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A NEW CRITERION FOR MULTIVALENT STARLIKE FUNCTIONS

Zhi-Gang Wang, Neng Xu

The main purpose of the present paper is to derive a new criterion for multivalent starlike functions by applying Jack's lemma.

1. INTRODUCTION

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{+\infty} a_n z^n \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}),$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^*(\rho)$ of p-valent starlike functions of order ρ in $\mathbb{U}$, if it satisfies the inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \rho \qquad (0 \leqq \rho < p\,;\ z \in \mathbb{U}).$$

For simplicity, we write

$$\mathcal{S}_p^*(0) =: \mathcal{S}_p^*.$$

In recent years, Nunokawa *et al.* [**2**], Xu [**3**], Yang [**4, 5, 6**], Yang and Xu [**7**] and other authors obtained some criteria for multivalent starlikeness. In the present paper, we derive a new criterion for multivalent starlike functions by applying Jack's lemma.

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2. PRELIMINARY RESULTS

In order to establish our main result, we need the following lemmas.

Lemma 1. (JACK's lemma [**1**]) *Let $\omega(z)$ be a non-constant analytic function in $\mathbb{U}$ with $w(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0\omega'(z_0) = k\omega(z_0),$$

where $k \geqq 1$ is a real number.

Lemma 2. (see [**2**]) *If $f \in \mathcal{A}_p$ satisfies the inequality*:

$$\Re\left(\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right) > \alpha \qquad (0 \leqq \alpha < 1;\ z \in \mathbb{U}),$$

then

$$f \in \mathcal{S}_p^*(p + \alpha - 1).$$

Lemma 3. (see [**2**]) *If $f \in \mathcal{A}_p$ satisfies the inequality*:

$$\Re\left(1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right) > \alpha \qquad (0 \leqq \alpha < 1;\ z \in \mathbb{U}),$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma(\alpha) + p - 1 \qquad (z \in \mathbb{U}),$$

where

$$\gamma(\alpha) := \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & \left(\alpha \neq \dfrac{1}{2}\right), \\ \dfrac{1}{2\log 2} & \left(\alpha = \dfrac{1}{2}\right). \end{cases}$$

Lemma 4. *Let*

(2.1) $$|\lambda - 1| < b \qquad (\lambda \in \mathbb{C}\ ;\ 0 < b \leqq 1).$$

Then

$$\Re\left(\frac{1}{\lambda}\right) > \frac{1}{1+b}\,.$$

Proof. From the condition (2.1), it easily follows that

(2.2) $$|\lambda - 1|^2 < b^2 \Longrightarrow \Re(\lambda) > \frac{1}{2}\left(1 + |\lambda|^2 - b^2\right) \ \text{ and } \ |\lambda|^2 < (1+b)^2.$$

We thus find from (2.2) that

$$\Re\left(\frac{1}{\lambda}\right) = \frac{\Re(\lambda)}{|\lambda|^2} > \frac{1}{2}\left(\frac{1-b^2}{|\lambda|^2} + 1\right) > \frac{1}{2}\left(\frac{1-b^2}{(1+b)^2} + 1\right) = \frac{1}{1+b}\,. \qquad \square$$

3. MAIN RESULT

We now give our main theorem below.

Theorem. *If* $f \in \mathcal{A}_p$ *satisfies the inequality*:

$$\left| \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} - 1 \right| < \gamma \qquad \left(0 < \gamma \leqq \frac{1}{2}\,;\ z \in \mathbb{U}\right), \tag{3.1}$$

then

$$f \in \mathcal{S}_p^*\,(p-\gamma) \subset \mathcal{S}_p^*.$$

Proof. Let

$$\omega(z) := \frac{(1-\gamma)\,\dfrac{f^{(p-1)}(z)}{zf^{(p)}(z)} - 1}{-\gamma} - 1 \qquad \left(0 < \gamma \leqq \frac{1}{2}\,; z \in \mathbb{U}\right). \tag{3.2}$$

Then the function ω is analytic in $\mathbb{U}$ with $\omega(0) = 0$. We now rewrite (3.2) as follows:

$$\frac{f^{(p-1)}(z)}{zf^{(p)}(z)} = \frac{(-\gamma)\,\omega(z) + 1 - \gamma}{1-\gamma}. \tag{3.3}$$

Differentiating both sides of (3.3) with respect to z logarithmically, we get

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} - 1 = \frac{(-\gamma)\,z\omega'(z)}{(-\gamma)\,\omega(z) + 1 - \gamma}. \tag{3.4}$$

Since $0 < \gamma \leqq \dfrac{1}{2}$, combining (3.1) and (3.4), we find that

$$\left| \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} - 1 \right| = \gamma \left| \frac{z\omega'(z)}{(-\gamma)\,\omega(z) + 1 - \gamma} \right| < \gamma. \tag{3.5}$$

Now, we can claim that $|\omega(z)| < 1$. Indeed, if not, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |\omega(z)| = |\omega(z_0)| = 1.$$

By Lemma 1, we have

$$z_0\omega'(z_0) = k\omega(z_0) = ke^{i\theta}$$

for $0 < \theta < 2\pi$, where $k \geqq 1$. With $z = z_0$, from (3.4), we have

$$\begin{aligned} \left| \frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)} - \frac{z_0 f^{(p+1)}(z_0)}{f^{(p)}(z_0)} - 1 \right| &= \gamma \left| \frac{k}{-\gamma + (1-\gamma)e^{-i\theta}} \right| \\ &\geqq \gamma \left| \frac{1}{-\gamma + (1-\gamma)e^{-i\theta}} \right|. \end{aligned} \tag{3.6}$$

It follows from (3.6) and $0 < \gamma \leqq \frac{1}{2}$ that

$$\left|\frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)} - \frac{z_0 f^{(p+1)}(z_0)}{f^{(p)}(z_0)} - 1\right|^2 \geqq \gamma^2 \left|\frac{1}{-\gamma + (1-\gamma)e^{-i\theta}}\right|^2$$

$$\geqq \frac{\gamma^2}{\big(-\gamma - (1-\gamma)\big)^2} = \gamma^2,$$

this contradicts to (3.5). Thus, we conclude that $|\omega(z)| < 1$, which implies that

$$\left|\frac{(1-\gamma)\,\dfrac{f^{(p-1)}(z)}{z f^{(p)}(z)} - 1}{-\gamma} - 1\right| < 1 \qquad (z \in \mathbb{U}),$$

or equivalently,

$$\left|\frac{f^{(p-1)}(z)}{z f^{(p)}(z)} - 1\right| < 1 - \frac{1-2\gamma}{1-\gamma} \qquad \left(0 < \gamma \leqq \frac{1}{2};\ z \in \mathbb{U}\right). \tag{3.7}$$

It now follows from (3.7) and Lemma 4 that

$$\Re\left(\frac{z f^{(p)}(z)}{f^{(p-1)}(z)}\right) > 1 - \gamma \qquad \left(0 < \gamma \leqq \frac{1}{2};\ z \in \mathbb{U}\right).$$

Thus, by Lemma 2, we know that

$$f \in \mathcal{S}_p^*\,(p-\gamma) \subset \mathcal{S}_p^*.$$ □

Our main result yields

Corollary 1. *If $f \in \mathcal{A}_p$ satisfies the inequality*:

$$\left|\frac{z f^{(p)}(z)}{f^{(p-1)}(z)} - \frac{z f^{(p+1)}(z)}{f^{(p)}(z)} - 1\right| < \gamma \qquad \left(0 < \gamma \leqq \frac{1}{2};\ z \in \mathbb{U}\right),$$

also let $f = z^{p-1} f_1$, where

$$f_1(z) = z + \sum_{n=p+1}^{+\infty} a_n z^{n-p+1} \qquad (p \in \mathbb{N} \setminus \{1\}),$$

then

$$f_1 \in \mathcal{S}_1^*\,(1-\gamma).$$

Corollary 2. *If $f \in \mathcal{A}_p$ satisfies the inequality*:

$$\left|\frac{z f^{(p+1)}(z)}{f^{(p)}(z)} - \frac{z\left(2f^{(p+1)}(z) + z f^{(p+2)}(z)\right)}{f^{(p)}(z) + z f^{(p+1)}(z)}\right| < \gamma \qquad \left(0 < \gamma \leqq \frac{1}{2};\ z \in \mathbb{U}\right), \tag{3.8}$$

then

$$f \in \mathcal{S}_p^*\left(\frac{1-2\delta}{2^{2-2\delta}(1-2^{2\delta-1})}+p-1\right) \subset \mathcal{S}_p^* \qquad \left(\delta := 1-\gamma;\ 0<\gamma<\frac{1}{2}\right),$$

and

$$f \in \mathcal{S}_p^*\left(\frac{1}{2\log 2}+p-1\right) \subset \mathcal{S}_p^* \qquad \left(\gamma = \frac{1}{2}\right).$$

Proof. It follows from (3.8) and the proof of our main theorem that

$$\Re\left(1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right) > 1-\gamma \qquad \left(0<\gamma \leqq \frac{1}{2};\ z\in\mathbb{U}\right).$$

By noting that

$$\frac{1}{2} \leqq 1-\gamma < 1$$

for $0<\gamma\leqq \frac{1}{2}$. Thus, by Lemma 3, we conclude that the assertions of Corollary 2 hold true. □

Finally, we give an example to illustrate our criterion for multivalent starlike functions.

Example. We consider the function h defined by:

$$h(z) = -\frac{2(1-\gamma)}{\gamma}z - \frac{2(1-\gamma)^2}{\gamma^2}\log\left(1-\frac{\gamma}{1-\gamma}z\right) = z^2 + \frac{2\gamma}{3(1-\gamma)}z^3 + \cdots \qquad \left(0<\gamma\leqq\frac{1}{2};\ z\in\mathbb{U}\right).$$

It is easy to verify that

$$\frac{zh''(z)}{h'(z)} = \frac{1}{1-\frac{\gamma}{1-\gamma}z}. \tag{3.9}$$

Differentiating both sides of equation (3.9) with respect to z logarithmically, we get

$$1+\frac{zh'''(z)}{h''(z)} - \frac{zh''(z)}{h'(z)} = \frac{\frac{\gamma}{1-\gamma}z}{1-\frac{\gamma}{1-\gamma}z}. \tag{3.10}$$

It follows from (3.10) that

$$\left|\frac{zh''(z)}{h'(z)} - \frac{zh'''(z)}{h''(z)} - 1\right| < \gamma.$$

By virtue of our criterion for multivalent starlike functions, we conclude that

$$h \in \mathcal{S}_2^*(2-\gamma) \qquad \left(0<\gamma\leqq\frac{1}{2}\right).$$

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School of Mathematics and Computing Science,
Changsha University of Science and Technology,
Changsha 410076, Hunan,
People's Republic of China
E–mail: zhigangwang@foxmail.com

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Department of Mathematics,
Changshu Institute of Technology,
Changshu 215500, Jiangsu,
People's Republic of China
E–mail: xun@cslg.edu.cn