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IDENTITIES INVOLVING RATIONAL SUMS BY INVERSION AND PARTIAL FRACTION DECOMPOSITION

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Identities appearing recently in: J. L. DÍAZ-BARRERO, J. GIBERGANS-BÁGUENA, P. G. POPESCU: *Some identities involving rational sums.* Appl. Anal. Discrete Math., **1** (2007), 397–402, are treated by inverting them; the resulting sums are evaluated using partial fraction decomposition, following WENCHANG CHU: *A binomial coefficient identity associated with Beukers' conjecture on Apéry numbers.* Electron. J. Combin., **11** (1): Note 15, 3 pp. (electronic), 2004. This approach produces a general formula, not only special cases.

1. INTRODUCTION

The following sums are evaluated in [**2**]:

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{1}{\binom{x+k}{k}}\sum_{1\le i\le j\le k}\frac{1}{x^2+(i+j)x+ij}=\frac{n}{(x+n)^3}, \tag{1}$$

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\mathsf{complicated}(k)=\frac{n}{(x+n)^4}. \tag{2}$$

Here, we present an alternative approach to such identities, which will produce a general formula. It is based on two principles: *inverse pairs* and *partial fraction decomposition.*

Other approaches might also work, but I have chosen the one that I find useful and appealing. Of course, it is not limited to the sums treated in this paper.

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2. INVERSE PAIRS

The following inverse relation is well-known:

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} a_k \longleftrightarrow a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} b_k.$$

The proof is easy, for example, using exponential generating functions.

So, if we want a “nice” answer, like $b_n = \dfrac{n}{(x+n)^2}$, we must compute

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} b_k$$

to find the “complicated” term.

A technical comment: we will treat $x = 0$ as a limiting case, otherwise we would have trouble with b_0, and we would have to artificially define it as 0.

The computation of

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} \frac{k}{(x+k)^2}$$

and similar sums will be treated in the next section.

3. PARTIAL FRACTION DECOMPOSITION

The following approach is based on [**1**]. Consider (for $n \ge 1$)

$$T := \frac{n!}{z(z-1)\cdots(z-n)} \frac{z}{(x+z)^2}$$

and perform partial fraction decomposition:

$$T = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \frac{k}{(x+k)^2} \frac{1}{z-k} + \frac{\lambda}{(x+z)^2} + \frac{\mu}{x+z}.$$

Now we multiply this relation by z and let $z \to \infty$ to find

$$0 = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} \frac{k}{(x+k)^2} + \mu.$$

This evaluates the sum:

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{k}{(x+k)^2} = (-1)^n \mu.$$

Now

$$\begin{aligned}(-1)^n\mu &= (-1)^n\,[(x+z)^{-1}]\,\frac{n!}{z(z-1)\cdots(z-n)}\,\frac{z}{(x+z)^2}\\ &= (-1)^n\,[(x+z)^1]\,\frac{n!}{(z-1)\cdots(z-n)}\\ &= [z^1]\,\frac{n!}{(1+x-z)\cdots(n+x-z)}\\ &= \frac{n!}{(1+x)\cdots(n+x)}\sum_{k=1}^{n}\frac{1}{k+x}.\end{aligned}$$

This produces the identity

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{1}{\binom{k+x}{x}}\sum_{j=1}^{k}\frac{1}{j+x} = \frac{n}{(x+n)^2}.$$

This instance was a warm-up for the general instance $b_n = \frac{n}{(x+n)^{d+1}}$, which is not much more complicated.

Analogous computations lead to

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{k}{(x+k)^{d+1}} = (-1)^n\mu$$

with

$$\begin{aligned}(-1)^n\mu &= (-1)^n\,[(x+z)^{-1}]\,\frac{n!}{z(z-1)\cdots(z-n)}\,\frac{z}{(x+z)^{d+1}}\\ &= (-1)^n\,[(x+z)^d]\,\frac{n!}{(z-1)\cdots(z-n)} = [z^d]\,\frac{n!}{(1+x-z)\cdots(n+x-z)}\\ &= \frac{n!}{(1+x)\cdots(n+x)}\,[z^d]\,\frac{1}{\left(1-\frac{z}{1+x}\right)\cdots\left(1-\frac{z}{n+x}\right)}\\ &= \frac{1}{\binom{x+n}{n}}\,[z^d]\,\exp\left(\log\frac{1}{1-\frac{z}{1+x}}+\cdots+\log\frac{1}{1-\frac{z}{n+x}}\right)\\ &= \frac{1}{\binom{x+n}{n}}[z^d]\exp\left(\sum_{k=1}^{n}\sum_{j\ge1}\frac{z^j}{j(k+x)^j}\right) = \frac{1}{\binom{x+n}{n}}\,[z^d]\,\exp\left(\sum_{j\ge1}\frac{s_{n,j}z^j}{j}\right)\\ &= \frac{1}{\binom{x+n}{n}}\,[z^d]\,\prod_{j\ge1}\sum_{\ell\ge0}\frac{s_{n,j}^{\ell}z^{j\ell}}{\ell!j^{\ell}}\,,\end{aligned}$$

where

$$s_{n,j}=\sum_{k=1}^{n}\frac{1}{(k+x)^j}. \tag{3}$$

Consequently

$$(-1)^n\mu=\frac{1}{\binom{x+n}{n}}\sum_{\ell_1+2\ell_2+3\ell_3+\cdots=d}\frac{s_{n,1}^{\ell_1}s_{n,2}^{\ell_2}\cdots}{\ell_1!\ell_2!\cdots 1^{\ell_1}2^{\ell_2}\ldots}.$$

Therefore, the following result holds.

Theorem 1.

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{k}{(x+k)^{d+1}}=\frac{1}{\binom{x+n}{n}}\sum_{\ell_1+2\ell_2+3\ell_3+\cdots=d}\frac{s_{n,1}^{\ell_1}s_{n,2}^{\ell_2}\cdots}{\ell_1!\ell_2!\cdots 1^{\ell_1}2^{\ell_2}\cdots}$$

where $s_{n,j}$ *is given by* (3).

For $d=2$, we recover the inverse form of (1)

$$\frac{s_{n,1}^2+s_{n,2}}{2}=\sum_{1\le i\le j\le n}\frac{1}{x^2+(i+j)x+ij},$$

as one can easily check. The other instance $d=3$, given in [**2**] evaluates here handily as $s_{n,1}^3/6+s_{n,1}s_{n,2}/2+s_{n,3}/3$.

REFERENCES

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