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A FIXED POINT THEOREM FOR MULTI-MAPS SATISFYING AN IMPLICIT RELATION ON METRIC SPACES

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We present a fixed point theorem for multi-valued mapping satisfying an implicit relation on metric spaces.

1. INTRODUCTION AND PRELIMINARIES

In 1922, the Polish mathematician Stefan Banach proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [**6**], Jungck introduced more generalized commuting mappings, called *compatible* mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems (see, [**3**], [**5**], [**7**]–[**13**].

Throughout this paper, let (X,d) be a metric space. Also $B(X)$ is the set of all non-empty bounded subsets of X. Denote for $A,B \in B(X)$

$$\begin{aligned} D(A,B) &= \inf\{d(a,b) : a \in A,\ b \in B\},\\ \delta(A,B) &= \sup\{d(a,b) : a \in A,\ b \in B\}. \end{aligned}$$

If A consists of a single point a, we write $\delta(A,B) = \delta(a,B)$. If B also consists of a single point b, we write $\delta(A,B) = d(a,b)$.

$$H(A,B) = \max\Big\{ \sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\Big\}$$

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for $A, B \in CB(X)$, where $CB(X)$ is the set of all non-empty closed and bounded subsets of X. Note that $D(A,B) \le H(A,B) \le \delta(A,B)$. Function H is a metric on $CB(X)$ and is called a HAUSDORFF metric. It is well known that if X is a complete metric space, then so is the metric space $(CB(X), H)$. The following definition is given by JUNGCK and RHOADES [**7**].

Definition 1. *The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if they commute at coincidence points, i.e., for each point u in X such that $Fu = \{Iu\}$, we have $FIu = IFu$. (Note that the equation $Fu = \{Iu\}$ implies that Fu is a singleton).*

2. IMPLICIT RELATION

Implicit relation on metric space have been used in many articles (see [**1**], [**2**], [**4**], [**9**], [**13**]).

Definition 2. *Let $\mathbb{R}^+$ be the set of all non-negative real numbers and let $\mathcal{T}$ be the set of all continuous functions $T : (\mathbb{R}^+)^5 \to \mathbb{R}$ satisfying the following conditions:*

(C_1) : *$T(t_1, \ldots, t_5)$ is non-decreasing in t_1 and non-increasing in $t_2, \ldots, t_5$.*

(C_2) : *There exists $h \in (0,1)$ such that*

$$T(u, v, v, u, v+u) \le 0 \quad or \quad T(u, v, u, v, v+u) \le 0$$

implies $u \le hv$.

(C_3) : *$T(u, 0, 0, u, u) > 0, T(u, 0, u, 0, u) > 0$ and $T(u, u, 0, 0, 2u) > 0$, for all $u > 0$.*

Now, we give some examples.

EXAMPLE 1. Let $T(t_1, \ldots, t_5) = t_1 - \alpha \max\{t_2, t_3, t_4\} - \beta t_5$, where $\alpha, \beta \ge 0$ and $\alpha + 2\beta < 1$. (C_1): Obvious. (C_2): Let $u > 0$ and $T(u, v, v, u, v+u) = u - \alpha \max\{u, v\} - \beta(v+u) \le 0$. Thus $u \le \max\{(\alpha+\beta)u + \beta v, (\alpha+\beta)v + \beta u\}$. Now if $u \ge v$, then $u \le (\alpha+\beta)u + \beta v \le (\alpha + 2\beta)u$, a contradiction. Thus $u < v$ and $u \le (\alpha+\beta)v + \beta u$ and so $u \le \dfrac{\alpha+\beta}{1-\beta} v$. Similarly, let $u > 0$ and $T(u, v, u, v, v+u) = u - \alpha \max\{u, v\} - \beta(v+u) \le 0$, then we have $u \le \dfrac{\alpha+\beta}{1-\beta} v$. If $u = 0$, then $u \le \dfrac{\alpha+\beta}{1-\beta} v$. Thus (C_2) is satisfying with $h = \dfrac{\alpha+\beta}{1-\beta} < 1$. (C_3): $T(u, 0, 0, u, u) = T(u, 0, u, 0, u) = u(1-\alpha-\beta) > 0$ and $T(u, u, 0, 0, 2u) = u(1-\alpha-2\beta) > 0$, for all $u > 0$. Therefore $T \in \mathcal{T}$.

EXAMPLE 2. Let $T(t_1, \ldots, t_5) = t_1 - m \max\{t_2, t_3, t_4, t_5/2\}$, where $0 \le m < 1$. (C_1): Obvious. (C_2): Let $u > 0$ and $T(u, v, v, u, v+u) = u - m \max\{u, v\} \le 0$. Thus $u \le m \max\{u, v\}$. Now if $u \ge v$, then $u \le mu$, a contradiction. Thus $u < v$ and $u \le mv$. Similarly, let $u > 0$ and $T(u, v, u, v, v+u) = u - m \max\{u, v\} \le 0$, then we have $u \le mv$. If $u = 0$, then $u \le mv$. Thus (C_2) is satisfying with $h = m < 1$. (C_3): $T(u, 0, 0, u, u) = T(u, 0, u, 0, u) = T(u, u, 0, 0, 2u) = u(1-m) > 0$, for all $u > 0$. Therefore $T \in \mathcal{T}$.

3. THE MAIN RESULT

Theorem 1. *Let F, G be mappings of a complete metric space (X, d) into $B(X)$ and f, g be mappings of X into itself satisfying:*

(i) $Fx \subseteq g(X)$, $Gx \subseteq f(X)$ *for every* $x \in X$,

(ii) *The pair* (F, f) *and* (G, g) *are weakly compatible,*

(iii) $T(\delta(Fx, Gy), d(fx, gy), D(fx, Fx), D(gy, Gy), D(fx, Gy)+D(gy, Fx)) \leq 0$

for every x, y in X, where $T \in \mathcal{T}$. Suppose that one of $g(X)$ or $f(X)$ is a closed subset of X, then there exists a unique $p \in X$ such that $\{p\} = \{fp\} = \{gp\} = Fp = Gp$.

Proof. Let x_0 be an arbitrary point in X. By (i), we choose a point x_1 in X such that $y_0 = gx_1 \in Fx_0$. For this point x_1 there exists a point x_2 in X such that $y_1 = fx_2 \in Gx_1$, and so on. Continuing in this manner we can define a sequence $\{x_n\}$ as follows

$$y_{2n} = gx_{2n+1} \in Fx_{2n}, \quad y_{2n+1} = fx_{2n+2} \in Gx_{2n+1},$$

for $n = 0, 1, 2, \ldots$. We prove that sequence $\{y_n\}$ is a Cauchy sequence. From (iii), we have

$$T\Big(\delta(Fx_{2n}, Gx_{2n+1}), d(fx_{2n}, gx_{2n+1}), D(fx_{2n}, Fx_{2n}), D(gx_{2n+1}, Gx_{2n+1}),$$
$$D(fx_{2n}, Gx_{2n+1}) + D(gx_{2n+1}, Fx_{2n})\Big) \leq 0.$$

Using (C_1) we get

$$T\Big(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),$$
$$d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})\Big) \leq 0$$

and so we get

$$T\Big(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),$$
$$d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\Big) \leq 0,$$

that is

$$T(u, v, v, u, v + u) \leq 0,$$

where $u = d(y_{2n}, y_{2n+1})$ and $v = d(y_{2n-1}, y_{2n})$. Hence, from (C_2), there exists $h \in (0, 1)$ such that

$$d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n}).$$

Similarly, from (iii), we have

$$T\Big(\delta(Fx_{2n+2}, Gx_{2n+1}), d(fx_{2n+2}, gx_{2n+1}), D(fx_{2n+2}, Fx_{2n+2}), D(gx_{2n+1}, Gx_{2n+1}),$$
$$D(fx_{2n+2}, Gx_{2n+1}) + D(gx_{2n+1}, Fx_{2n+2})\Big) \leq 0.$$

Thus we have

$$T\Big(d(y_{2n+2},y_{2n+1}),d(y_{2n+1},y_{2n}),d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}),\\ d(y_{2n+1},y_{2n+1})+d(y_{2n},y_{2n+2})\Big)\leq 0.$$

Using (C_1) we have

$$T\Big(d(y_{2n+2},y_{2n+1}),d(y_{2n+1},y_{2n}),d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}),\\ d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})\Big)\leq 0,$$

That is

$$T(u,v,u,v,v+u)\leq 0,$$

where $u=d(y_{2n+2},y_{2n+1})$ and $v=d(y_{2n+1},y_{2n})$. Hence, from (C_2), we have

$$d(y_{2n+2},y_{2n+1})\leq hd(y_{2n+1},y_{2n}).$$

Therefore,

$$d(y_n,y_{n+1})\leq hd(y_{n-1},y_n)\leq\cdots\leq h^n d(y_0,y_1),$$

Thus

$$\begin{aligned}d(y_n,y_m)&\leq d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\cdots+d(y_{m-1},y_m)\\&\leq h^n d(y_0,y_1)+h^{n+1}d(y_0,y_1)+\cdots+h^m d(y_0,y_1)\\&=\frac{h^n-h^m}{1-h}d(y_0,y_1)\\&\leq\frac{h^n}{1-h}\,d(y_0,y_1)\to 0.\end{aligned}$$

Hence the sequence $\{y_n\}$, is a Cauchy sequence in X. By completeness X there exist $p\in X$ such that

$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}y_{2n}=\lim_{n\to\infty}gx_{2n+1}=p\in\lim_{n\to\infty}Fx_{2n},$$

and

$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}y_{2n+1}=\lim_{n\to\infty}fx_{2n+2}=p\in\lim_{n\to\infty}Gx_{2n+1}.$$

Suppose that $g(X)$ is closed, then for some $v\in X$ we have $p=gv\in g(X)$. If set x_{2n},v replacing x,y respectively, in inequality (iii) we get

$$T\Big(\delta(Fx_{2n},Gv),d(fx_{2n},gv),D(fx_{2n},Fx_{2n}),D(gv,Gv),\\ D(fx_{2n},Gv)+D(gv,Fx_{2n})\Big)\leq 0.$$

From (C$_1$), we have

$$T\Big(\delta(y_{2n}, Gv), d(y_{2n-1}, gv), d(y_{2n-1}, y_{2n}), D(gv, Gv),$$
$$D(y_{2n-1}, Gv) + d(gv, y_{2n})\Big) \leq 0.$$

Letting $n \to \infty$, we have

$$T\Big(\delta(p, Gv), d(p, gv), d(p, p), D(p, Gv), D(p, Gv) + d(p, p)\Big) \leq 0.$$

Thus from C_1 we get,

$$T\Big(\delta(p, Gv), 0, 0, \delta(p, Gv), \delta(p, Gv)\Big) \leq 0.$$

That is, $T(u, 0, 0, u, u) \leq 0$, hence from (C$_3$), we get $u = \delta(p, Gv) = 0$. Hence $Gv = \{p\} = \{gv\}$. From weak compatibility of (G, g) , we have $Ggv = gGv$, hence $Gp = \{gp\}$. If set x_{2n}, p replacing x, y respectively, in inequality (iii) we get

$$T\Big(\delta(Fx_{2n}, Gp), d(fx_{2n}, gp), D(fx_{2n}, Fx_{2n}), D(gp, Gp),$$
$$D(fx_{2n}, Gp) + D(gp, Fx_{2n})\Big) \leq 0.$$

From (C$_1$), we have

$$T\Big(d(y_{2n}, gp), d(y_{2n-1}, gp), d(y_{2n-1}, y_{2n}), d(gp, gp), d(y_{2n-1}, gp) + d(gp, y_{2n})\Big) \leq 0.$$

Letting $n \to \infty$, we get

$$T\Big(d(p, gp), d(p, gp), d(p, p), d(gp, gp), d(p, gp) + d(gp, p)\Big) \leq 0.$$

That is, $T(u, u, 0, 0, 2u) \leq 0$, hence from (C$_3$), we have $u = d(p, gp) = 0$. Hence $gp = p$. Therefore, $Gp = \{p\}$. Since $Gp \subseteq f(X)$, then there exists $w \in X$ such that $\{fw\} = Gp = \{gp\} = \{p\}$. Now if set w, p replacing x, y respectively , in inequality (iii) we get

$$T\Big(\delta(Fw, Gp), d(fw, gp), D(fw, Fw), D(gp, Gp), D(fw, Gp) + D(gp, Fw)\Big) \leq 0.$$

and so we have

$$T\Big(\delta(Fw, p), 0, \delta(p, Fw), 0, \delta(p, Fw)\Big) \leq 0.$$

That is, $T(u, 0, u, 0, u) \leq 0$, hence from (C$_3$), we have $u = \delta(Fw, p) = 0$. Hence $Fw = \{p\} = Gp = \{fw\} = \{gp\}$. Since $Fw = \{fw\}$ and the pair (F, f) is weakly compatible, then we obtain $Fp = Ffw = fFw = \{fp\}$. Therefore, we obtain $Fp = Gp = \{fp\} = \{gp\} = \{p\}$.

The proof is similar when $f(X)$ is assumed to be a closed subset of X.

To see that p is unique, suppose that $\{q\} = \{gq\} = \{fq\} = Fq = Gq$. If $p \neq q$, then

$$T\Big(\delta(Fp, Gq), d(fp, gq), D(fp, Fp), D(gq, Gq), D(fp, Gq) + D(gq, Fp)\Big) \leq 0,$$

therefore $T(d(p,q), d(p,q), 0, 0, 2d(p,q) \leq 0$, that is $d(p,q) = 0$. It follows that $p = q$. □

Corollary 1. *Let F, G be mappings of a complete metric space (X,d) into $B(X)$ such that satisfying*:

(iv) $$T\Big(\delta(Fx, Gy), d(x,y), D(x, Fx), D(y, Gy), D(x, Gy) + D(y, Fx)\Big) \leq 0$$

for every x, y in X. Then there exists a unique $p \in X$ such that $\{p\} = Fp = Gp$.

Proof. By Theorem 1, it is enough defined f, g be identity mappings. □

If we combine Theorem 1 with Example 1 we have the following corollary.

Corollary 2. *Let F, G be mappings of a complete metric space (X,d) into $B(X)$ and f, g be mappings of X into itself satisfying*:

(i) *$Fx \subseteq g(X)$, $Gx \subseteq f(X)$ for every $x \in X$,*

(ii) *The pair (F, f) and (G, g) are weakly compatible,*

(iii) $\delta(Fx, Gy) \leq \alpha \max\{d(fx, gy), D(fx, Fx), D(gy, Gy)\} + \beta\big(D(fx, Gy) + D(gy, Fx)\big)$

for every x, y in X, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$. Suppose that one of $g(X)$ or $f(X)$ is a closed subset of X, then there exists a unique $p \in X$ such that $\{p\} = \{fp\} = \{gp\} = Fp = Gp$.

Example 3. Let $X = [0,1]$ endowed with the Euclidean metric d. Define $F, G : X \to B(X)$ and $f, g : X \to X$ as follows:

$$Fx = \{1/2\}, \qquad Gx = \begin{cases} \{1/2\}, & x \in [0, 1/2] \\ (3/8, 1/2], & x \in (1/2, 1] \end{cases},$$

$$fx = \begin{cases} \dfrac{1}{2}, & x \in [0, 1/2] \\ \dfrac{x+1}{4}, & x \in (1/2, 1] \end{cases}, \qquad gx = \begin{cases} 1 - x, & x \in [0, 1/2] \\ 0, & x \in (1/2, 1] \end{cases}.$$

It is clear that $Fx = \{1/2\} \subseteq g(X) = \{0\} \cup [1/2, 1]$, $Gx = (3/8, 1/2] = f(X)$ and $g(X)$ is closed subset of X. Now we consider the following cases:

Case 1. If $x \in [0, 1/2]$ and $y \in [0, 1/2]$, then

$$\delta(Fx, Gy) = 0 \leq \frac{1}{3}\, d(fx, gy).$$

Case 2. If $x \in [0, 1/2]$ and $y \in (1/2, 1]$, then

$$\delta(Fx, Gy) = \frac{1}{8} \leq \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3} d(fx, gy).$$

Case 3. If $x \in (1/2, 1]$ and $y \in [0, 1/2]$, then

$$\delta(Fx, Gy) = 0 \leq \frac{1}{3} d(fx, gy).$$

Case 4. If $x \in (1/2, 1]$ and $y \in (1/2, 1]$, then

$$\delta(Fx, Gy) = \frac{1}{8} \leq \frac{1}{3} \cdot \frac{3}{8} \leq \frac{1}{3} d(fx, gy).$$

Therefore, we obtain

$$\begin{aligned} \delta(Fx, Gy) &\leq \frac{1}{3} d(fx, gy) \\ &\leq \frac{1}{3} \max\left\{ d(fx, gy), D(fx, Fx), D(gy, Gy), \frac{D(fx, Gy) + D(gy, Fx)}{2} \right\} \end{aligned}$$

for all $x, y \in X$. That is, the condition (iii) of Theorem 1 is satisfied with

$$T(t_1, \ldots, t_5) = t_1 - \frac{1}{3} \max\left\{ t_2, t_3, t_4, \frac{1}{2} t_5 \right\}.$$

Also, the coincidence points of F and f are $1/2$ and 1, and it is clear that F and f are commuting at $1/2$ and 1. Similarly, the only coincidence point of G and g is $1/2$, and G and g are commuting at $1/2$. Thus F and f as well as G and g are weakly compatible. Consequently all conditions of Theorem 1 are satisfied and so these mappings have a unique common fixed point on X. On the other hand, if $x_n = \dfrac{1}{2} - \dfrac{1}{2^n}$, so that $\delta(Ggx_n, gGx_n) \to 1/8 \neq 0$ even though $Gx_n, \{gx_n\} \to \{1/2\}$, that is, the mappings G and g are not compatible. Therefore the fixed point results, which have condition of compatibility, are not applicable to this example. For example the results in [**6**], [**8**]–[**10**] and some others.

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