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ENERGY OF A GRAPH IS NEVER THE SQUARE ROOT OF AN ODD INTEGER

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The energy $E(G)$ of a graph G is the sum of the absolute values of the eigenvalues of G. Bapat and Pati (Bull. Kerala Math. Assoc., **1** (2004), 129–132) proved that (a) $E(G)$ is never an odd integer. We now show that (b) $E(G)$ is never the square root of an odd integer. Furthermore, if r and s are integers such that $r \geq 1$ and $0 \leq s \leq r-1$ and q is an odd integer, then $E(G)$ cannot be of the form $(2^s\,q)^{1/r}$, a result that implies both (a) and (b) as special cases.

1. INTRODUCTION

In this paper we are concerned with simple finite graphs. Let G be such a graph and let its order be n. If $\mathbf{A}$ is the adjacency matrix of G, then the eigenvalues of $\mathbf{A}$, denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$, are said to be the eigenvalues of the graph G and to form its spectrum [bf 2].

The *energy* of the graph G is defined as

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \ .$$

This definition was put forward by one of the authors [**3**] and was motivated by earlier results in theoretical chemistry [**4**]. Nowadays the energy of graphs is a much studied quantity in the mathematical literature (see, for example, the recent papers [**5**, **6**]).

In 2004 Bapat and Pati [**1**] communicated an interesting (yet simple) result:

Theorem 1. *The energy of a graph cannot be an odd integer.*

In what follows we demonstrate the validity of a slightly more general result of the same kind:

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Theorem 2. *The energy of a graph cannot be the square root of an odd integer.*

In order to prove Theorem 2 we need some preliminaries.

2. PRELIMINARIES

Using the terminology and notation from the book [**2**], we define two operations with graphs.

By $V(G)$ and $E(G)$ are denoted the vertex and edge sets, respectively, of the graph G.

Let G_1 and G_2 be two graphs with disjoint vertex sets of orders n_1 and n_2, respectively.

The *product* of G_1 and G_2, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ such that two vertices $(x_1, x_2) \in V(G_1 \times G_2)$ and $(y_1, y_2) \in V(G_1 \times G_2)$ are adjacent if and only if $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$.

The *sum* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ such that two vertices $(x_1, x_2) \in V(G_1 + G_2)$ and $(y_1, y_2) \in V(G_1 + G_2)$ are adjacent if and only if either $(x_1, y_1) \in E(G_1)$ and $x_2 = y_2$ or $(x_2, y_2) \in E(G_2)$ and $x_1 = y_1$.

The above specified two graph products have the following spectral properties (see [**2**], p. 70).

Let $\lambda_i^{(1)},\ i = 1, \ldots, n_1$, and $\lambda_j^{(2)},\ j = 1, \ldots, n_2$, be, respectively, the eigenvalues of the graphs G_1 and G_2.

Lemma 1. *The eigenvalues of* $G_1 \times G_2$ *are* $\lambda_i^{(1)} \lambda_j^{(2)}, i = 1, \ldots, n_1\,;\ j = 1, \ldots, n_2$.

Lemma 2. *The eigenvalues of* $G_1 + G_2$ *are* $\lambda_i^{(1)} + \lambda_j^{(2)},\ i = 1, \ldots, n_1\,;\ j = 1, \ldots, n_2$.

The eigenvalues of a graph are zeros of the characteristic polynomial and the characteristic polynomial is a monic polynomial with integer coefficients. Therefore we have:

Lemma 3. *If an eigenvalue of a graph is a rational number, then it is an integer.*

3. PROOF OF THEOREM 2

Consider a graph G and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be its positive eigenvalues. Then in view of the fact that the sum of all eigenvalues of any graph is equal to zero

$$E(G) = 2 \sum_{i=1}^{m} \lambda_i.$$

Denote $\lambda_1 + \lambda_2 + \cdots + \lambda_m$ by λ. By Lemma 1 λ is an eigenvalue of some graph H isomorphic to the sum of m disjoint copies of the graph G. By Lemma 2 λ^2 is an eigenvalues of the product of two disjoint copies of the graph H.

Suppose now that $E(G) = \sqrt{q}$, where q is some integer. Then $2\,\lambda = \sqrt{q}$, i.e., $\lambda^2 = q/4$. If q would be an odd integer, then $q/4$ would be a nonintegral rational number in contradiction to Lemma 3.

Theorem 2 follows. □

We have an immediate extension of Theorem 2:

Observation. *The energy of a graph cannot be the square root of the double of an odd integer.*

However, this observation is just a special case of a somewhat more general result.

4. GENERALIZING THEOREM 2

Let H be the same graph as in the preceding section. Thus λ an eigenvalue of H. Let H^* be the product of r disjoint copies of H. Then by Lemma 2 λ^r is an eigenvalue H^*.

Suppose now that $E(G) = q^{1/r}$, where q is some integer. Then $2\,\lambda = q^{1/r}$, i.e., $\lambda^r = q/2^r$. If q would not be divisible by 2^r, then λ^r would be a nonintegral rational number in contradiction to Lemma 3. Therefore we have:

Theorem 3. *Let r and s be integers such that $r \geq 1$ and $0 \leq s \leq r-1$ and q be an odd integer. Then $E(G)$ cannot be of the form $(2^s\,q)^{1/r}$.*

For $r = 1$ and $s = 0$ Theorem 3 reduces to Theorem 1. For $r = 2$ and $s = 0$ Theorem 3 reduces to Theorem 2. The Observation pertains to $r = 2$ and $s = 1$.

REFERENCES

1. R. B. Bapat, S. Pati: *Energy of a graph is never an odd integer.* Bull. Kerala Math. Assoc., **1** (2004), 129–132.
2. D. Cvetković, M. Doob, H. Sachs: *Spectra of Graphs – Theory and Application.* Academic Press, New York, 1980; 2nd revised ed.: Barth, Heidelberg, 1995.
3. I. Gutman: *The energy of a graph.* Ber. Math. Statist. Sekt. Forshungsz. Graz, **103** (1978), 1–22.
4. I. Gutman, O. E. Polansky: *Mathematical Concepts in Organic Chemistry.* Springer–Verlag, Berlin, 1986.
5. H. Hua, M. Wang: *Unicyclic graphs with given number of pendent vertices and minimal energy.* Lin. Algebra Appl., **426** (2007), 478–489.
6. V. Nikiforov: *Graphs and matrices with maximal energy.* J. Math. Anal. Appl., **327** (2007) 735–738.

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