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COMPARISON INEQUALITIES BETWEEN POLYNOMIALS WITH CONSTRAINTS ON THEIR ZEROS

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The goal of this paper is to obtain some comparison inequalities for a linear operator between polynomials in the plane. The polynomials under study have constraints on their zeros and the estimates obtained turn out to be generalizations of many classical inequalities for polynomials, including the well known Erdős-Lax inequality and inequality of the Ankeny-Rivlin theorem.

1. INTRODUCTION AND PRELIMINARIES

Let P_n be the class of all complex polynomials $P(z) := \sum_{v=0}^{n} a_v z^v$ of degree at most n and $P'(z)$ is the derivative of $P(z)$. A classical majorization result due to Bernstein [**4**] is that, for two polynomials $f(z)$ and $h(z)$ with degree of $f(z)$ not exceeding that of $h(z)$ and $h(z) \neq 0$ for $|z| > 1$, the majorization $|f(z)| \leq |h(z)|$ on the unit circle $|z| = 1$ implies the majorization of their derivatives $|f'(z)| \leq |h'(z)|$ on $|z| = 1$. In particular, this majorization result allows to establish the famous Bernstein inequality [**3**] for the sup-norm on the unit circle: for $P \in P_n$, it is true that

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

The inequality (1) is best possible with equality holding for polynomials $P(z) = \alpha z^n,\ \alpha$ being a complex number.

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Since their appearance a century ago, these inequalities of Bernstein have attracted substantial attention and have been the starting point of a considerable body of literature on polynomial approximations. Over a period, these inequalities were generalized and extended in several directions, in different norms, and for different classes of functions (see, for example, Gardner et al. [**6**], Marden [**11**], Milovanović et al. [**12**], and Rahman and Schmeisser [**16**]). However, with regard to the maximum modulus of $P(z)$ on the circle $|z| = R,\ R \geq 1$, we have another classical result, known as the Bernstein-Walsh lemma ([**16**], Corollary 12.1.3), which asserts that, if $f,\ h \in P_n$ with $deg\ f \leq deg\ h$ and $h(z) \neq 0$ for $|z| > 1$, the majorization $|f(z)| \leq |h(z)|$ on the unit circle $|z| = 1$ implies that $|f(z)| < |h(z)|$ for $|z| > 1$, unless $f(z) = e^{i\theta} g(z)$, $\theta \in \mathbb{R}$. From this, one can deduce that if $P \in P_n$, then for $R \geq 1$,

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \tag{2}$$

Equality holds in (2) if and only if $P(z)$ has all its zeros at the origin. It was shown by Govil, Qazi and Rahman [**8**] that the inequalities (1) and (2) are equivalent in the sense that any of these inequalities can be derived from the other. For the class of polynomials $P \in P_n$, not vanishing in the interior of the unit circle, the above inequalities have been replaced by:

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{3}$$

and

$$\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|, \quad R \geq 1. \tag{4}$$

Both of the above inequalities are sharp and hold with equality for polynomials with all their zeros on the unit circle. As is well known, inequality (3) was conjectured by Erdős and later proved by Lax [**10**], while inequality (4) is due to Ankeny and Rivlin [**1**]. It is topical in geometric function theory to study the extremal problems of functions of a complex variable and generalize the classical polynomial inequalities in various directions. Although the literature on polynomial inequalities is vast and growing, over the years, many authors have produced an abundance of various versions and generalizations of the above inequalities by introducing various operators that preserve such types of inequalities between polynomials (see, for example, [**5**], [**9**], [**13**], and [**14**]). It is an interesting problem, as pointed out by Professor Q. I. Rahman to characterize all such operators, and as part of this characterization, Rahman [**15**] (see also Rahman and Schmeisser [[**16**], page 538-551]) introduced a class B_n of operators B that maps $P \in P_n$ into $B[P] \in P_n$. The study of such operators preserving inequalities between polynomials in geometric function theory is a problem of interest both in mathematics and in application areas such as physical systems. In addition to having numerous applications, this study has been the inspiration for much more research, both from a theoretical and practical point of view. Recently, in 2021, Rather et al. [**17**] considered the generalized B_n operator N, which carries $P \in P_n$ into $N[P] \in P_n$ defined by

$$N[P](z) := \sum_{\nu=0}^{m} \lambda_\nu \left(\frac{nz}{2}\right)^\nu \frac{P^{(\nu)}(z)}{\nu!}, \tag{5}$$

where $\lambda_\nu;\ \nu = 0, 1, 2, ..., m$, are such that all the zeros of

$$\phi(z) = \sum_{\nu=0}^{m} C(n,\nu)\lambda_\nu z^\nu,\ m \leq n, \tag{6}$$

lie in the half plane

$$|z| \leq \left|z - \frac{n}{2}\right|. \tag{7}$$

It can be easily seen that if we take $\lambda_\nu = 0$ in (5) and (6) for $3 \leq \nu \leq m$, the operator N reduces to the B-operator. In the same paper, Rather et al. [**17**] established certain results concerning the upper bound of $|N[P]|$ for $|z| \geq 1$. More precisely, they proved the following results:

Theorem 1. *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P \in P_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then*

$$|N[P](z)| \leq |N[f](z)| \quad \textit{for } |z| \geq 1. \tag{8}$$

Equality in (8) holds for $P(z) = e^{i\gamma} f(z),\ \gamma \in \mathbb{R}$.

Theorem 2. *If $P \in P_n$, and $P(z) \neq 0$ in $|z| < 1$, then*

$$|N[P](z)| \leq \frac{1}{2}\left\{\left|N[\psi_n](z)\right| + |\lambda_0|\right\} \max_{|z|=1} |P(z)| \quad \textit{for } |z| \geq 1, \tag{9}$$

where $\psi_n(z) = z^n$. Equality in (9) holds for $P(z) = az^n + b,\ |a| = |b| \neq 0$.

The Erdős-Lax type inequalities and their extensions are seminal in the field of classical analysis, and here, we are interested in establishing some new inequalities in the uniform norm for the operator N, giving generalizations and refinements of the above results and related inequalities. In the process, the authors thought of a more general problem of investigating the dependence of $\big|N[P(Rz)] - \alpha N[P(rz)] + \beta\big\{\big(\frac{1+R}{1+r}\big)^n - |\alpha|\big\} N[P(rz)]\big|$ for $|z| = 1$ on the maximum and minimum of $|P(z)|$ for every $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r \geq 1$, and develop a unified method for arriving at these results.

2. MAIN RESULTS

Our first result in this direction will be a comparison inequality between complex polynomials involving the operator N, when the zeros of one of the polynomials are restricted. The obtained inequality gives compact generalizations of (1) and (2) and includes Theorem 1 as a special case. More precisely, we first prove the following result:

Theorem 3. *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P \in P_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then for every $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right|$$

$$\leq \left|N[f(Rz)] - \alpha N[f(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[f(rz)]\right| \quad \textit{for } |z| \geq 1. \tag{10}$$

Equality in (10) *holds for $P(z) = e^{i\gamma}f(z)$, $\gamma \in \mathbb{R}$.*

The following result immediately follows from Theorem 3, if we take $r = 1$.

Corollary 4. *If $f(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P \in P_n$ such that $|P(z)| \leq |f(z)|$ for $|z| = 1$, then for every $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq 1$, we have*

$$\left|N[P(Rz)] - \alpha N[P(z)] + \beta\left\{\left(\frac{1+R}{2}\right)^n - |\alpha|\right\}N[P(z)]\right|$$

$$\leq \left|N[f(Rz)] - \alpha N[f(z)] + \beta\left\{\left(\frac{1+R}{2}\right)^n - |\alpha|\right\}N[f(z)]\right| \quad \textit{for } |z| \geq 1. \tag{11}$$

Equality in (11) *holds for $P(z) = e^{i\gamma}f(z)$, $\gamma \in \mathbb{R}$.*

Remark 5. *For $\beta = 0$, Theorem 3 reduces to a result recently proved by Mir in [**13**]. If we take $\alpha = \beta = 0$ in* (11)*, we get Theorem 1.*

If in Theorem 3, we take $f(z) = Mz^n$, where $M = \max_{|z|=1} |P(z)|$, then we get the following result.

Corollary 6. *If $P \in P_n$, then for every $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right|$$

$$\leq \left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\left|N[\varphi_n(z)]\right| \max_{|z|=1}|P(z)| \quad \textit{for } |z| \geq 1, \tag{12}$$

where $\varphi_n(z) = z^n$. Equality in (12) *holds for $P(z) = \gamma z^n$, $\gamma \neq 0$.*

If in (12), after substituting the value of $N[\varphi_n(z)]$, we get for every $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$\begin{aligned} &\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\ (13)\quad &\leq \left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right||z|^n\left|\sum_{\nu=0}^{m}\lambda_\nu\ C(n,\nu)\left(\frac{n}{2}\right)^\nu\right|\max_{|z|=1}|P(z)|, \end{aligned}$$

for $|z| \geq 1$, where $\lambda_\nu;\ 0 \leq \nu \leq m$, are such that all the zeros of $\phi(z)$ defined by (6) lie in the half plane (7). Taking $\lambda_\nu = 0,\ \nu = 1, 2, 3, ..., m$, in (13) and noting that $N[P](z) = \lambda_0 P(z)$, we get the following result.

Corollary 7. *If $P \in P_n$, then for every $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\begin{aligned} &\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}P(rz)\right| \\ (14)\quad &\leq \left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right||z|^n\max_{|z|=1}|P(z)| \quad \text{for } |z| \geq 1. \end{aligned}$$

Equality in (14) holds for $P(z) = \gamma z^n,\ \gamma \neq 0$.

If in (14), we take $\alpha = r = 1$, $\beta = 0$ and divide both sides of it by $R - 1$ and make $R \to 1$, we get

$$|P'(z)| \leq |z|^{n-1}\max_{|z|=1}|P(z)| \quad \text{for } |z| \geq 1,$$

which in particular yields (1), whereas (2) is a special case of (14), if we take $\alpha = \beta = 0$.

Next, we establish an estimate for the lower bound on $|z| \geq 1$ of $\big|N[P(Rz)] - \alpha N[P(rz)] + \beta\{(\frac{1+R}{1+r})^n - |\alpha|\}N[P(rz)]\big|$ in the form of the following result:

Theorem 8. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \leq 1$, then for every $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\begin{aligned} &\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\ (15)\quad &\geq \left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\Big|N[\varphi_n(z)]\Big|m \quad \text{for } |z| \geq 1, \end{aligned}$$

where $\varphi_n(z) = z^n$ and $m = \min_{|z|=1}|P(z)|$.

If in (15), we take $\beta = 0$, we get for every $|\alpha| \leq 1$ and $R \geq r \geq 1$,

$$|N[P(Rz)] - \alpha N[P(rz)]| \geq |R^n - \alpha r^n||N[\varphi_n(z)]|\min_{|z|=1}|P(z)| \quad \text{for } |z| \geq 1,$$

which generalizes the result of Aziz and Dawood ([**2**], Theorem 1).

Theorem 9. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \geq 1$, then for every $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r \geq 1$, we have*

$$\begin{aligned} &\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\ &\quad \leq \frac{1}{2}\left[\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\left|N[\varphi_n(z)]\right|\right. \\ &\quad \left. + |\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\right]\max_{|z|=1}|P(z)| \quad \text{for } |z| \geq 1, \end{aligned} \tag{16}$$

where $\varphi_n(z) = z^n$. Equality in (16) *holds for $P(z) = \gamma z^n + \delta$ with $|\gamma| = |\delta| \neq 0$.*

Remark 10. *For $\alpha = \beta = 0$, Theorem 9 in particular gives Theorem 2 and for suitable choices of λ_ν; $0 \leq \nu \leq m$, it yields inequalities* (3) *and* (4) *as well.*

We now prove the following more refined result which besides strengthens Theorem 9, also provides extensions of Theorem 2 and some results of Aziz and Dawood [**2**].

Theorem 11. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \geq 1$, then for every $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r \geq 1$, we have*

$$\begin{aligned} &\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\ &\quad \leq \frac{1}{2}\left[\left(\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\left|N[\varphi_n(z)]\right|\right.\right. \\ &\quad \left. + |\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\right)\max_{|z|=1}|P(z)| \\ &\quad - \left(\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\left|N[\varphi_n(z)]\right|\right. \\ &\quad \left.\left. - |\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\right)m\right] \quad \text{for } |z| \geq 1, \end{aligned} \tag{17}$$

where $\varphi_n(z) = z^n$ and $m = \min_{|z|=1}|P(z)|$. Equality in (17) *holds for $P(z) = \gamma z^n + \delta$ with $|\gamma| = |\delta| \neq 0$.*

If in (17), after substituting the value of $N[\varphi_n(z)]$, we get for every $|\alpha| \leq 1$,

$|\beta| \leq 1$ and $R \geq r \geq 1$,

$$\begin{aligned}
&\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\
&\quad \leq \frac{1}{2}\Bigg[\Bigg(\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right||z|^n\left|\sum_{\nu=0}^{m}\lambda_\nu\ C(n,\nu)\left(\frac{n}{2}\right)^\nu\right| \\
&\quad + |\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\Bigg)\max_{|z|=1}|P(z)| \\
&\quad - \Bigg(\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right||z|^n\left|\sum_{\nu=0}^{m}\lambda_\nu\ C(n,\nu)\left(\frac{n}{2}\right)^\nu\right| \\
&\quad - |\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\Bigg)m\Bigg] \text{ for } |z| \geq 1,
\end{aligned} \tag{18}$$

where $m = \min_{|z|=1}|P(z)|$ and $\lambda_\nu;\ 0 \leq \nu \leq m$, are such that all the zeros of $\phi(z)$ defined by (6) lie in the half plane (7). Taking $\lambda_\nu = 0,\ \nu = 1, 2, 3, ..., m$, in (18) and noting that $N[P](z) = \lambda_0 P(z)$, we get the following result which is of independent interest, because besides giving generalizations and refinements of (3) and (4), it also provides generalizations of some results of Aziz and Dawood [**2**].

Corollary 12. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \geq 1$, then for $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r \geq 1$, we have*

$$\begin{aligned}
&\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}P(rz)\right| \\
&\quad \leq \frac{1}{2}\Bigg[\Bigg(\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right||z|^n \\
&\quad + \left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\Bigg)\max_{|z|=1}|P(z)| \\
&\quad - \Bigg(\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right||z|^n \\
&\quad - \left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\Bigg)m\Bigg], \text{ for } |z| \geq 1,
\end{aligned} \tag{19}$$

where $m = \min_{|z|=1}|P(z)|$. Equality in (19) holds for $P(z) = \gamma z^n + \delta$ with $|\gamma| = |\delta| \neq 0$.

Remark 13. *Taking $\alpha = r = 1$ and $\beta = 0$ in (19) and divide both sides of it by $R - 1$ and let $R \to 1$, we get in particular a result of Aziz and Dawood ([**2**], Theorem 2), whereas by taking $\alpha = \beta = 0$ and $r = 1$ in (19), it yields a result of Aziz and Dawood ([**2**], Theorem 3).*

A polynomial $P \in P_n$ is said to be self-inversive if $P(z) = \delta Q(z)$, where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ and $|\delta| = 1$. Finally, we prove the following result for self-inversive polynomials.

Theorem 14. *If $P \in P_n$ is self-inversive, then for $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r \geq 1$, we have*

$$\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right|$$

$$\leq \frac{1}{2}\left[\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\left|N[\varphi_n(z)]\right|\right.$$

$$\left. + |\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\right] \max_{|z|=1}|P(z)| \quad \text{for } |z| \geq 1, \tag{20}$$

where $\varphi_n(z) = z^n$. Equality in (20) *holds for $P(z) = z^n + 1$.*

Remark 15. *For $\alpha = \beta = 0$, the above result in particular reduces to a result of Rather et al. ([**17**], Theorem 1.4).*

3. AUXILIARY RESULTS

In order to prove our main results, we need the following lemmas.

Lemma 16. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \leq 1$, then for every $R \geq r \geq 1$, and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{1+R}{1+r}\right)^n |P(rz)|.$$

Proof. The proof of this lemma is similar to the proof of Lemma 2.1 of Govil et al. [**7**], and hence we omit the details. □

If we take $r = s = 1$ and $\sigma = \frac{n}{2}$ in Theorem 1.1 of Rather et al. [**17**], we get the following:

Lemma 17. *If all the zeros of polynomial $P \in P_n$ lie in $|z| \leq 1$, then all the zeros of $N[P(z)]$ defined by* (5) *also lie in $|z| \leq 1$.*

Lemma 18. *If $P \in P_n$, and $P(z) \neq 0$ in $|z| < 1$, then for every complex numbers $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right|$$

$$\leq \left|N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]\right| \quad \text{for } |z| \geq 1, \tag{21}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

Proof. Since $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, therefore, $|P(z)| = |Q(z)|$ for $|z| = 1$. Also, since $P(z) \neq 0$ $|z| < 1$ and hence $Q(z) \neq 0$ in $|z| > 1$, it follows by the maximum modulus principle that $|P(z)| \leq |Q(z)|$ for $|z| \geq 1$. Thus, inequality (21) holds trivially for $R = r$ according to Theorem 1.1. Therefore, we now presume that $R > r$. If λ is any complex number such that $|\lambda| > 1$, it follows by Rouché's theorem that the polynomial $T(z) = P(z) + \lambda Q(z)$ has all its zeros in $|z| \leq 1$. On applying Lemma 16 to the polynomial $T(z)$, we get for $R > r \geq 1$ and for each $0 \leq \theta < 2\pi$,

$$|T(Re^{i\theta})| \geq \left(\frac{1+R}{1+r}\right)^n |T(re^{i\theta})|. \tag{22}$$

Since $T(Re^{i\theta}) \neq 0$ and $\frac{1+R}{1+r} > 1$, for every $R > r \geq 1$, it follows from (22) that

$$\begin{aligned}|T(Re^{i\theta})| &> \left(\frac{1+R}{1+r}\right)^n |T(Re^{i\theta})| \\ &\geq |T(re^{i\theta})|,\end{aligned}$$

which is equivalent to

$$|T(Rz)| > |T(rz)| \quad \text{for } |z| = 1 \text{ and } R > r \geq 1. \tag{23}$$

If α is any complex number with $|\alpha| \leq 1$, we have

$$\begin{aligned}|T(Rz) - \alpha T(rz)| &\geq |T(Rz)| - |\alpha||T(rz)| \\ &\geq \left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}|T(rz)| \quad \text{for } |z| = 1.\end{aligned} \tag{24}$$

Since $T(Rz)$ has all its zeros in $|z| \leq \frac{1}{R} < 1$. Therefore, it follows from inequality (23) by direct application of Rouché's theorem that the polynomial $T(Rz) - \alpha T(rz)$ has all its zeros in $|z| < 1$. Again from inequality (24), by direct application of Rouché's theorem, it follows that all the zeros of the polynomial $T(Rz) - \alpha T(rz) + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}|T(rz)|$ lie in $|z| < 1$, for any complex number β with $|\beta| \leq 1$, and $R > r \geq 1$. Applying Lemma 17 and noting that N is a linear operator, we conclude that all the zeros of the polynomial

$$H(z) := N[T(Rz)] - \alpha N[T(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[T(rz)]$$

lie in $|z| < 1$, for every $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R > r \geq 1$. Replacing $T(z)$ by $P(z) + \lambda Q(z)$, we conclude that all the zeros of the polynomial

$$\begin{aligned}H(z) := {} & N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)] \\ & + \lambda\left[N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]\right]\end{aligned}$$

lie in $|z| < 1$, for all complex numbers $\alpha,\ \beta$ with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R > r \geq 1$. This implies,

$$\begin{aligned}
&\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\
&\leq \left|N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]\right| \quad \text{for } |z| \geq 1.
\end{aligned} \tag{25}$$

If inequality (25) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$\begin{aligned}
&\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\
&\qquad > \left|N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]\right|.
\end{aligned}$$

Taking

$$\lambda = -\frac{N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]}{N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]},$$

so that $|\lambda| > 1$, and with this choice of λ, we have $H(z_0) = 0$ for $|z_0| \geq 1$, which is clear contradiction to the fact that $H(z) \neq 0$ for $|z| \geq 1$. Thus for all complex numbers $\alpha,\ \beta$ with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R > r \geq 1$, the inequality (25) holds and this proves Lemma 18 completely. □

Lemma 19. *If $P \in P_n$, then for $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq r \geq 1$, we have*

$$\begin{aligned}
&\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\
&\qquad + \left|N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]\right| \\
&\qquad \leq \left[\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\left|N[\varphi_n(z)]\right|\right. \\
&\qquad \left. + \left|1 - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right||\lambda_0|\right]\max_{|z|=1}|P(z)| \quad \text{for } |z| \geq 1,
\end{aligned} \tag{26}$$

where $\varphi_n(z) = z^n$ and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

Proof. Let $M = \max_{|z|=1}|P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. If λ is any complex number with $|\lambda| > 1$, then by Rouché's theorem the polynomial $G(z) = P(z) + \lambda M$

has no zeros in $|z|<1$. Applying Lemma 18 to $G(z)$, we have for $|\alpha|\leq 1$, $|\beta|\leq 1$ and $R\geq r\geq 1$,

$$\begin{aligned}&\left|N[G(Rz)]-\alpha N[G(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[G(rz)]\right|\\&\quad\leq\left|N[H(Rz)]-\alpha N[H(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[H(rz)]\right| \text{ for } |z|\geq 1,\end{aligned}$$

where $H(z)=z^n\overline{G(\frac{1}{\bar z})}=Q(z)-\overline{\lambda}z^nM$.
On substituting $G(z)$, $H(z)$ and using the fact that N is linear with $N[1]=\lambda_0$, we get from above inequality for $|\alpha|\leq 1$, $|\beta|\leq 1$, $|z|\geq 1$ and $R\geq r\geq 1$,

$$\begin{aligned}&\Bigg|N[P(Rz)]-\alpha N[P(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[P(rz)]\\&\qquad+\lambda_0\lambda M\left[1-\alpha+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}\right]\Bigg|\\&\leq\Bigg|N[Q(Rz)]-\alpha N[Q(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[Q(rz)]\\&\qquad+\overline{\lambda}M\left[R^n-\alpha r^n+\beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}\right]N[\varphi_n(z)]\Bigg|,\end{aligned}\tag{27}$$

where $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$.
Choosing the argument of λ suitably, which is possible by Corollary 6 such that

$$\begin{aligned}&\Bigg|N[Q(Rz)]-\alpha N[Q(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[Q(rz)]\\&\qquad+\overline{\lambda}M\left[R^n-\alpha r^n+\beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}\right]N[\varphi_n(z)]\Bigg|\\&=|\lambda|M\left|R^n-\alpha r^n+\beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}\right|\Big|N[\varphi_n(z)]\Big|\\&\qquad-\left|N[Q(Rz)]-\alpha N[Q(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[Q(rz)]\right|,\end{aligned}$$

we get from (27) the required result on making $|\lambda|\to 1$. This completes the proof of Lemma 19. $\square$

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 3. The result holds trivially for $R = r$, by virtue of Theorem 1, so we now assume that $R > r$. By Rouché's theorem, the polynomial $T(z) = P(z) - \lambda f(z)$, with $|\lambda| > 1$, has all its zeros in $|z| \leq 1$. On applying Lemmas 16 and 17 and proceeding similarly as in the proof of Lemma 18, the result follows. Hence, we omit the details.

Proof of Theorem 8. Let $m = \min_{|z|=1} |P(z)|$. In case $m = 0$, there is nothing to prove. Assume that $m > 0$, so that all the zeros of $P(z)$ lie in $|z| < 1$ and we have, $m|z|^n \leq |P(z)|$ for $|z| = 1$. Applying Theorem 3 with $f(z)$ replaced by mz^n, we obtain for every $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right|$$
$$\geq \left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\Big|N[\varphi_n(z)]\Big|m \quad \text{for } |z| \geq 1,$$

which is inequality (15). This completes the proof of Theorem 8.

Proof of Theorem 9. Since $P(z) \neq 0$ in $|z| < 1$, therefore by Lemma 18, we have

$$\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right|$$
$$\leq \left|N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]\right|,$$

for $|z| \geq 1$ and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.
Equivalently,

$$2\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right|$$
$$\leq \left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right|$$
$$+ \left|N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]\right|,$$

which on applying Lemma 19, gives the desired result. This completes the proof of Theorem 9.

Proof of Theorem 11. If $P(z)$ has a zero on $|z| = 1$, the result follows from Theorem 9. Henceforth, we assume that $P(z)$ has all its zeros in $|z| > 1$, so that $m = \min_{|z|=1} |P(z)| > 0$. Now for every complex number λ with $|\lambda| < 1$, we have $|\lambda m| < m \leq |P(z)|$ for $|z| = 1$. Therefore, by Rouché's theorem, the polynomial $G(z) = P(z)+\lambda mz^n$ does not vanish in $|z| \leq 1$. Hence $H(z) = z^n\overline{G(\frac{1}{\bar{z}})} = Q(z)+\bar{\lambda}m$, where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, has all its zeros in $|z| \leq 1$ and $|G(z)| = |H(z)|$ for $|z| = 1$.

On applying Lemma 18 and noting that N is a linear operator with $N[1]=\lambda_0$, we get for every $|\alpha|\le 1$, $|\beta|\le 1$, $R\ge r\ge 1$ and $|z|\ge 1$,

$$\begin{aligned}&\left|N[G(Rz)]-\alpha N[G(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[G(rz)]\right|\\&\qquad\le\left|N[H(Rz)]-\alpha N[H(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[H(rz)]\right|.\end{aligned}$$

Equivalently,

$$\begin{aligned}&\Bigg|N[P(Rz)]-\alpha N[P(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[P(rz)]\\&\qquad+\lambda m\left[R^n-\alpha r^n+\beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}\right]N[\varphi_n(z)]\Bigg|\\&\le\Bigg|N[Q(Rz)]-\alpha N[Q(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[Q(rz)]\\&\qquad+\lambda_0\overline{\lambda}m\left[1-\alpha+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}\right]\Bigg|\quad\text{for }|z|\ge 1,\end{aligned}\tag{28}$$

where $\varphi_n(z)=z^n$ and $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$. Choosing the argument of λ in left hand side of (28) such that

$$\begin{aligned}&\Bigg|N[P(Rz)]-\alpha N[P(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[P(rz)]\\&\qquad+\lambda m\left[R^n-\alpha r^n+\beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}\right]N[\varphi_n(z)]\Bigg|\\&=\left|N[P(Rz)]-\alpha N[P(rz)]+\beta\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}N[P(rz)]\right|\\&\qquad+m|\lambda|\left|R^n-\alpha r^n+\beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n-|\alpha|\right\}\right|\Big|N[\varphi_n(z)]\Big|,\end{aligned}$$

we get for $|\alpha| \le 1$, $|\beta| \le 1$ and $R \ge r \ge 1$,

$$\begin{aligned}
&\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\
&+ m|\lambda|\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\left|N[\varphi_n(z)]\right| \\
&\le \left|N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]\right| \\
&+ m|\lambda_0||\lambda|\left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right| \text{ for } |z| \ge 1.
\end{aligned}$$

This gives by using Lemma 19 and making $|\lambda| \to 1$,

$$\begin{aligned}
&\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\
&\le \frac{1}{2}\Bigg[\Bigg(\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\left|N[\varphi_n(z)]\right| \\
&+ |\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\Bigg)\max_{|z|=1}|P(z)| \\
&- \Bigg(\left|R^n - \alpha r^n + \beta r^n\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\left|N[\varphi_n(z)]\right| \\
&- |\lambda_0|\left|1 - \alpha + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}\right|\Bigg)m\Bigg],
\end{aligned}$$

which is the desired inequality and this completes the proof of Theorem 11.

Proof of Theorem 14. Recall that $P \in P_n$ is self-inversive, therefore $P(z) = \delta Q(z)$, where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$ and $|\delta| = 1$. It gives for every $|\alpha| \le 1$, $|\beta| \le 1$ and $R \ge r \ge 1$,

$$\begin{aligned}
&\left|N[P(Rz)] - \alpha N[P(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[P(rz)]\right| \\
&= \left|N[Q(Rz)] - \alpha N[Q(rz)] + \beta\left\{\left(\frac{1+R}{1+r}\right)^n - |\alpha|\right\}N[Q(rz)]\right| \quad \text{for all } z.
\end{aligned}$$

The above inequality combined with Lemma 19 yields (20). This completes the proof of Theorem 14.

5. CONCLUSION

In this paper, we continue the study about the comparison inequalities for a linear operator between polynomials in the plane, following up on a study started by various authors in the recent past. More specifically, we establish some new Erdős-Lax type inequalities for a constrained polynomial.

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