbers $\mathbb{N}$, the set of real numbers $\mathbb{R}$ and the set of positive real numbers, $\mathbb{R}^+$. The following notation and results will be useful in the subsequent sections of this paper. A generalized binomial coefficient $\binom{\lambda}{\mu}$ $(\lambda, \mu \in \mathbb{C})$ is defined, in terms of the familiar (Euler's) gamma function, by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} \qquad (\lambda, \mu \in \mathbb{C}),$$

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which, in the special case when $\mu = n \ \ (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$, yields

$$\binom{\lambda}{0} := 1 \quad \text{and} \quad \binom{\lambda}{n} := \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \quad (n \in \mathbb{N}),$$

where $(\lambda)_\nu \ \ (\lambda, \nu \in \mathbb{C})$ is the Pochhammer symbol defined, also in terms of the Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists. The generalized $p-$order harmonic numbers, $H_n^{(p)}(\alpha, \beta)$ are defined as the partial sums of the modified Hurwitz zeta function

$$\zeta(p, \alpha, \beta) = \sum_{n \geq 0} \frac{1}{(\alpha n + \beta)^p}.$$

The classical Hurwitz zeta function

$$\zeta(p, a) = \sum_{n \geq 0} \frac{1}{(n+a)^p}$$

for $\mathrm{Re}(p) > 1$ and by analytic continuation to other values of $p \neq 1$, where any term of the form $(n+a) = 0$ is excluded. Therefore

$$H_n^{(p)}(\alpha, \beta) = \sum_{j=1}^{n} \frac{1}{(\alpha j + \beta)^p}$$

and the "ordinary" $p-$order harmonic numbers $H_n^{(p)} = H_n^{(p)}(1, 0)$. Many functions can be expanded through the generalized $p-$order harmonic numbers, such as the Dirichlet Beta cases

$$\beta(1) = \sum_{n \geq 0} \frac{n!}{2^{n+1}\left(\frac{3}{2}\right)_n} = \sum_{n \geq 1} \frac{(n-1)!}{2^{n+1}\left(\frac{1}{2}\right)_n}$$

and

$$\beta(2) = \sum_{n \geq 0} \frac{n!\,(1 + H_n(2, 1))}{2^{n+1}\left(\frac{3}{2}\right)_n} = \sum_{n \geq 1} \frac{(n-1)!\,h_n}{2^{n+1}\left(\frac{1}{2}\right)_n},$$

here $\beta(2)$ is Catalan's constant and $h_n = H_{2n} - \frac{1}{2}H_n$. Two special cases of the Legendre-Chi function are

$$\chi_1(x) = \sum_{n \geq 0} \frac{(-1)^n\, n!\, x^{2n+1}}{(1-x^2)^{n+1}\left(\frac{3}{2}\right)_n}$$

and

$$\chi_2(x)=\sum_{n\geq 0}\frac{n!}{\left(\frac{3}{2}\right)_n}\left(1+\frac{(-1)^n x^{2n+1}H_n(2,1)}{(1-x^2)^{n+1}}\right).$$

The Catalan constant

$$G=\beta(2)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\approx 0.91597$$

is a special case of the Dirichlet beta function

$$\begin{aligned}\beta(z) &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^z},\ \text{for } \operatorname{Re}(z)>0 \\ &= \frac{1}{(-2)^{2z}(z-1)!}\left(\psi^{(z-1)}\left(\frac{1}{4}\right)-\psi^{(z-1)}\left(\frac{3}{4}\right)\right),\end{aligned} \tag{2}$$

with functional equation

$$\beta(1-z)=\left(\frac{2}{\pi}\right)^z\sin\left(\frac{\pi z}{2}\right)\Gamma(z)\,\beta(z)$$

extending the Dirichlet Beta function to the left hand side of the complex plane $\operatorname{Re}(z)\leq 0$. The Lerch transcendent,

$$\Phi(z,t,a)=\sum_{m=0}^{\infty}\frac{z^m}{(m+a)^t}$$

is defined for $|z|<1$ and $\operatorname{Re}(a)>0$ and satisfies the recurrence:

$$\Phi(z,t,a)=z\ \Phi(z,t,a+1)+a^{-t}.$$

It is known that the Lerch transcendent extends by analytic continuation to a function $\Phi(z,t,a)$ which is defined for all complex t, $z\in\mathbb{C}-[1,\infty)$ and $a>0$, which can be represented, [**15**], by the integral formula

$$\Phi(z,t,a)=\frac{1}{\Gamma(t)}\int_0^{\infty}\frac{x^{t-1}e^{-(t-1)x}}{e^x-z}dx=\frac{1}{\Gamma(t)}\int_0^1\frac{x^{a-1}\ln\left(\frac{1}{x}\right)}{1-xz}dx$$

for $\operatorname{Re}(t)>0$. The Lerch transcendent generalizes the Hurwitz zeta function at $z=1$,

$$\Phi(1,t,a)=\zeta(t,a)=\sum_{m=0}^{\infty}\frac{1}{(m+a)^t}$$

and the polylogarithm, or de-Jonquière's function, when $a=1$,

$$\operatorname{Li}_t(z):=\sum_{m=1}^{\infty}\frac{z^m}{m^t},\ t\in\mathbb{C}\text{ when }|z|<1;\ \operatorname{Re}(t)>1\text{ when }|z|=1.$$

The polylogarithm of negative integer order arises in the sums of the form

$$\sum_{j\geq 1} j^n z^j = \mathrm{Li}_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{i=0}^{n-1} \left\langle \begin{array}{c} n \\ i \end{array} \right\rangle z^{n-i}$$

where the Eulerian number $\left\langle \begin{array}{c} n \\ i \end{array} \right\rangle = \sum_{j=0}^{i+1} (-1)^j \left(\begin{array}{c} n+1 \\ j \end{array} \right) (i-j+1)^n$. The Legendre-Chi function is a special case of the Lerch transcendent

$$\chi_p(x) = 2^{-p}\, x\ \Phi\left(x^2, p, \frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)^p}$$

and is related to the polylogarithm by

$$\chi_p(x) = \frac{1}{2}\left(\mathrm{Li}_p(x) - \mathrm{Li}_p(-x)\right) = \mathrm{Li}_p(x) - 2^{-p}\mathrm{Li}_p(x^2).$$

There are many special values of the LCF, from Lewin [**19**]

$$\chi_2(x) = \frac{1}{2}\int_0^x \ln\left(\frac{1+t}{1-t}\right)\frac{dt}{t}$$

and

$$\chi_2(x)\left(\frac{1-x}{1+x}\right) + \chi_2(x) = \frac{3}{4}\zeta(2) + \frac{1}{2}\ln x \ln\left(\frac{1+x}{1-x}\right),$$

hence,

$$\chi_2\left(\sqrt{5}-2\right) = \frac{1}{4}\zeta(2) - \frac{3}{4}\ln^2(\phi)$$

where the golden ratio $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$. The Dirichlet lambda function, $\lambda(p)$ is

$$2\lambda(p) = 2\chi_p(1) = \zeta(p) + \eta(p) \tag{3}$$

where

$$\eta(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = \left(1-2^{1-p}\right)\zeta(p)$$

is the alternating zeta function and $\eta(1) = \ln 2$. In the case of the summation of harmonic numbers, we know that the famous Euler identity states, for $m \in \mathbb{N} \geq 2$,

$$EU(m) = \sum_{n=1}^{\infty} \frac{H_n}{n^m} = \frac{1}{2}(m+2)\zeta(m+1) - \frac{1}{2}\sum_{j=1}^{m-2} \zeta(m-j)\,\zeta(j+1), \tag{4}$$

for odd powers of the denominator, Georghiou and Philippou [**14**] established the identity:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^{2m+1}} = \frac{1}{2}\sum_{r=2}^{2m} (-1)^r\, \zeta(r)\, \zeta(2m+2-r), \quad m \geq 1.$$

We know that for $n \geq 1$, $\psi(n+1) - \psi(1) = H_n$ with $\psi(1) = -\gamma$, where γ is the Euler Mascheroni constant and $\psi(n)$ is the digamma function. For real values of x, $\psi(x)$ is the digamma (or psi) function defined by

$$\psi(x) := \frac{d}{dx}\{\log \Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)} \ ,$$

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n}\right) = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{x+n-1}\right),$$

leading to the telescoping sum:

$$\psi(1+x) - \psi(x) = \sum_{n=1}^{\infty}\left(\frac{1}{x+n-1} - \frac{1}{x+n}\right) = \frac{1}{x}.$$

The polygamma function

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1}\, k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}}$$

and has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k\, k!}{z^{k+1}}.$$

The connection of the polygamma function with harmonic numbers is,

$$\begin{aligned} (5) \qquad H_z^{(m+1)} &= \zeta\,(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}\,(z+1)\,,\ z \neq \{-1,-2,-3,\ldots\}\,. \\ &= \frac{(-1)^m}{m!}\int_0^1 \frac{(1-t^z)}{1-t}\ln^m t\ dt \end{aligned}$$

and the multiplication formula is

$$(6) \qquad \psi^{(k)}(pz) = \delta_{p,0}\ln p + \frac{1}{p^{k+1}}\sum_{j=0}^{p-1}\psi^{(k)}(z+\frac{j}{p})$$

for p a positive integer and $\delta_{p,k}$ is the Kronecker delta. Let $a \in \mathbb{R}^+$ and $H_n^{(p)}$ denote harmonic numbers of order p, and put $BW\,(a,p.q) = \sum_{n=1}^{\infty} \frac{H_{an}^{(p)}}{n^q}$. We have, from the work of [**4**] that for odd weight $(p+q)$

$$BW\,(1,p.q)=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q}$$

$$=(-1)^p\sum_{j=1}^{\left[\frac{p}{2}\right]}\begin{pmatrix} p+q-2j-1\\ p-1\end{pmatrix}\zeta\,(p+q-2j)\,\zeta\,(2j)$$

$$+\frac{1}{2}\left(1+(-1)^{p+1}\right)\zeta\,(p)\,\zeta\,(q)$$

$$+(-1)^p\sum_{j=1}^{\left[\frac{p}{2}\right]}\begin{pmatrix} p+q-2j-1\\ q-1\end{pmatrix}\zeta\,(p+q-2j)\,\zeta\,(2j) \tag{7}$$

$$+\frac{\zeta\,(p+q)}{2}\left(1+(-1)^{p+1}\begin{pmatrix} p+q-1\\ p\end{pmatrix}+(-1)^{p+1}\begin{pmatrix} p+q-1\\ q\end{pmatrix}\right),$$

where $[z]$ is the integer part of z. Let us denote

$$S\,(a,p,q):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\ H_{an}^{(p)}}{n^q},$$

then in the case where p and q are both positive integers and $p+q$ is an odd integer, Flajolet and Salvy [**12**] gave the identity:

$$\begin{aligned} 2S\,(1,p,q) &= (1-(-1)^p)\,\zeta\,(p)\,\eta\,(q)+2\,(-1)^p\sum_{i+2k=q}\begin{pmatrix} p+i-1\\ p-1\end{pmatrix}\zeta\,(p+i)\ \ \eta\,(2k) \\ &+ \eta\,(p+q)-2\sum_{j+2k=p}\begin{pmatrix} q+j-1\\ q-1\end{pmatrix}(-1)^j\,\eta\,(q+j)\ \ \eta\,(2k)\,, \end{aligned} \tag{8}$$

where $\eta\,(0)=\frac{1}{2}$, $\eta\,(1)=\ln 2$, $\zeta\,(1)=0$, and $\zeta\,(0)=-\frac{1}{2}$ in accordance with the analytic continuation of the Riemann zeta function. In particular

$$2S\,(1,1,q)=(1+q)\,\eta\,(1+q)-\zeta\,(1+q)-2\sum_{j=1}^{\frac{q}{2}-1}\eta\,(2j)\ \ \zeta\,(1+q-2j)\,. \tag{9}$$

It is interesting to note that recently [**1**], established that for $p\in\mathbb{N}\backslash\{1\}$

$$S\,(1,p,1)=\frac{1}{2}p\zeta\,(p+1)-\frac{1}{2}\sum_{j=1}^{p}\eta\,(j)\ \ \eta\,(p-j+1)\,. \tag{10}$$

From [**26**] we have the identities

$$
\begin{aligned}
BW\left(\frac{1}{2},p.q\right) &= \sum_{n=1}^{\infty}\frac{H_{\frac{n}{2}}^{(p)}}{n^q}\\
&= (-1)^q\sum_{r=1}^{p}2^{p-1}\begin{pmatrix} p-1+q-r\\ p-r\end{pmatrix}\begin{pmatrix} BW\left(1,r,p+q-r\right)\\ -S\left(1,r,p+q-r\right)\end{pmatrix}\\
&+ (-1)^{q+1}\sum_{r=2}^{p}\frac{1}{2^{q-r}}\begin{pmatrix} p-1+q-r\\ p-r\end{pmatrix}\zeta\left(r\right)\zeta\left(p+q-r\right)\\
&+ (-1)^{q+1}\sum_{k=2}^{q-1}\frac{(-1)^k}{2^{q-k}}\begin{pmatrix} p-1+q-k\\ q-k\end{pmatrix}\zeta\left(k\right)\zeta\left(p+q-k\right),
\end{aligned} \tag{11}
$$

$$
S\left(\frac{1}{2},p.q\right) = BW\left(\frac{1}{2},p.q\right) - 2^{1-q}BW\left(1,p.q\right) \tag{12}
$$

and

$$
2\sum_{n=1}^{\infty}\frac{H_{n-\frac{1}{2}}^{(p)}}{n^q} = BW\left(\frac{1}{2},p.q\right) - S\left(\frac{1}{2},p.q\right). \tag{13}
$$

The next lemma relates the sum of the double and unitary argument of harmonic numbers in closed form and will be useful in the following section.
Lemma 1. Let $m \in \mathbb{N} \geq 2, p \in \mathbb{N}$ then

$$
\begin{aligned}
HE\left(p,m\right) &= \sum_{n=1}^{\infty}\frac{H_{2n}^{(p)}}{(2n-1)^m}\\
&= (-1)^{m+1}\begin{pmatrix} p+m-2\\ p-1\end{pmatrix}\ln 2\\
&+ \frac{1}{2}\left(BW\left(1,p,m\right)+S\left(1,p,m\right)\right)\\
&+ \sum_{r=2}^{p}\frac{(-1)^m}{2^r}\begin{pmatrix} p+m-1-r\\ p-r\end{pmatrix}\zeta\left(r\right)\\
&+ \sum_{k=2}^{m}\frac{(-1)^{m-k}}{2}\begin{pmatrix} p+m-1-k\\ p-1\end{pmatrix}\lambda\left(k\right).
\end{aligned}
$$

For the unitary argument

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_{n}^{(p)}}{(2n+1)^m} &= 2^{p-1}\left(BW\left(1,p,m\right)+S\left(1,p,m\right)\right)-2^{p-1}\eta\left(p\right)\lambda\left(m\right)\\
&- \frac{1}{2}\left(BW\left(\frac{1}{2},p,m\right)+S\left(\frac{1}{2},p,m\right)\right)
\end{aligned}
$$

where $BW(p,m)$ is the Borwein identity (7), $S(p,m)$ is evaluated from (8) and $\lambda(j)$ is defined by (3).

Proof.

$$\begin{aligned}
HE(p,m) &= \sum_{n=1}^{\infty} \frac{H_{2n}^{(p)}}{(2n-1)^m} = \frac{1}{2}\left(\sum_{n=2}^{\infty} \frac{H_n^{(p)}}{(n-1)^m} - \sum_{n=2}^{\infty} \frac{(-1)^{n+1} H_n^{(p)}}{(n-1)^m}\right) \\
&= \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{\frac{1}{(n+1)^p} + H_n^{(p)}}{n^m} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(\frac{1}{(n+1)^p} + H_n^{(p)}\right)}{n^m}\right) \\
&= \frac{1}{2}\left(BW(p,m) + S(p,m)\right) + \sum_{n=1}^{\infty} \frac{1}{(2n)^p (2n-1)^m}.
\end{aligned}$$

Expanding in partial fraction form gives us

$$HE(p,m) = \frac{1}{2}\left(BW(p,m) + S(p,m)\right) + \sum_{n=1}^{\infty} \frac{(-1)^{m+1}}{2n(2n-1)} \binom{p+m-2}{p-1}$$

$$\sum_{n=1}^{\infty}\left(\sum_{r=2}^{p} \frac{(-1)^m}{2^r n^r} \binom{p+m-1-r}{p-r} + \sum_{k=2}^{m} \frac{(-1)^{m-k}}{(2n-1)^k} \binom{p+m-1-k}{p-1}\right)$$

and the result follows. For the unitary argument the proof follows the same pattern as above. $\square$

The following partial fraction decomposition holds.

Lemma 2. For $y \in \mathbb{R}, m, p \in \mathbb{N}\backslash\{0\}$ we have

$$\frac{1}{(2n-1)^p (2n+y)^{m+1}} = \frac{(-1)^{p+1}}{(1+y)^{m+p-1}} \binom{m+p-1}{m} \frac{1}{(2n-1)(2n+y)}$$

$$+(-1)^p \sum_{r=2}^{m+1} \frac{\binom{m+p-r}{m+1-r}}{(1+y)^{m+p+1-r} \quad (2n+y)^r} + (-1)^p \sum_{k=2}^{p} \frac{(-1)^k \binom{m+p-k}{m}}{(1+y)^{m+p+1-k} \quad (2n-1)^k}.$$

Proof. Follows simply by expansion. $\square$

The following lemma will also be useful in the evaluation of integrals of the type (1).

Lemma 3.

Let $m \in \mathbb{N} \geq 2$. then

$$\begin{aligned}
HM(m) &= \sum_{n=1}^{\infty} \frac{H_n}{(2n-1)^m} = m\lambda(m+1) + \left(2(-1)^{m+1} - \lambda(m)\right)\ln 2 \\
&+ \sum_{j=2}^{m} (-1)^{m+j} \lambda(j) - \frac{2}{(m-1)} \sum_{k=1}^{m-2} (m-k-1)\lambda(k+1)\lambda(m-k) \qquad (14)
\end{aligned}$$

where $\lambda(m)$ is given by (3).

Proof. We have, from [**24**], theorem 1, for x a real number, $x \neq -1, -2, -3, \cdots$

$$(15)\ (-1)^m (m-1)! \sum_{n=1}^{\infty} \frac{H_n}{(n+x)^m} = (\psi(x)+\gamma)\,\psi^{(m-1)}(x) - \frac{1}{2}\psi^{(m)}(x) + \sum_{k=1}^{m-2} \binom{m-2}{k} \psi^{(m)}(x)\psi^{(m-k-1)}(x).$$

Choosing $x = \frac{1}{2}$, we have $\psi(\frac{1}{2}) + \gamma = -2\ln 2$ and from (5), $(-1)^m \psi^{(m)}(\frac{1}{2}) = -2^m m! \lambda(m+1)$, therefore substituting into the above equation (16) and simplifying leads to

$$(16)\quad \sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^m} = m\lambda(m+1) - 2\lambda(m)\ln 2 - \tfrac{2}{(m-1)} \sum_{k=1}^{m-2} (m-k-1)\,\lambda(k+1)\,\lambda(m-k).$$

Identity (16) corrects a minor error in the paper [**20**], now reordering the counter in (16) we obtain the identity (14). □

Lemma 4. We define

$$\begin{aligned} W(3) &:= \sum_{n\geq 1} \frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3} = \sum_{n\geq 1} \frac{(-1)^{n+1}}{2^{2n}} \left(\frac{2}{(4n-3)^3} + \frac{2}{(4n-2)^3} + \frac{1}{(4n-1)^3} \right) \\ &= \sum_{n\geq 1} \frac{1}{2^{4n}} \begin{pmatrix} \frac{8}{(8n-7)^3} + \frac{8}{(8n-6)^3} + \frac{4}{(8n-5)^3} \\ -\frac{2}{(8n-3)^3} - \frac{2}{(8n-2)^3} - \frac{1}{(8n-1)^3} \end{pmatrix} \\ &= \frac{1}{256} \left(2\Phi\left(-\frac{1}{4}, 3, \frac{1}{4}\right) + 2\Phi\left(-\frac{1}{4}, 3, \frac{1}{2}\right) + \Phi\left(-\frac{1}{4}, 3, \frac{3}{4}\right) \right). \end{aligned}$$

where $\Phi(\cdot,\cdot,\cdot)$ is the Lerch transcendent.

Proof. Consider the polylogarithms

$$\mathrm{Li}_3\left(\frac{1\pm i}{2}\right) = \sum_{n\geq 1} \frac{\exp\left(\pm\frac{in\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3} = \sum_{n\geq 1} \frac{\cos\left(\frac{n\pi}{4}\right) \pm i \sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3},$$

adding produces

$$\begin{aligned} \mathrm{Li}_3\left(\frac{1+i}{2}\right) + \mathrm{Li}_3\left(\frac{1-i}{2}\right) &= 2\sum_{n\geq 1} \frac{\cos\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\ n^3} \\ &= \frac{35}{32}\zeta(3) + \frac{1}{24}\ln^3 2 - \frac{5}{16}\zeta(2)\ln 2, \end{aligned}$$

by a result from [**16**]. By the circular nature of the trigonometric function, the BBP type sum for

$$
\begin{aligned}
\sum_{n\geq 1}\frac{\cos\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\,n^3} &= \sum_{n\geq 1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{(4n-3)^3}-\frac{1}{(4n-1)^3}-\frac{1}{(4n)^3}\right)\\
&= \frac{1}{256}\left(2\Phi\left(-\frac{1}{4},3,\frac{1}{4}\right)-\Phi\left(-\frac{1}{4},3,\frac{3}{4}\right)-\Phi\left(-\frac{1}{4},3,1\right)\right)\\
&= \sum_{n\geq 1}\frac{1}{2^{4n}}\left(\frac{8}{(8n-7)^3}+\frac{1}{(8n-1)^3}+\frac{1}{(8n)^3}\right)\\
&\quad -\sum_{n\geq 1}\frac{1}{2^{4n}}\left(\frac{4}{(8n-5)^3}+\frac{4}{(8n-4)^3}+\frac{2}{(8n-3)^3}\right).
\end{aligned}
$$

By a similar argument we have

$$
\operatorname{Li}_3\left(\frac{1+i}{2}\right)-\operatorname{Li}_3\left(\frac{1-i}{2}\right)=2i\sum_{n\geq 1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\,n^3}
$$

and the

$$
\begin{aligned}
\operatorname{Im}\left(\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right) &= \sum_{n\geq 1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\,n^3}\\
&= \sum_{n\geq 1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{(4n-3)^3}+\frac{2}{(4n-2)^3}+\frac{1}{(4n-1)^3}\right)\\
&= \sum_{n\geq 1}\frac{1}{2^{4n}}\left(\begin{array}{c}\frac{8}{(8n-7)^3}+\frac{8}{(8n-6)^3}+\frac{4}{(8n-5)^3}\\[2mm] -\frac{2}{(8n-3)^3}-\frac{2}{(8n-2)^3}-\frac{1}{(8n-1)^3}\end{array}\right)\\
&= \frac{1}{256}\left(2\Phi\left(-\frac{1}{4},3,\frac{1}{4}\right)+2\Phi\left(-\frac{1}{4},3,\frac{1}{2}\right)+\Phi\left(-\frac{1}{4},3,\frac{3}{4}\right)\right).
\end{aligned}
$$

□

Since the Legendre-Chi function can be expressed as the difference of two polylogarithmic functions then we expect that integrals of the type (1) may be represented as Euler sums and therefore in terms of special functions such as the Riemann zeta function. A search of the current literature has not found many examples for the representation of the integral (1) and certainly not a systematic study of (1). Many papers, [**11**], [**13**], [**27**], [**28**] examined some polylogarithmic integrals in terms of Euler sums. Some other important sources of information on Legendre-Chi functions are the works of [**2**], [**6**], [**8**], [**9**], [**10**] and the excellent books [**19**], [**29**] and [**30**]. Other useful references related to the representation of Euler sums in terms of special functions include [**1**], [**3**], [**21**], [**22**], [**23**], [**24**], [**25**].

Some examples are highlighted, most of which are not amenable to a computer mathematical package.

2. MAIN RESULTS

Theorem 1. Let $a \in \mathbb{R} \geq -2,\ b \in \mathbb{R}^{+}, \delta = 1,\ (m,p,q) \in \mathbb{N} \cup \{0\},$ the integral of the product of the Legendre-Chi, polylogarithm and log functions,

$$\begin{aligned}
P(a,b,m,p,q) &= \int_0^1 x^a\ \chi_p(x)\ \mathrm{Li}_q\left(x^b\right)\ \ln^m(x)\,dx \\
&= \sum_{j=0}^{m} \frac{(-b)^{q-j-1}(-1)^m\, m!\,(q)_{m-j}}{(m-j)!} \sum_{n\geq 1} \frac{H^{(j+1)}_{\frac{2n+a}{b}}}{(2n-1)^p\,(2n+a)^{q+m-j}} \\
&- \sum_{j=1}^{m} \frac{(-b)^{q-j-1}(-1)^m\, m!\,(q)_{m-j}}{(m-j)!} \sum_{n\geq 1} \frac{\zeta(j+1)}{(2n-1)^p\,(2n+a)^{q+m-j}} \\
&+ \sum_{r=2}^{q} (-b)^{q-r}(-1)^m\,(q+1-r)_m\ \zeta(r) \\
&\times \sum_{n\geq 1} \frac{1}{(2n-1)^p\,(2n+a)^{q+m+1-r}}
\end{aligned}$$

where $H^{(m+1)}_{\frac{2n+a}{b}}$ are harmonic numbers of order $(m+1)$, $\zeta(\cdot)$ is the Riemann zeta function, $\chi_p(x)$ is the Legendre-Chi function, $\mathrm{Li}_q(\cdot)$ is the polylogarithm function and $(q)_j$ is the Pochhammer symbol.

Proof. From the Taylor series expansion of the LCF function and the polylogarithm functions we have

$$\begin{aligned}
\int_0^1 x^a\ \chi_p(x)\ \mathrm{Li}_q\left(x^b\right)\ dx &= \sum_{n\geq 1} \frac{1}{(2n-1)^p} \sum_{j\geq 1} \frac{1}{j^q} \int_0^1 x^{2n+bj+a-1}\ dx \\
&= \sum_{n\geq 1}\sum_{j\geq 1} \frac{1}{(2n-1)^p\ \ j^q\,(2n+bj+a)}.
\end{aligned}$$

By partial fraction expansion, from Lemma 2

$$\int_0^1 x^a\ \chi_p(x)\ \mathrm{Li}_q\left(x^b\right)\ dx = \sum_{n\geq 1}\frac{1}{(2n-1)^p}\sum_{j\geq 1}\left(\begin{array}{c}\sum_{r=2}^{q}\frac{(-b)^{q-r}}{j^r(2n+a)^{q+1-r}}\\ -\frac{(-b)^{q-1}}{j(2n+a)^{q-1}(2n+bj+a)}\end{array}\right)$$

$$= \sum_{n\geq 1}\frac{(-b)^{q-1}H_{\frac{2n+a}{b}}}{(2n-1)^p(2n+a)^q} + \sum_{r=2}^{q}\sum_{n\geq 1}\frac{(-b)^{q-r}\zeta(r)}{(2n-1)^p(2n+a)^{q+1-r}}.$$

Differentiating $'m'$ times under the integral sign, with respect to the real parameter a, we obtain

$$P(a,b,m,p,q) = (-b)^{q-1}(-1)^m m!\begin{pmatrix} m+q-1\\ q-1\end{pmatrix}\sum_{n\geq 1}\frac{H_{\frac{2n+a}{b}}}{(2n-1)^p(2n+a)^{q+m}}$$
$$+\sum_{j=1}^{m}(-b)^{q-j-1}(-1)^{m+j}(m-j)!\begin{pmatrix} m\\ j\end{pmatrix}\begin{pmatrix} m+q-1-j\\ q-1\end{pmatrix}$$
$$\times\sum_{n\geq 1}\frac{\psi^{(m)}\left(\frac{2n+b+a}{b}\right)}{b^j(2n-1)^p(2n+a)^{q+m+1-j}}$$
$$+\sum_{r=2}^{q}(-b)^{q-r}(-1)^m m!\begin{pmatrix} m+q-r\\ q-r\end{pmatrix}\zeta(r)\sum_{n\geq 1}\frac{1}{(2n-1)^p(2n+a)^{q+m+1-r}},$$

simplifying by the use of the Polygamma identity (5) and applying the Pochhammer symbol, we obtain

$$P(a,b,m,p,q) = \int_0^1 x^a\ \chi_p(x)\ \mathrm{Li}_q\left(x^b\right)\ \ln^m(x)\,dx$$
$$= \sum_{j=0}^{m}\frac{(-b)^{q-j-1}(-1)^m m!\,(q)_{m-j}}{(m-j)!}\sum_{n\geq 1}\frac{H^{(m+1)}_{\frac{2n+a}{b}}}{(2n-1)^p(2n+a)^{q+m-j}}$$
$$- \sum_{j=1}^{m}\frac{(-b)^{q-j-1}(-1)^m m!\,(q)_{m-j}}{(m-j)!}\sum_{n\geq 1}\frac{\zeta(j+1)}{(2n-1)^p(2n+a)^{q+m-j}}$$
$$+\sum_{r=2}^{q}(-b)^{q-r}(-1)^m(q+1-r)_m\ \zeta(r)\sum_{n\geq 1}\frac{1}{(2n-1)^p(2n+a)^{q+m+1-r}}$$

and Theorem 1 follows. □

There are some interesting special cases of Theorem 1 and we present these in the next corollaries before listing some illustrative examples.

Corollary 1. Let the conditions of Theorem 1 hold then, for $p=0$

$$\begin{aligned} P(a,b,m,0,q) &= \int\limits_0^1 x^a \ \chi_0(x) \ \mathrm{Li}_q\left(x^b\right) \ \ln^m(x)\,dx \\ &= \frac{(-1)^m m!}{2^{m+1}}\left(\zeta(m+1)\,\zeta(q) - \sum_{n\geq 1}\frac{H^{(m+1)}_{\frac{bn+a}{2}}}{n^q}\right), \end{aligned}$$

where $\chi_0(x) = \frac{x}{1-x^2}.$.

Proof. Applying the same technique as in Theorem 1, we have

$$\begin{aligned} P(a,b,m,0,q) &= \int\limits_0^1 x^a \ \chi_0(x) \ \mathrm{Li}_q\left(x^b\right) \ \ln^m(x)\,dx \\ &= \sum_{n\geq 1}\sum_{j\geq 1}\frac{(-1)^m m!}{j^q(2n+bj+a)^{m+1}} = \sum_{j\geq 1}\frac{1}{j^q}\sum_{n\geq 1}\frac{(-1)^m m!}{(2n+bj+a)^{m+1}} \\ &= (-1)^m\sum_{j\geq 1}\frac{1}{j^q}\psi^{(m)}\left(\frac{bj+a}{2}+1\right) \\ &= \frac{(-1)^m m!}{2^{m+1}}\left(\zeta(m+1)\,\zeta(q) - \sum_{j\geq 1}\frac{H^{(m+1)}_{\frac{bj+a}{2}}}{j^q}\right), \end{aligned}$$

replacing the counter $'j'$ for $'n'$ the proof is completed. □

Corollary 2. Let the conditions of Theorem 1 hold then, for $q=0$

$$P(a,b,m,p,0) = \int\limits_0^1 x^a \ \chi_p(x) \ \mathrm{Li}_0\left(x^b\right) \ \ln^m(x)\,dx$$

$$= \frac{(-1)^{m+1} m!}{b^{m+1}}\left(\sum_{n\geq 1}\frac{H^{(m+1)}_{\frac{2n+a}{b}}}{(2n-1)^p} - \zeta(m+1)\,\lambda(p)\right),$$

where $\mathrm{Li}_0\left(x^b\right) = \frac{x^b}{1-x^b}.$.

Proof. Following the same steps as in corollary 2, we have

$$P(a,b,m,p,0) = \frac{1}{b^{m+1}}\sum_{n\geq 1}\frac{\psi^{(m)}\left(\frac{2n+a}{b}+1\right)}{(2n-1)^p},$$

replacing the polygamma function with harmonic numbers

$$P(a,b,m,p,0)=\frac{(-1)^{m+1}m!}{b^{m+1}}\sum_{n\geq 1}\left(\frac{H^{(m+1)}_{\frac{2n+a}{b}}-\zeta(m+1)}{(2n-1)^{p}}\right)$$

and proof is complete. □

Some illustrative examples of Theorem 1 and its corollaries follow. To evaluate the resultant Euler sums for these examples we need a mixture of identities, including (4), (5), (6), (7), (8), (9), (10), (11), (12), (13), Lemma 1, Lemma 2, Lemma 3 and Lemma 4.

Examples. 1. Let $a=-1, b=1$, then

$$\begin{aligned}
P(-1,1,m,p,q) &= \int_0^1 x^{-1}\ \chi_p(x)\ \operatorname{Li}_q(x)\ \ln^m(x)\,dx \\
&= \sum_{j=0}^{m}\frac{(-1)^{q+m-j-1}m!\,(q)_{m-j}}{(m-j)!}\sum_{n\geq 1}\frac{H^{(m+1)}_{2n-1}}{(2n-1)^{m+p+q-j}} \\
&- \sum_{j=1}^{m}\frac{(-1)^{q+m-j-1}m!\,(q)_{m-j}}{2(m-j)!}\zeta(j+1)\,\lambda(m+p+q-j) \\
&+ \sum_{r=2}^{q}(-1)^{m+q-r}(q+1-r)_m\ \zeta(r)\,\lambda(m+p+q+1-r).
\end{aligned}$$

The Euler sums $\sum_{n\geq 1}\frac{H^{(m)}_{2n-1}}{(2n-1)^p}$ can be evaluated from Lemma 1. If we let $m=p=q=2$,

$$\begin{aligned}
P(-1,1,2,2,2) &= \int_0^1 x^{-1}\ \chi_2(x)\ \operatorname{Li}_2(x)\ \ln^2(x)\,dx \\
&= \frac{67}{8}\zeta(5)\,\zeta(2)-\frac{889}{64}\zeta(7).
\end{aligned}$$

2. For $a=0, b=\frac{1}{2}, m=p=q=1$

$$\begin{aligned}
P\left(0,\frac{1}{2},1,1,1\right) &= \int_0^1 \chi_1(x)\ \operatorname{Li}_1\left(x^{\frac{1}{2}}\right)\ \ln(x)\,dx \\
&= \sum_{n\geq 1}\frac{H_{4n}}{(2n)^2(2n-1)}+\sum_{n\geq 1}\frac{\zeta(2)-H^{(2)}_{4n}}{n(2n-1)}.
\end{aligned}$$

Here we require the Euler sums

$$\sum_{n\geq 1}\frac{H_{4n}}{n^2}=\frac{67}{8}\zeta(3)-2\pi G,\sum_{n\geq 1}\frac{H_{4n}}{n(2n-1)}=\frac{\pi}{2}+\frac{5}{4}\zeta(2)-\frac{1}{2}\ln^2 2$$

$$\sum_{n\geq 1}\frac{H_{4n}^{(2)}}{n\left(2n-1\right)}=2\zeta\left(3\right)+G-\frac{1}{2}\pi G+\frac{\pi}{4}-\frac{13}{16}\zeta\left(2\right)-\frac{1}{8}\zeta\left(2\right)\ln 2-\frac{1}{4}\ln 2,$$

which may be evaluated by the techniques described in [**5**], [**7**], [**18**], [**26**], [**31**], [**32**] therefore

$$P\left(0,\frac{1}{2},1,1,1\right)=\frac{1}{2}\pi G-2G-\frac{61}{32}\zeta\left(3\right)-\frac{3\pi}{4}+\zeta\left(2\right)+\frac{9}{4}\zeta\left(2\right)\ln 2+\frac{1}{2}\ln 2+\frac{1}{4}\ln^2 2.$$

3. For $a=0,b=2,m=p=1,q=5$

$$\begin{aligned}P\left(0,2,1,1,5\right) &= \int_0^1 \chi_1\left(x\right)\ \mathrm{Li}_5\left(x^2\right)\ \ln\left(x\right)dx\\ &= -\frac{5}{4}\sum_{n\geq 1}\frac{H_n}{n^6\left(2n-1\right)}+\frac{1}{4}\sum_{n\geq 1}\frac{\zeta\left(2\right)-H_n^{(2)}}{n^5\left(2n-1\right)}\\ &- \sum_{r=2}^{5}\left(-2\right)^{5-r}\binom{6-r}{5-r}\zeta\left(r\right)\sum_{n\geq 1}\frac{1}{\left(2n-1\right)\left(2n\right)^{7-r}}.\end{aligned}$$

The Euler sums may be evaluated from Lemma 1, therefore

$$\begin{aligned}P\left(0,2,1,1,5\right) &= \frac{5}{2}\zeta\left(7\right)-\frac{23}{12}\zeta\left(6\right)+\frac{3}{4}\zeta^2\left(3\right)-\zeta\left(2\right)\zeta\left(5\right)-\zeta\left(5\right)\ln 2-12\zeta\left(3\right)\ln 2\\ &+ 50\zeta\left(3\right)+4\zeta\left(4\right)\ln 2-4\zeta\left(2\right)\zeta\left(3\right)-\frac{1}{2}\zeta\left(4\right)\zeta\left(3\right)+\frac{21}{2}\zeta\left(5\right)\\ &+ 32\zeta\left(2\right)\ln 2-9\zeta\left(4\right)+8\zeta\left(2\right)+80\ln^2 2-12\ln 2.\end{aligned}$$

4. For $a=-\frac{1}{2},b=\frac{1}{2},m=p=q=1$

$$\begin{aligned}&\int_0^1 x^{-\frac{1}{2}}\ \chi_1\left(x\right)\ \mathrm{Li}_5\left(x^{\frac{1}{2}}\right)\ \ln\left(x\right)dx\\ &=4\sum_{n\geq 1}\frac{\zeta\left(2\right)-H_{4n-1}^{(2)}}{\left(2n-1\right)\left(4n-1\right)}-4\sum_{n\geq 1}\frac{H_{4n-1}}{\left(2n-1\right)\left(4n-1\right)^2}.\end{aligned}$$

Here we require various Euler sums and we just highlight the result

$$\begin{aligned}\sum_{n\geq 1}\frac{H_{4n}^{(2)}}{\left(2n-1\right)\left(4n-1\right)} &= \frac{\pi}{2}-\frac{11}{16}\zeta\left(2\right)+G-\frac{1}{4}\pi G-\frac{\pi^3}{48}-\frac{7}{4}\ln 2\\ &+ \frac{7}{16}\zeta\left(2\right)\ln 2+\frac{1}{2}G\ln 2+2W\left(3\right)-\frac{1}{16}\pi\ln 2\end{aligned}$$

from which

$$\begin{aligned}\int_0^1 x^{-\frac{1}{2}}\chi_1(x)\,\mathrm{Li}_5\left(x^{\frac{1}{2}}\right)\ln(x)\,dx &= \frac{15}{4}\zeta(2)-8G+\pi G+\frac{\pi^3}{32}-\frac{15}{4}\zeta(2)\ln 2\\ &+ 4\ln 2+\frac{21}{4}\zeta(3)+\frac{1}{8}\pi\ln^2 2-\frac{3}{2}\ln^2 2\\ &+ \frac{1}{2}\pi\ln 2-4W(3)-2\pi\end{aligned}$$

where $W(3)$ is defined in Lemma 4.
5. For $a=2, b=2, m=4, p=0, q=5$

$$\int_0^1 x^2\ \chi_0(x)\ \mathrm{Li}_5\left(x^2\right)\ \ln^4(x)\,dx=\frac{3}{4}\left(\zeta^2(5)-\sum_{n\ge1}\frac{H_{n+1}^{(5)}}{n^5}\right)$$

and evaluating the Euler sum gives

$$P(2,2,4,0,5)=\frac{3}{8}\zeta^2(5)-\frac{3}{8}\zeta(10)+\frac{15}{2}\zeta(4)+\frac{105}{2}\zeta(2)-\frac{189}{2}.$$

6. For $a=-1, b=2, m=p=3, q=0$

$$\int_0^1 x^{-1}\ \chi_3(x)\ \mathrm{Li}_0\left(x^2\right)\ \ln^3(x)\,dx=-\frac{3}{8}\left(\sum_{n\ge1}\frac{H_{n-\frac{1}{2}}^{(4)}}{(2n-1)^3}-\frac{1}{2}\zeta(3)\lambda(4)\right),$$

and using Lemma 1

$$\int_0^1 x^{-1}\ \chi_3(x)\ \mathrm{Li}_0\left(x^2\right)\ \ln^3(x)\,dx=\frac{135}{32}\zeta(4)\zeta(3)-\frac{45}{32}\zeta(5)\zeta(2)-\frac{381}{128}\zeta(7).$$

7. For the degenerate case $a=2, b=2, m\in\mathbb{N},\ \delta=\frac{1}{t}$ for $1\le t\le -1,\ p=q=0$, then by expansion

$$\begin{aligned}\int_0^1 x^2\ \chi_0(x)\,\mathrm{Li}_0\left(\frac{1}{t}x^2\right)\ \ln^m(x)\,dx &= \int_0^1\frac{x^5}{(x^2-1)(x^2-t)}\ \ln^m(x)\,dx\\ &= \sum_{n\ge1}\sum_{j=1}^{n}\frac{1}{t^j}\int_0^1 x^{2n+3}\ \ln^m(x)\,dx=\sum_{n\ge1}\sum_{j=1}^{n}\frac{(-1)^m\,m!}{2^{m+1}t^j\,(n+2)^{m+1}}\\ &= \frac{(-1)^{m+1}\,m!}{2^{m+1}(1-t)}\sum_{n\ge1}\frac{1-\frac{1}{t^n}}{(n+2)^{m+1}}\\ &= \frac{(-1)^{m+1}\,m!}{2^{m+1}(1-t)}\left(-1-\frac{1}{2^{m+1}}+\zeta(m+1)-\frac{1}{t}\Phi\left(\frac{1}{t},m+1,3\right)\right)\\ &= \frac{(-1)^{m+1}\,m!}{2^{m+1}(1-t)}\left(t-1+\zeta(m+1)-t^2\mathrm{Li}_{m+1}\left(\frac{1}{t}\right)\right).\end{aligned}$$

Next we investigate the case of the integral in question containing a polylogarithmic function with negative argument.

Theorem 2. Let $a \in \mathbb{R} \geq -2,\ b \in \mathbb{R}^{+}, \delta = -1,\ \ (m, p, q) \in \mathbb{N} \cup \{0\}$, then

$$Q(a, b, m, p, q) = \int_0^1 x^a\ \chi_p(x)\ \operatorname{Li}_q\left(-x^b\right)\ \ln^m(x)\, dx$$

$$= \sum_{j=0}^{m} (-b)^{q-1} (-1)^m (m+1-j)_j\, (q)_{m-j} \sum_{n\geq 1} \frac{2H^{(j+1)}_{\frac{2n+a}{2b}} - 2^{j+1} H^{(j+1)}_{\frac{2n+a}{b}}}{b^j 2^{j+1} (2n-1)^p (2n+a)^{q+m-j}}$$

$$+ \sum_{j=1}^{m} (-b)^{q-1} (-1)^m (m+1-j)_j\, (q)_{m-j}\, \eta(j) \sum_{n\geq 1} \frac{1}{(2n-1)^p (2n+a)^{q+m-j}}$$

$$- \sum_{r=2}^{q} (-b)^{q-r} (-1)^m (q+1-r)_m\ \ \eta(r) \sum_{n\geq 1} \frac{1}{(2n-1)^p (2n+a)^{q+m+1-r}}$$

where $\eta(\cdot)$ is the alternating zeta function.

Proof. From the Taylor series expansion of the LCF function and the polylogarithm functions we have

$$\begin{aligned}
\int_0^1 x^a\ \chi_p(x)\ \operatorname{Li}_q\left(-x^b\right)\ d &= \sum_{n\geq 1} \frac{1}{(2n-1)^p} \sum_{j\geq 1} \frac{(-1)^j}{j^q} \int_0^1 x^{2n+bj+a-1}\ dx \\
&= \sum_{n\geq 1} \sum_{j\geq 1} \frac{(-1)^j}{(2n-1)^p\ \ j^q (2n+bj+a)}.
\end{aligned}$$

By partial fraction expansion, from Lemma 2

$$\int_0^1 x^a\ \chi_p(x)\ \operatorname{Li}_q\left(-x^b\right)\ dx = \sum_{n\geq 1} \frac{1}{(2n-1)^p} \sum_{j\geq 1} (-1)^j \left(\begin{array}{c} \sum_{r=2}^{q} \frac{(-b)^{q-r}}{j^r (2n+a)^{q+1-r}} \\ + \frac{(-b)^{q-1}}{j(2n+a)^{q-1}(2n+bj+a)} \end{array} \right)$$

$$= \sum_{n\geq 1} \frac{(-1)^{q-1} \left(\Phi\left(-1, 1, \frac{2n+a+b}{b}\right) - \ln 2\right)}{(2n-1)^p (2n+a)^q} - \sum_{r=2}^{q} \sum_{n\geq 1} \frac{(-b)^{q-r}\, \eta(r)}{(2n-1)^p (2n+a)^{q+1-r}}$$

$$= \sum_{n\geq 1} \frac{(-1)^{q-1} \left(\frac{1}{2}\psi\left(\frac{2n+a+b}{b}\right) - \frac{1}{2}\psi\left(\frac{2n+a+b}{b} + \frac{1}{2}\right) - \ln 2\right)}{(2n-1)^p (2n+a)^q}$$

$$- \sum_{r=2}^{q} \sum_{n\geq 1} \frac{(-b)^{q-r}\, \eta(r)}{(2n-1)^p (2n+a)^{q+1-r}}.$$

Differentiating $'m'$ times under the integral sign, with respect to the real parameter

a, we obtain

$$\begin{aligned}
&Q(a,b,m,p,q)\\
&=\sum_{j=0}^{m}(-b)^{q-1}(-1)^{m}(m+1-j)_{j}(q)_{m-j}\sum_{n\geq 1}\frac{2H^{(j+1)}_{\frac{2n+a}{2b}}-2^{j+1}H^{(j+1)}_{\frac{2n+a}{b}}}{b^{j}2^{j+1}(2n-1)^{p}(2n+a)^{q+m-j}}\\
&+\sum_{j=1}^{m}(-b)^{q-1}(-1)^{m}(m+1-j)_{j}(q)_{m-j}\,\eta(j)\sum_{n\geq 1}\frac{1}{(2n-1)^{p}(2n+a)^{q+m-j}}\\
&-\sum_{r=2}^{q}(-b)^{q-r}(-1)^{m}(q+1-r)_{m}\;\;\eta(r)\sum_{n\geq 1}\frac{1}{(2n-1)^{p}(2n+a)^{q+m+1-r}},
\end{aligned}$$

where $\Phi(\cdot,\cdot,\cdot)$ is the Lerch transcendent and $(q)_m$ is the Pochhammer symbol, the proof of the theorem is finished. □

There are some interesting special cases of Theorem 2 and we present these in the next corollaries before listing some illustrative examples.

Corollary 3. Let $a=-1,\ b=1,\ (m.p.q)\in\mathbb{N}\cup\{0\}$, then

$$\begin{aligned}
Q(-1,1,m,p,q) &= \int_0^1 x^{-1}\;\chi_p(x)\;\operatorname{Li}_q(-x)\;\ln^m(x)\,dx\\
= &\sum_{j=0}^{m}(-1)^{q+m-1}(m+1-j)_{j}(q)_{m-j}\sum_{n\geq 1}\frac{2H^{(j+1)}_{n-\frac{1}{2}}-2^{j+1}H^{(j+1)}_{2n-1}}{2^{j+1}(2n-1)^{m+p+q-j}}\\
+ &\frac{1}{2}\sum_{j=1}^{m}(-1)^{m+q-1}(m+1-j)_{j}(q)_{m-j}\,\eta(j+1)\,\lambda(m+p+q-j)\\
- &\frac{1}{2}\sum_{r=2}^{q}(-1)^{m+q-r}(q+1-r)_{m}\;\;\eta(r)\,\lambda(m+p+q+1-r)
\end{aligned}$$

where $\eta(\cdot)$ is the Dirichlet eta function and the Euler sums $\sum_{n\geq 1}\frac{H^{(j+1)}_{n-\frac{1}{2}}}{(2n-1)^{m}}$ can be evaluated by (13).

Corollary 4. Let the conditions of Theorem 2 hold and put $p=0$, then

$$\begin{aligned}
Q(a,b,m,0,q) &= \int_0^1 x^{a}\;\chi_0(x)\;\operatorname{Li}_q\left(-x^{b}\right)\;\ln^m(x)\,dx\\
&= \frac{(-1)^{m}m!}{2^{m+1}}\left(\sum_{n\geq 1}\frac{H^{(m+1)}_{\frac{bn+a}{2}}}{n^{q}}-\zeta(m+1)\,\eta(q)\right).
\end{aligned}$$

Proof. The proof follows the same pattern as in Corollary 1. □

Corollary 5. Let the conditions of Theorem 1 hold then, for $q=0$

$$Q(a,b,m,p,0)=\int_0^1 x^a\ \chi_p(x)\ \mathrm{Li}_0\left(-x^b\right)\ \ln^m(x)\,dx$$

$$=\frac{(-1)^m m!}{(2b)^{m+1}}\left(\sum_{n\geq 1}\frac{2^{m+1}H^{(m+1)}_{\frac{2n+a}{b}}-2H^{(m+1)}_{\frac{2n+a}{2b}}}{(2n-1)^p}-2^m\eta(m+1)\lambda(p)\right),$$

where $\mathrm{Li}_0\left(-x^b\right)=-\frac{x^b}{1+x^b.}$.

Proof. The proof follows the same pattern as in Corollary 2. □

Some illustrative examples of Theorem 2 and its corollaries follow. To evaluate the resultant Euler sums for these examples we need a mixture of identities, including (4), (5), (6), (7), (8), (9), (10), (11), (12), (13), Lemma 1, Lemma 2, Lemma 3 and Lemma 4.

Examples. 1. Let $a=-1, b=1, m=1, p=3, q=2$ then

$$Q(-1,1,1,3,2)=\int_0^1 x^{-1}\ \chi_3(x)\ \mathrm{Li}_2(-x)\ \ln(x)\,dx$$

$$=2\sum_{n\geq 1}\frac{H_{n-\frac{1}{2}}-H_{2n-1}}{(2n-1)^6}+\frac{1}{2}\sum_{n\geq 1}\frac{H^{(2)}_{n-\frac{1}{2}}-2H^{(2)}_{2n-1}}{(2n-1)^5}+\eta(2)\lambda(5).$$

The Euler sums can be evaluated by the results in Lemma 1, so that finally

$$\int_0^1 x^{-1}\ \chi_3(x)\ \mathrm{Li}_2(-x)\ \ln(x)\,dx\ =\frac{381}{128}\zeta(7)-\frac{13}{8}\zeta(5)\zeta(2).$$

2. Let $a=3, b=1, m=p=q=1$ then

$$\begin{aligned}Q(3,1,1,1,1)&=\int_0^1 x^3\ \chi_1(x)\ \mathrm{Li}_1(-x)\ \ln(x)\,dx\\&=2\sum_{n\geq 1}\frac{H_{2n+3}-H_{n+\frac{3}{2}}}{(2n-1)(2n+3)^2}+2\sum_{n\geq 1}\frac{4H^{(2)}_{2n+3}-2H^{(2)}_{n+\frac{3}{2}}-4\eta(2)}{4(2n-1)(2n+3)}\end{aligned}$$

and evaluating the Euler sums gives

$$Q(3,1,1,1,1)=\frac{43}{192}\zeta(2)+\frac{13}{12}\ln 2-\frac{829}{864}-\frac{7}{64}\zeta(3).$$

3. Let $a=0, b=m=p=1, q=3$ then

$$\begin{aligned} Q(0,1,1,1,3) &= \int_0^1 \chi_1(x)\ \mathrm{Li}_3(-x)\ \ln(x)\,dx \\ &= \frac{3}{16}\sum_{n\geq 1}\frac{H_{2n}-H_n+\frac{1}{4}\eta(3)}{n^4(2n-1)}+\frac{1}{16}\sum_{n\geq 1}\frac{2H_{2n}^{(2)}-H_n^{(2)}-\frac{3}{8}\eta(2)}{n^3(2n-1)} \end{aligned}$$

and evaluating the Euler sums gives

$$\begin{aligned} Q(0,1,1,1,3) = {} & \frac{5}{8}\zeta(5)-\frac{7}{16}\zeta(2)\zeta(3)-\frac{1}{8}\zeta(3)\ln 2+\frac{7}{16}\zeta(3)-\mathrm{Li}_4(\frac{1}{2})-\frac{1}{24}\ln^2 2 \\ & +\frac{1}{4}\zeta(2)\ln^2 2+\frac{3}{2}\ln^2 2-\frac{5}{2}\zeta(2)\ln 2-4\ln 2+\frac{45}{32}\zeta(4)+\frac{7}{4}\zeta(2). \end{aligned}$$

4. Let $a=-\frac{1}{2}, b=\frac{1}{2}, m=p=q=1$, then

$$\begin{aligned} Q(0,1,1,1,3) &= \int_0^1 x^{-\frac{1}{2}}\chi_1(x)\,\mathrm{Li}_1\left(-x^{\frac{1}{2}}\right)\ \ln(x)\,dx \\ &= 4\sum_{n\geq 1}\frac{H_{4n-1}-H_{2n-\frac{1}{2}}}{(2n-1)(4n-1)^2}+\sum_{n\geq 1}\frac{2H_{2n-\frac{1}{2}}^{(2)}-H_{4n-1}^{(2)}-2\eta(2)}{(2n-1)(4n-1)}. \end{aligned}$$

From the multiplication identity (6) we can write

$$\begin{aligned} Q(0,1,1,1,3) &= 4\sum_{n\geq 1}\frac{2\ln 2-\frac{1}{4n}-H_{4n}+H_{2n}}{(2n-1)(4n-1)^2} \\ &+ \sum_{n\geq 1}\frac{7H_{4n}^{(2)}+\frac{1}{(4n)^2}-2H_{2n}^{(2)}-4\zeta(2)-2\eta(2)}{(2n-1)(4n-1)}. \end{aligned}$$

Evaluating the various Euler sums and highlighting

$$\begin{aligned} \sum_{n\geq 1}\frac{H_{4n}}{(4n-1)^2} &= \frac{1}{2}W(3)-\frac{1}{2}G+\frac{\pi}{8}+\frac{3}{8}\zeta(2)-\frac{7\pi^3}{256}-\frac{3}{4}\ln 2 \\ &+ \frac{1}{4}G\ln 2-\frac{1}{64}\pi\ln^2 2+\frac{21}{32}\zeta(3), \end{aligned}$$

we obtain

$$\begin{aligned} Q(0,1,1,1,3) &= 12W(3)-4G-2\pi+\pi G+8G\ln 2+\frac{19}{4}\zeta(2)-\frac{11\pi^3}{32} \\ &+ 4\ln 2+\frac{1}{2}\pi\ln 2-\frac{21}{4}\zeta(2)\ln 2-\frac{5}{2}\ln^2 2-\frac{3}{8}\pi\ln^2 2+\frac{7}{4}\zeta(3). \end{aligned}$$

5. Let $a=2, b=1, m=q=2, p=1$, then

$$Q(2,1,2,1,2)=\int_0^1 x^2\chi_1(x)\,\mathrm{Li}_2(-x)\,\ln^2(x)\,dx=6\sum_{n\geq1}\frac{H_{2n+2}-H_{n+1}}{(2n-1)(2n+2)^4}$$
$$+2\sum_{n\geq1}\frac{2H^{(2)}_{2n+2}-H^{(2)}_{n+1}-3\eta(2)}{(2n-1)(2n+2)^3}+\frac{1}{2}\sum_{n\geq1}\frac{4H^{(3)}_{2n+2}-H^{(3)}_{n+1}-4\eta(3)}{(2n-1)(2n+2)^2}.$$

After a shift in index and some simplification, we require the evaluation of a number of Euler sums including,

$$\sum_{n\geq1}\frac{H_{2n}}{n(2n-3)} = \frac{1}{3}\zeta(2)+\frac{8}{9}\ln 2-\frac{1}{3}\ln^2 2-\frac{5}{9},$$
$$\sum_{n\geq1}\frac{H^{(2)}_{2n}}{n(2n-3)} = \frac{2}{3}\zeta(3)-\frac{3}{4}\zeta(2)\ln 2-\frac{17}{36}\zeta(2)+\frac{20}{27}\ln 2-\frac{13}{54},$$
$$\sum_{n\geq1}\frac{H^{(3)}_{2n}}{n(2n-3)} = \frac{19}{24}\zeta(4)-\frac{29}{72}\zeta(3)-\frac{1}{2}\zeta(3)\ln 2-\frac{67}{216}\zeta(2)+\frac{56}{81}\ln 2-\frac{35}{324}.$$

To show the process required for the above three identities, we now give a detailed description of another required Euler identity, namely

(17)
$$\sum_{n\geq1}\frac{H^{(3)}_{n}}{n(2n-3)}=\frac{2}{3}\zeta(3)\ln 2-\frac{7}{16}\zeta(4)-\frac{8}{9}\zeta(3)-\frac{49}{27}\zeta(2)+\frac{448}{81}\ln 2-\frac{16}{81}+\frac{8}{3}L(3),$$

where

$$L(3)=\sum_{n\geq1}\frac{(-1)^{n+1}H_n}{n^3}=\frac{11}{4}\zeta(4)-\frac{7}{4}\zeta(3)\ln 2+\frac{1}{2}\zeta(2)\ln^2 2-\frac{1}{12}\ln^4 2-2Li_4\left(\frac{1}{2}\right).$$

By the integral definition of harmonic numbers

$$\sum_{n\geq1}\frac{H^{(3)}_{n}}{n(2n-3)}=\frac{1}{2}\int_0^1\frac{\ln^2 x}{(1-x)}\sum_{n\geq1}\frac{(1-x^n)}{n(2n-3)}dx$$
$$=\frac{1}{2}\int_0^1\frac{\ln^2 x}{(1-x)}\left(2x+2\ln 2-2-\ln(1-x)-2x^{\frac{3}{2}}\tanh^{-1}\left(\sqrt{x}\right)\right)dx$$

(18)
$$=\frac{2}{3}\zeta(3)\ln 2+\frac{1}{12}\zeta(4)-\frac{2}{3}-\frac{1}{3}\int_0^1\frac{\ln^2 x}{(1-x)}\left(x^{\frac{3}{2}}\tanh^{-1}\left(\sqrt{x}\right)\right)dx.$$

Now we apply a Taylor series expansion to the integrand so that,

$$\int_0^1\frac{x^{\frac{3}{2}}\tanh^{-1}(\sqrt{x})}{(1-x)}\ln^2 x\,dx=\int_0^1\sum_{n\geq1}h_n\,x^{\frac{2n-1}{2}+\frac{3}{2}}\ln^2 x\,dx=\sum_{n\geq1}\frac{2h_n}{(n+2)^3},$$

where $h_n = H_{2n} - \frac{1}{2}H_n$. By a change of index and using (4), (7) and (8) we have

$$\int_0^1 \frac{x^{\frac{3}{2}} \tanh^{-1}(\sqrt{x})}{(1-x)} \ln^2 x \, dx = \frac{35}{4}\zeta(4) + \frac{8}{3}\zeta(3) + \frac{49}{9}\zeta(2) - \frac{448}{27}\ln 2 - \frac{38}{27} - 8L(3)$$

and substituting into (18) we obtain (17). Finally

$$\begin{aligned} Q(2,1,2,1,2) &= \int_0^1 x^2 \chi_1(x) \ \mathrm{Li}_2(-x) \ \ln^2(x) \, dx \\ &= \frac{307}{324} - \frac{23}{108}\zeta(2) - \frac{40}{81}\ln 2 - \frac{7}{27}\zeta(2)\ln 2 + \frac{89}{144}\zeta(4) + \frac{1}{27}\ln^2 2 \\ &- \frac{5}{72}\zeta(3) + \frac{1}{8}\zeta(2)\zeta(3) - \frac{7}{18}\zeta(3)\ln 2 - \frac{5}{24}\zeta(5) - \frac{1}{3}L(3). \end{aligned}$$

6. Let $a = 2, b = 1, m = q = 3, p = 0$, then

$$\begin{aligned} Q(2,1,3,0,3) &= \int_0^1 x^2 \chi_0(x) \ \mathrm{Li}_3(-x) \ \ln^3(x) \, dx \\ &= \frac{3}{8}\left(\zeta(4)\eta(3) - \sum_{n\geq 1} \frac{(-1)^{n+1} H_{\frac{n}{2}}^{(4)}}{n^3}\right) \\ &= \frac{135}{32}\zeta(4)\zeta(3) - \frac{483}{128}\zeta(7) - \frac{15}{16}\zeta(5)\zeta(2) \\ &+ \frac{21}{32}\zeta(4) + \frac{9}{16}\zeta(3) + \frac{15}{16}\zeta(2) - 3. \end{aligned}$$

7. Let $a = 0, b = 4, m = q = 1, p = 0$, then

$$\begin{aligned} Q(0,4,1,0,1) &= \int_0^1 \chi_0(x) \ \mathrm{Li}_1\left(-x^4\right) \ \ln(x) \, dx \\ &= \frac{1}{4}\left(\zeta(2)\eta(1) - \sum_{n\geq 1} \frac{(-1)^{n+1} H_{2n}^{(2)}}{n^2}\right) \\ &= \frac{1}{8}\pi G - \frac{1}{2}\zeta(3) + \frac{9}{32}\zeta(2)\ln 2. \end{aligned}$$

8. Let $a = -1, b = 1, q = 0$, then for any even weight $(m+p)$, we can explicitly evaluate

$$\begin{aligned} Q(-1,1,m,p,0) &= \int_0^1 x^{-1} \chi_p(x) \ \mathrm{Li}_0(-x) \ \ln^m(x) \, dx \\ &= \frac{(-1)^m m!}{2^{m+1}} \left(\sum_{n\geq 1} \frac{2^{m+1} H_{2n-1}^{(m+1)} - 2H_{n-\frac{1}{2}}^{(m+1)}}{(2n-1)^p} - 2^m \eta(m+1)\lambda(p) \right), \end{aligned}$$

when $m=2,\ p=6$ we have

$$\begin{aligned}&\int_0^1 x^{-1}\chi_6(x)\ \mathrm{Li}_0(-x)\ \ln^2(x)\,dx\\&=\frac{28105}{512}\zeta(9)-\frac{3207}{128}\zeta(7)\zeta(2)-\frac{675}{64}\zeta(5)\zeta(4)-\frac{189}{128}\zeta(6)\zeta(3).\end{aligned}$$

9. Let $a=0,b=2,m=1,p=1,q=0$ then

$$\begin{aligned}Q(0,2,1,1,0) &= \int_0^1 \chi_1(x)\ \mathrm{Li}_0\left(-x^2\right)\ \ln(x)\,dx\\ &= \frac{1}{4}\sum_{n\geq1}(-1)^{n+1}\left(\begin{array}{c}\frac{6\zeta(2)}{(2n+1)^2}+\frac{16\ln2}{(2n+1)^3}+\frac{8H_n}{(2n+1)^3}\\ -\frac{2}{(2n+1)^2}\left(\zeta(2)-H_n^{(2)}\right)\end{array}\right)\\ &= \frac{3\pi^3}{64}-G\ln2-\frac{1}{4}\zeta(2)+\frac{\pi}{16}\ln^2 2.+\ln2-2W(3).\end{aligned}$$

10. Consider the degenerate case $a=2,b=2,\delta=\frac{1}{2},p=q=0$, then

$$\begin{aligned}\int_0^1 x^2\ \chi_0(x)\ \mathrm{Li}_0\left(-\frac{1}{2}x^2\right)\ \ln^m(x)\,dx &= \int_0^1\left(\frac{x^5}{x^4-x^2+2}\right)\ \ln^m(x)\,dx\\ &= -\sum_{n\geq1}\frac{J_n}{2^n}\int_0^1\sum_{n\geq1}x^{2n+3}\ \ln^m(x)\,dx\end{aligned}$$

where

$$J_n=\sum_{r=1}^{\left[\frac{(n-1)}{2}\right]}2^r\left(\begin{array}{c}n-1-r\\ r\end{array}\right)=\sum_{r=1}^{\left[\frac{n}{2}\right]}\frac{2^r n}{n-r}\left(\begin{array}{c}n-r\\ r\end{array}\right)$$

are the Jacobsthal numbers, or in Binet form, $J_n=\frac{1}{3}\left(2^n-(-1)^n\right)$. Therefore

$$\begin{aligned}\int_0^1 x^2\ \chi_0(x)\ \mathrm{Li}_0\left(-\frac{1}{2}x^2\right)\ \ln^m(x)\,dx &= (-1)^{m+1}m!\sum_{n\geq1}\frac{J_n}{2^{n+m+1}(n+2)^{m+1}}\\ &= \frac{(-1)^{m+1}m!}{3\cdot2^{m+1}}\sum_{n\geq1}\frac{\left(2^n-(-1)^n\right)}{2^n(n+2)^{m+1}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{m+1}m!}{3\cdot 2^{m+1}}\left(\sum_{n\geq 1}\frac{1}{(n+2)^{m+1}}+\sum_{n\geq 1}\frac{(-1)^{n+1}}{2^n(n+2)^{m+1}}\right)\\
&= \frac{(-1)^{m+1}m!}{3\cdot 2^{m+1}}\left(\zeta(m+1)-1-\frac{1}{2^{m+1}}+\frac{1}{2}\Phi\left(-\frac{1}{2},m+1,3\right)\right)\\
&= \frac{(-1)^{m+1}m!}{3\cdot 2^{m+1}}\left(\zeta(m+1)-3-4\mathrm{Li}_{m+1}\left(-\frac{1}{2}\right)\right).
\end{aligned}
$$

11. Finally consider the instructive degenerate case $a=-\frac{3}{2}, b=2, m\in\mathbb{N},\ \delta=\frac{1}{t}$ for $1\leq t\leq -1,\ p=q=0$, then by expansion

$$
J(m,t)=\int_0^1 x^{-\frac{3}{2}}\ \chi_0(x)\ \mathrm{Li}_0\left(-\frac{1}{t}x^2\right)\ \ln^m(x)\,dx=\int_0^1\frac{x^{\frac{3}{2}}}{(x^2-1)(x^2+t)}\ \ln^m(x)\,dx
$$

$$
\begin{aligned}
&= \sum_{n\geq 1}\sum_{j=1}^{n}\frac{(-1)^j}{t^j}\int_0^1 x^{2n-\frac{1}{2}}\ \ln^m(x)\,dx=\sum_{n\geq 1}\sum_{j=1}^{n}\frac{(-1)^{m+j}m!2^{m+1}}{t^j(4n+1)^{m+1}}\\
&= \frac{(-1)^m m!}{(1+t)}\sum_{n\geq 1}\frac{1}{\left(2n+\frac{1}{2}\right)^{m+1}}\left(\frac{(-1)^n}{t^n}-1\right)\\
&= \frac{(-1)^m m!}{2^{m+1}(1+t)}\left(2^{2m+2}-\zeta\left(m+1,\frac{1}{4}\right)-\frac{1}{t}\Phi\left(-\frac{1}{t},m+1,\frac{5}{4}\right)\right).
\end{aligned}
$$

The Hurwitz zeta function $\zeta\left(m+1,\frac{1}{4}\right)=\frac{(-1)^{m+1}}{m!}\psi^{(m)}\left(\frac{1}{4}\right)$, so that in terms of Euler and Bernoulli numbers, see [**17**], we have.
For $m=2u, u\in\mathbb{N}$

$$
\zeta\left(2u+1,\frac{1}{4}\right)=-\frac{1}{(2u)!}\psi^{(2u)}\left(\frac{1}{4}\right)=\frac{2^{2u-1}}{(2u)!}\left((2u)!2^{2u+1}\lambda(2u+1)+\pi^{2u+1}|E_{2u}|\right)
$$

where $\lambda(\cdot)$ is the dirichlet lamba function defined by (3) and E_{2u} are the Euler numbers defined by

$$
\sec(z)=\sum_{n\geq 0}\frac{|E_{2u}|}{(2n)!}z^{2n};\ \ |z|<\frac{\pi}{2}.
$$

For $m=2u-1, u\in\mathbb{N}$

$$
\zeta\left(2u+1,\frac{1}{4}\right)=\frac{1}{(2u-1)!}\psi^{(2u-1)}\left(\frac{1}{4}\right)=\frac{2^{4u-2}}{(2u)!}\left(\pi^{2u}\left(2^{2u}-1\right)|B_{2u}|+2(2u)!\beta(2u)\right),
$$

where $\beta(2u)$ is the Dirichlet beta function, defined by (2) and B_{2u} are the Bernoulli numbers defined by

$$
\tan(z)=\sum_{n\geq 0}\frac{2^{2n}\left(2^{2n}-1\right)|B_{2u}|}{(2n)!}z^{2n-1};\ \ |z|<\frac{\pi}{2}.
$$

Finally

$$J(2u,t) = \int_0^1 x^{-\frac{3}{2}}\ \chi_0(x)\ \mathrm{Li}_0\left(-\frac{1}{t}x^2\right)\ \ln^{2u}(x)\,dx$$
$$= \frac{(2u)!}{2^{2u+1}(1+t)}\left(\begin{array}{c} 2^{4u+2} - \frac{2^{2u-1}}{(2u)!}\left(\begin{array}{c}(2u)!2^{2u+1}\lambda(2u+1)\\ \\ +\pi^{2u+1}|E_{2u}|\end{array}\right)\\ \\ -\frac{1}{t}\Phi\left(-\frac{1}{t},2u+1,\frac{5}{4}\right)\end{array}\right),$$

and

$$J(2u-1,t) = \int_0^1 x^{-\frac{3}{2}}\ \chi_0(x)\ \mathrm{Li}_0\left(-\frac{1}{t}x^2\right)\ \ln^{2u-1}(x)\,dx$$
$$= -\frac{(2u-1)!}{2^{2u}(1+t)}\left(2^{4u} - \frac{2^{4u-2}}{(2u)!}\left(\pi^{2u}\left(2^{2u}-1\right)|B_{2u}| + 2(2u)!\beta(2u)\right) - \frac{1}{t}\Phi\left(-\frac{1}{t},2u,\frac{5}{4}\right)\right).$$

$$J(5,2) = \int_0^1 x^{-\frac{3}{2}}\ \chi_0(x)\ Li_0\left(-\frac{1}{2}x^2\right)\ ln^5(x)\,dx$$
$$= \frac{5}{16}\Phi\left(-\frac{1}{2},6,\frac{5}{4}\right) - 2560 + 1280\beta(6) + 1260\zeta(6),$$

and

$$J(4,5) = \int_0^1 x^{-\frac{3}{2}}\ \chi_0(x)\ Li_0\left(-\frac{1}{5}x^2\right)\ ln^4(x)\,dx$$
$$= 128 - \frac{5}{25}\pi^5 - \frac{1}{40}\Phi\left(-\frac{1}{3},5,\frac{5}{4}\right) - 62\zeta(7).$$

Concluding Remarks: We have carried out a systematic study of the product of the Legendre Chi function, polylogarithm function and log function in terms of Euler sums. We believe most of our results are new in the literature and given many examples which most are not amenable to a mathematical computer package.

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