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JENSEN'S FUNCTIONAL AND POLYNOMIALS

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We investigate some estimates of the Jensen's functional

$$J(p) = \int_0^{2\pi} \log |p(e^{i\theta})| \, \frac{\mathrm{d}\theta}{2\pi},$$

if the polynomial $p(z)$ has a concentration at low degrees measured by ℓ_2-norm. We consider the constants involved, both from a theoretical and a numerical point of view.

1. INTRODUCTION

Let $p(z) = \sum\limits_{j=0}^{n} a_j z^j \not\equiv 0$ be a polynomial with complex coefficient and with ℓ_2-norm: $|p|_{\ell_2} = \left(\sum\limits_{j=0}^{n} |a_j|^2\right)^{1/2}$. Next, let d be a real number such that $0 < d < 1$. We say that $p(z)$ has a concentration d of degrees at most k, measured by the ℓ_2-norm, if the following inequality holds

$$\left(\sum_{j\leqslant k} |a_j|^2\right)^{1/2} \geqslant d \cdot \left(\sum_{j=0}^{n} |a_j|^2\right)^{1/2} = d \cdot |p|_{\ell_2}. \tag{1}$$

There are other ways of measuring such a concentration. For instance,

$$\sum_{j\leqslant k} |a_j| \geqslant d \cdot \sum_{j\geqslant 0} |a_j|, \tag{2}$$

or

$$\sum_{j\leqslant k} |a_j| \geqslant d \cdot \|p\|_\infty, \tag{3}$$

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where $\|p\|_\infty = \max_\theta |p(e^{i\theta})|$. It is not hard to see that the condition (1) is of course more general, since both (2) and (3) imply (1), that is, (2) $\Rightarrow$ (3) $\Rightarrow$ (1). This concept was originally introduced by P. Enflo [**6**], where it was used in order to obtain, for products of polynomials, estimates from below independent of the degrees; the concentration of the polynomial at low degrees measured by ℓ_1-norm under ℓ_1-norm plays an important role in the construction of an operator on a Banach space with no non-trivial invariant subspace [**6**].

2. MAIN RESULTS

We investigate here some asymptotic estimates for the best lower bound of the Jensen's functional $J(p) = \int_0^{2\pi} \log \left| \frac{p(e^{i\theta})}{|p|_{\ell_2}} \right| \frac{\mathrm{d}\theta}{2\pi}$ for a polynomials satisfying (1).

In the sequel, we shall normalize p under ℓ_2-norm and assume that

$$|p|_{\ell_2} = 1. \tag{4}$$

Theorem 1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \not\equiv 0$ *be a polynomial which satisfies* (1) *and* (4). *Then there exists a function*

$$f_{d,k}(t) = t \log d \left(\frac{1 - \left(\frac{t+1}{t-1}\right)^2}{1 - \left(\frac{t+1}{t-1}\right)^{2(k+1)}} \right)^{1/2} - \frac{1}{2}\, t^2, \quad t > 1$$

such that

$$J(p) = \int_0^{2\pi} \log |p(e^{i\theta})| \, \frac{\mathrm{d}\theta}{2\pi} \geqslant f_{d,k}(t) \text{ for each } t > 1.$$

Proof. In the proof of the Theorem 1 we use the following well known relations:

1° $j \in \mathbb{N}_0 \Rightarrow a_j = \int_0^{2\pi} \frac{p(re^{i\theta})}{r^j e^{ij\theta}} \, \frac{\mathrm{d}\theta}{2\pi}$;

2° $|a_j| \leqslant |p(z_0)| \cdot \frac{1}{r^j}, \forall j$, where $|p(z_0)| = \max\{|p(z)| : |z| = r\}$;

3° The classical Jensen's inequality and the known transformation:

$$\log |p(z_0)| \leqslant \int_0^{2\pi} \log \left| p\left(\frac{z_0 + e^{i\theta}}{1 + \overline{z_0} e^{i\theta}}\right) \right| \frac{\mathrm{d}\theta}{2\pi} = \int_0^{2\pi} \frac{1 - r^2}{|1 - \overline{z_0} e^{i\theta}|^2} \log |p(e^{i\theta})| \, \frac{\mathrm{d}\theta}{2\pi}$$

where $|z_0| = r$;

4° If $0 < r < 1$ then $\frac{1-r}{1+r} \leqslant \frac{1 - r^2}{|1 - \overline{z_0} e^{i\theta}|^2} \leqslant \frac{1+r}{1-r}$, where $|z_0| = r$;

5° $\displaystyle\int_0^{2\pi} \log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi} = \int\limits_{\log|p|<0} + \int\limits_{\log|p|>0}.$

Now, according to (1), 1° and 2°, we have

$$d \leqslant \Big(\sum_{j\leqslant k}|a_j|^2\Big)^{1/2} \leqslant \Bigg(\sum_{j\leqslant k}\bigg|\int_0^{2\pi}\frac{p(re^{i\theta})}{r^j e^{ij\theta}}\,\frac{\mathrm{d}\theta}{2\pi}\bigg|^2\Bigg)^{1/2} \leqslant \Big(\sum_{j\leqslant k}|p(z_0)|^2\cdot\frac{1}{r^{2j}}\Big)^{1/2} = |p(z_0)|\cdot\left(\frac{1-\frac{1}{r^{2(k+1)}}}{1-\frac{1}{r^2}}\right)^{1/2}. \tag{5}$$

From (5) it follows:

$$\begin{aligned}\log d &\leqslant \log|p(z_0)| + \frac{1}{2}\log\frac{1-\frac{1}{r^{2(k+1)}}}{1-\frac{1}{r^2}}\\ &\leqslant \int_0^{2\pi}\frac{1-r^2}{|1-\overline{z_0}\,e^{i\theta}|^2}\log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi} + \frac{1}{2}\log\frac{1-\frac{1}{r^{2(k+1)}}}{1-\frac{1}{r^2}}\\ &= \int\limits_{\log|p|>0} + \int\limits_{\log|p|<0} + \frac{1}{2}\log\frac{1-\frac{1}{r^{2(k+1)}}}{1-\frac{1}{r^2}}.\end{aligned} \tag{6}$$

Since $p(z)$ satisfies (4), then

$$\begin{aligned}\int\limits_{\log|p|>0}\log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi} &= \frac{1}{2}\int\limits_{\log|p|>0}\log|p(e^{i\theta})|^2\,\frac{\mathrm{d}\theta}{2\pi} < \frac{1}{2}\int\limits_{\log|p|>0}|p(e^{i\theta})|^2\,\frac{\mathrm{d}\theta}{2\pi}\\ &\leqslant \frac{1}{2}\int_0^{2\pi}|p(e^{i\theta})|^2\,\frac{\mathrm{d}\theta}{2\pi} = \frac{1}{2}\,||p||_{L_2}^2 = \frac{1}{2}|p|_{\ell_2}^2 = \frac{1}{2}.\end{aligned} \tag{7}$$

Using 3°, 5° and the relation (7) we get:

$$\begin{aligned}\log|p(z_0)| &\leqslant \int_0^{2\pi}\frac{1-r^2}{|1-\overline{z_0}\,e^{i\theta}|^2}\log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi}\\ &= \int\limits_{\log|p|>0}\frac{1-r^2}{|1-\overline{z_0}\,e^{i\theta}|^2}\log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi} + \int\limits_{\log|p|<0}\frac{1-r^2}{|1-\overline{z_0}\,e^{i\theta}|^2}\log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi}\\ &\leqslant \frac{1}{2}\cdot\frac{1+r}{1-r} + \frac{1-r}{1+r}\int_0^{2\pi}\log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi}.\end{aligned}$$

Then the last relation and relation (6) yield

$$\int_0^{2\pi} \log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi} \geqslant \frac{1+r}{1-r}\log d\left(\frac{1-\dfrac{1}{r^2}}{1-\dfrac{1}{r^{2(k+1)}}}\right)^{1/2} - \frac{1}{2}\left(\frac{1+r}{1-r}\right)^2.$$

Finally, putting $t = \dfrac{1+r}{1-r}$ in the previous inequality, we obtain the statement of Theorem 1.

Remark. Taking for instance $r = \dfrac{1}{3}$, we obtain the rough estimate:

$$\int_0^{2\pi} \log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi} \geqslant 2\log\frac{2\sqrt{2}\,d}{\sqrt{3^{2(k+1)}-1}} - 2$$

which is a generalization of the classical Jensen's inequality, for polynomials satisfying (1) and (4). From this it follows that there exists

$$C(d,k) := \inf\left\{\int_0^{2\pi} \log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi} : p \text{ satisfies (1) and (4)}\right\}.$$

For $k = 0$, the function is $f_{d,0}(t) = t\log d - \dfrac{1}{2}t^2$, the maximum of which is $\log d - \dfrac{1}{2}$. Hence, $C(d,0) \geqslant \log d - \dfrac{1}{2}$. In the meantime, in this case $C(d,0) = \log d$. Otherwise, we do not know the precise value of the best constant $C(d,k)$, for $k > 0$, but we obtain some asymptotic estimates, when $k \to +\infty$.

The precise value of $C(d,k)$ is known only in two following cases:

1) For the Hurwitz polynomials (that is polynomials with real positive coefficient, such that the roots have negative real part) which satisfies (2), the best constant $C^H_{d,k}$ was determined by Rigler-Trimble-Varga [**16**]: $C^H_{d,k} = \log\dfrac{\rho}{(\rho+1)2^{n-1}}$, where n is unique integer satisfying $\dfrac{1}{2^n}\sum_{j=0}^{k}\binom{n}{j} \leqslant d < \dfrac{1}{2^{n-1}}\sum_{j=0}^{k}\binom{n-1}{j}$, and $\rho = \dfrac{\binom{n-1}{k}}{\sum_{j=0}^{k}\binom{n-1}{j} - d\,2^{n-1}}$.

2) For functions in H^∞ which satisfy (3) the best constant $C^{H^\infty}_{d,1}$ was determined by Beauzamy [3]: $C^{H^\infty}_{d,1}$ is the unique number $c < 0$ solution of the equation $e^c(1-2c) = d$.

The following theorem provides a lower bound for the best constant $C(d,k)$:

Theorem 2. *The best constant $C(d,k)$ satisfies*

$$\limsup_{k\to+\infty}\frac{C(d,k)}{-2k} \leqslant 1,$$

that is $-2k \leqslant C(d,k)$ *asymptotically when* $k \to +\infty$*, for each fixed* $d \in]0,1[$.

Proof. It is obvious that $\lim_{t\to 1+} f_{d,k}(t) = -\infty$ and $\lim_{t\to +\infty} f_{d,k}(t) = -\infty$; therefore a maximum exists.

We write the function $f_{d,k}(t)$ in the form:

$$f_{d,k}(t) = \varphi_{d,k}(t) - \frac{t}{2}\log\left(1 - \left(\frac{t-1}{t+1}\right)^{2(k+1)}\right)$$

where

$$\varphi_{d,k}(t) = t\log d - kt\log(t+1) + kt\log(t-1) + \frac{t}{2}\log\left(1 - \left(\frac{t-1}{t+1}\right)^{2}\right) - \frac{1}{2}\,t^2.$$

Since $0 < \frac{t-1}{t+1} < 1$, we have $\log\left(1 - \left(\frac{t-1}{t+1}\right)^{2(k+1)}\right) < 0$ and then

$$f_{d,k}(t) > \varphi_{d,k}(t), \quad t > 1.$$

Now, we shall prove that the function $\varphi_{d,k}(t)$ takes its maximum value at point t_k such that $t_k \to +\infty$, when $k \to +\infty$.

Let

$$\begin{aligned} g_k(t) &= kt\log(t-1) - kt\log(t+1),\\ h_d(t) &= t\log d + \frac{t}{2}\log\left(1 - \left(\frac{t-1}{t+1}\right)^{2}\right) - \frac{1}{2}\,t^2. \end{aligned}$$

Now, taking the first and the second derivatives of $g_k(t)$ and $h_d(t)$ with respect to t, we get:

$$\begin{aligned} g_k'(t) &= k\log(t-1) - k\log(t+1) + \frac{2kt}{t^2-1},\\ g_k''(t) &= -\frac{4k}{(t^2-1)^2}, \end{aligned} \tag{8}$$

$$\begin{aligned} h_d'(t) &= \log d + \frac{1}{2}\log(1-u^2) - \frac{2t}{t^2-1}\cdot\frac{u^2}{1-u^2} - t,\\ h_d''(t) &= \frac{4}{(t^2-1)^2}\cdot\frac{u^2}{1-u^2}\cdot\frac{1-u^2-2t}{1-u^2} - 1, \end{aligned}$$

where is $u = \frac{t-1}{t+1}$, $0 < u < 1$.

Setting $A(t) = 1 - u^2 - 2t$ we obtain $A'(t) = -2\left(u^2\cdot\frac{2}{t^2-1} + 1\right) < 0$. Since $\lim_{t\to 1+} A(t) < 0$, it follows that $A(t) < 0$ for each $t > 1$. From this and from (8), we get $\varphi''_{d,k}(t) < 0$ for each $t > 1$.

Since $\lim\limits_{t\to 1+} \varphi'_{d,k}(t) = +\infty$ and $\lim\limits_{t\to +\infty} \varphi'_{d,k}(t) = -\infty$, there exists $t_k > 1$ such that $\varphi'_{d,k}(t) = 0$. From the equality $\varphi'_{d,k}(t) = 0$ for $t = t_k$, we get

$$\begin{aligned} k &= \frac{(t^2-1)h'_d(t)}{(t^2-1)\log(t+1) - (t^2-1)\log(t-1) - 2t} \\ &= \frac{h'_d(t)}{\log(t+1) - \log(t-1) - \dfrac{2t}{t^2-1}} \end{aligned}$$

from which we easily deduce that $t_k \to +\infty$ if and only if $k \to +\infty$.

Indeed, writing $\log(t\pm 1) = \log t + \log\left(1 \pm \frac{1}{t}\right)$, $(t\pm 1)^2 = t^2\left(1\pm\frac{1}{t}\right)^2$ and substituting the Taylor expansion of order 3 when $t \to +\infty$, we obtain:

$$\begin{aligned} \log(t+1) - \log(t-1) - \frac{2t}{t^2-1} &= -\frac{4}{3t^3}(1+o(1)) \sim -\frac{4}{3t^3}, \\ h'_d(t) &= -t(1+o(1)) \sim -t. \end{aligned}$$

It follows that $k \sim \dfrac{3t^4}{4}$ when $t \to +\infty$.

From this it follows that the remaining term $\dfrac{t}{2}\log\left(1 - \left(\dfrac{t-1}{t+1}\right)^{2(k+1)}\right)$ can be neglected, for $t = t_k$, when $k \to +\infty$. So we have shown that at the point t_k, the value of $f_{d,k}(t)$ and the value of $\varphi_{d,k}(t)$ are asymptotically the same. All we have to do now is to compute $f_{d,k}(t_k)$:

$$\begin{aligned} f_{d,k}(t_k) \sim \varphi_{d,k}(t_k) &= g_k(t_k) + h_d(t_k) \\ &= kt_k\log\left(1 - \frac{1}{t_k}\right) - kt_k\log\left(1+\frac{1}{t_k}\right) + t_k\log d \\ &\quad + \frac{t_k}{2}\log\left(\left(1+\frac{1}{t_k}\right)^2 - \left(1-\frac{1}{t_k}\right)^2\right) - t_k\log\left(1+\frac{1}{t_k}\right) - \frac{1}{2}t_k^2 \\ &= -2k\left(1 - \frac{t_k}{2k}\log d + \frac{1}{3t_k^2} - \frac{t_k}{4k}\log 4 + \frac{t_k}{4k}\log t_k \right. \\ &\quad \left. + \frac{1}{2k} - \frac{1}{4kt_k^2} + \frac{1}{6kt_k^2} + \frac{t_k^2}{4k} + o\left(\frac{1}{t_k^2}\right)\right) \\ &= -2k(1+o(1)) \sim -2k, \end{aligned}$$

because $t \sim \left(\dfrac{4k}{3}\right)^{1/4}$ when $k \to +\infty$.

This proves the asymptotic estimate: $C(d,k) \geqslant f_{d,k}(t_k) \sim -2k$, $k \to +\infty$, for each fixed $d \in]0,1[$, that is $\limsup\limits_{k\to+\infty} \dfrac{C(d,k)}{-2k} \leqslant 1$.

In the sequel we obtain an upper bound for the best constant $C(d,k)$:

Theorem 3. *The best constant* $C(d,k)$ *satisfies*

$$\liminf_{k\to+\infty}\frac{C(d,k)}{-2k}\geqslant \log 2,$$

that is $C(d,k)\leqslant -2k\log 2$, *asymptotically, when* $k\to+\infty$, *for each fixed* $d\in\,]0,1/2]$.

Proof. Now, let us consider the polynomial $p(z)=\frac{1}{|p|_{\ell_2}}\cdot p_1(z)$, where $p_1(z)=\left(\frac{1+z}{2}\right)^{2k+1}$. The polynomial $p(z)$ satisfies (4) and by the properties of the binomial coefficients it has a concentration $d\leqslant 1/\sqrt{2}$ at degrees k, measured by the ℓ_2-norm. Indeed, from

$$\left(\sum_{j\leqslant k}\left|\frac{1}{|p_1|_{\ell_2}\cdot 2^{2k+1}}\binom{2k+1}{j}\right|^2\right)^{1/2}\geqslant d\cdot|p|_{\ell_2}=d\cdot 1$$
$$=d\cdot\left(\sum_{j=0}^{2k+1}\left|\frac{1}{|p_1|_{\ell_2}\cdot 2^{2k+1}}\binom{2k+1}{j}\right|^2\right)^{1/2},$$

it follows

$$\sum_{j\leqslant k}\left|\frac{1}{|p_1|_{\ell_2}\cdot 2^{2k+1}}\binom{2k+1}{j}\right|^2\geqslant 2\cdot d^2\sum_{j\leqslant k}\left|\frac{1}{|p_1|_{\ell_2}\cdot 2^{2k+1}}\binom{2k+1}{j}\right|^2,$$

that is $d\leqslant 1/\sqrt{2}$.

But the constant term is $\frac{1}{|p_1|_{\ell_2}}\cdot\frac{1}{2^{2k+1}}$, the only root is -1, so Jensen's formula says that:

$$\int_0^{2\pi}\log|p(e^{i\theta})|\,\frac{\mathrm{d}\theta}{2\pi}= -\,(2k+1)\log 2-\log|p_1|_{\ell_2}$$
$$=(-2k\log 2)(1+o(1))\sim -2k\log 2\,,\ k\to+\infty.$$

Hence, we have asymptotically: $C(d,k)\leqslant -2k\log 2$, when $k\to+\infty$, for each fixed $d\in\,]0,1/\sqrt{2}\,]$.

According to theorems 2 and 3 it follows that for a fixed $d\in\,]0,1/\sqrt{2}\,]$

$$-2\leqslant\liminf_{k\to+\infty}\frac{C(d,k)}{k}\leqslant\frac{C(d,k)}{k}\leqslant\limsup_{k\to+\infty}\frac{C(d,k)}{k}\leqslant -2\log 2.$$

The above result can be compared with [**16**], (2.18).

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