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NEW INTEGRAL AND SERIES REPRESENTATIONS OF THE GENERALIZED MATHIEU SERIES

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By using some recently investigated Fourier sine integral representations for the Mathieu type series (see [**4**]), new integral and series representations are derived here for certain general families of Mathieu type series.

1. INTRODUCTION AND PRELIMINARIES

The following familiar infinite series

$$S(r) = \sum_{n=1}^{+\infty} \frac{2n}{\left(n^2+r^2\right)^2} \qquad \left(r \in \mathbb{R}^+\right) \tag{1.1}$$

is named after Emile Leonard Mathieu (1835-1890), who investigated it in his 1890 work [**7**] on elasticity of solid bodies. Alternating version of (1.1)

$$\widetilde{S}(r) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{2n}{\left(n^2+r^2\right)^2} \qquad \left(r \in \mathbb{R}^+\right) \tag{1.2}$$

was recently introduced by Pogany et al. in [**8**]. Integral representations of (1.1) and (1.2) are given by (see [**5**] and [**8**])

$$S(r) = \frac{1}{r} \int_0^{+\infty} \frac{t \sin(rt)}{e^t - 1}\, \mathrm{d}t, \tag{1.3}$$

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$$\widetilde{S}(r) = \frac{1}{r}\int_0^{+\infty} \frac{t\sin(rt)}{e^t+1}\,\mathrm{d}t. \tag{1.4}$$

Several interesting problems and solutions dealing with integral representations and bounds for the following slight generalization of the MATHIEU series with a fractional power

$$S_\mu(r) = \sum_{n=1}^{+\infty} \frac{2n}{(n^2+r^2)^\mu} \qquad \left(r\in\mathbb{R}^+;\ \mu>1\right) \tag{1.5}$$

can be found in the works by DIANANDA [**3**], TOMOVSKI and TRENCEVSKI [**12**] and CERONE and LENARD [**2**]. Motivated essentially by the works of CERONE and LENARD [**2**] (and QI [**10**]), SRIVASTAVA and TOMOVSKI in [**11**] defined a family of generalized MATHIEU series

$$S_\mu^{(\alpha,\beta)}(r;a) = S_\mu^{(\alpha,\beta)}\left(r;\{a_n\}_{n=1}^{+\infty}\right) = \sum_{n=1}^{+\infty} \frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu} \qquad \left(r,\alpha,\beta,\mu\in\mathbb{R}^+\right), \tag{1.6}$$

where it is tacitly assumed that the positive sequence

$$a = \{a_n\}_{n=1}^{+\infty} = \{a_1,a_2,a_3,\ldots\}\left(\lim_{n\to+\infty} a_n = +\infty\right)$$

is so chosen that the infinite series in definition (1.6) converges, that is, that the following auxiliary series $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{a_n^{\mu\alpha-\beta}}$ is convergent. Comparing the definitions (1.1), (1.5) and (1.6), we see that $S_2(r) = S(r)$ and $S_\mu(r) = S_\mu^{(2,1)}(r,\{k\})$. Furthermore, the special cases $S_2^{(2,1)}(r;\{a_k\})$, $S_\mu(r) = S_\mu^{(2,1)}(r;\{k\})$, $S_\mu^{(2,1)}(r;\{k^\gamma\})$ and $S_\mu^{(\alpha,\alpha/2)}(r;\{k\})$ were investigated by QI [**10**], DIANANDA [**3**], TOMOVSKI [**13**] and CERONE-LENARD [**2**]. Let $\widetilde{S}_\mu^{(\alpha,\beta)}(r;a)$ be an alternating variant of (1.6). In [**8**], [**14**] several integral representations were obtained for (1.6) and its alternating variant in terms of the generalized hypergeometric functions and the BESSEL function of first kind.

2. INTEGRAL FORMULA FOR SUMMATION OF THE POWER SERIES $\displaystyle\sum_{k=1}^{+\infty}\frac{2k}{(k^2+r^2)^{\mu+1}}x^k$

The following FOURIER sine integral formula is well-known (see for example [**1**], page 71)

$$\int_0^{+\infty} \frac{x^{2\nu}}{(x^2+r^2)^{\mu+1}}\sin(xy)\,\mathrm{d}x \tag{2.1}$$

$$= \frac{1}{2}\, r^{2\nu-2\mu} \frac{\Gamma(1+\nu)\,\Gamma(\mu-\nu)}{\Gamma(\mu+1)}\, y\; {}_1F_2\left(\nu+1; \nu+1-\mu, \frac{3}{2}; \frac{r^2y^2}{4}\right)$$

$$+ 4^{\nu-\mu-1}\sqrt{\pi}\, \frac{\Gamma(\nu-\mu)}{\Gamma\left(\mu-\nu+\frac{3}{2}\right)}\, y^{2\mu-2\nu+1}\; {}_1F_2\left(\mu+1; \mu-\nu+\frac{3}{2}, \mu-\nu+1; \frac{r^2y^2}{4}\right),$$

where $(-1 < \Re(\nu) < \Re(\mu)+1,\ \Re(r) > 0)$.

For $\nu = 1/2$ we find from (2.1) that

$$\text{(2.2)}\quad \int_0^{+\infty} \frac{2x}{(x^2+r^2)^{\mu+1}}\, \sin(xy)\, \mathrm{d}x = \sqrt{\pi}\, r^{1-2\mu} \frac{\Gamma\left(\mu-\frac{1}{2}\right)}{2\Gamma(\mu+1)}\, y\; {}_0F_1\left(-; \frac{3}{2}-\mu; \frac{r^2y^2}{4}\right)$$

$$+ \frac{\sqrt{\pi}}{2^{2\mu}} \frac{\Gamma\left(\frac{1}{2}-\mu\right)}{\Gamma(\mu+1)}\, y^{2\mu}\; {}_0F_1\left(-; \mu+\frac{1}{2}; \frac{r^2y^2}{4}\right) \qquad \left(\Re(\mu) > -\frac{1}{2},\ \Re(r) > 0\right).$$

Both functions ${}_1F_2$ and ${}_0F_1$ (the normalized Bessel functions) denote special cases of the generalized hypergeometric functions ${}_pF_q$ (see [**1**]) with $p < q$, of the so-called class of the Bessel-type generalized hypergeometric functions (see [**6**]).

Using the relation (see [**9**], Vol. 1; p. 651)

$$\text{(2.3)}\qquad \sum_{k=1}^{+\infty} G(k)\, x^k = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t)\, \frac{1-x\sin t}{1-2x\cos t+x^2}\, \mathrm{d}t, \qquad |x| < 1$$

where G is the sine Fourier transform of f, we get for $|x| < 1$

$$\text{(2.4)}\quad \sum_{k=1}^{+\infty} \frac{2k}{(k^2+r^2)^{\mu+1}}\, x^k$$

$$= \frac{\sqrt{2}}{2}\, r^{1-2\mu} \frac{\Gamma\left(\mu-\frac{1}{2}\right)}{\Gamma(\mu+1)} \int_0^{+\infty} t\; {}_0F_1\left(-; \frac{3}{2}-\mu; \frac{r^2t^2}{4}\right) \frac{1-x\sin t}{1-2x\cos t+x^2}\, \mathrm{d}t$$

$$+ \frac{\sqrt{2}}{2^{2\mu}} \frac{\Gamma\left(\frac{1}{2}-\mu\right)}{\Gamma(\mu+1)} \int_0^{+\infty} t^{2\mu}\; {}_0F_1\left(-; \mu+\frac{1}{2}; \frac{r^2t^2}{4}\right) \frac{1-x\sin t}{1-2x\cos t+x^2}\, \mathrm{d}t.$$

Especially, for $\mu = 1$ we get the following integral representation:

$$\sum_{k=1}^{+\infty} \frac{2k}{(k^2+r^2)^2}\, x^k$$

$$= \frac{1}{r} \int_0^{+\infty} \left({}_0F_1\left(-; \frac{1}{2}; \frac{r^2t^2}{4}\right) - (rt)\; {}_0F_1\left(-; \frac{3}{2}; \frac{r^2t^2}{4}\right)\right) t\, \frac{1-x\sin t}{1-2x\cos t+x^2}\, \mathrm{d}t$$

$$= \frac{1}{r}\int_0^{+\infty}\left(\sum_{n=0}^{+\infty}\frac{\left(\frac{r^2t^2}{4}\right)^n 2^n}{(2n-1)!!} - (rt)\sum_{n=0}^{+\infty}\frac{\left(\frac{r^2t^2}{4}\right)^n 2^n}{(2n+1)!!}\right) t\,\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t$$

$$= \frac{1}{r}\int_0^{+\infty}\left(\sum_{n=0}^{+\infty}\frac{(rt)^{2n}}{(2n)!} - \sum_{n=0}^{+\infty}\frac{(rt)^{2n+1}}{(2n+1)!}\right) t\,\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t$$

$$= \frac{1}{r}\int_0^{+\infty}\big(\operatorname{ch}(rt) - \operatorname{sh}(rt)\big) t\,\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t$$

$$= \frac{1}{r}\int_0^{+\infty} te^{-rt}\,\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t.$$

3. INTEGRAL FORMULAS FOR SUMMATIONS OF THE POWER SERIES WHOSE TERMS CONTAIN MATHIEU TYPE SERIES

Now in the evaluation of the more general sums like $\sum_{k=1}^{+\infty} S(k)$, $\sum_{k=1}^{+\infty} S_{\mu+1}(k)$, $\sum_{k=1}^{+\infty} S_{\mu}^{(\alpha,0)}\left(k;\left\{n^{2/\alpha}\right\}\right)$, $\sum_{k=1}^{+\infty} S_{\mu}^{(\alpha,\beta)}\left(k;\left\{n^{2/\alpha}\right\}\right)$ and their alternating variants we shall make use of the following Fourier sine integral representations for $S(r)$, $S_{\mu+1}(r)$, $S_{\mu}^{(\alpha,0)}\left(r;\left\{n^{2/\alpha}\right\}\right)$ and $S_{\mu}^{(\alpha,\beta)}\left(r;\left\{n^{2/\alpha}\right\}\right)$ proven recently by Elezovic et al. in [**4**].

We present here some of them:

$$\int_0^{+\infty}\sin(xr)\,S(r)\,\mathrm{d}r = \frac{1}{2}\,\mathrm{P.V.}\left(\int_0^{+\infty}\frac{t}{e^t-1}\ln\left|\frac{t+x}{t-x}\right| dt\right) \qquad (x>0), \tag{3.1}$$

where the Cauchy Principal Value (P.V.) of the last integral is assumed to exist,

$$\int_0^{+\infty}\sin(xr)\,S_{\mu+1}(r)\,\mathrm{d}r = \frac{\sqrt{\pi}\,2^{\frac{1}{2}-\mu}}{\Gamma(\mu+1)}\left(\int_0^{x}\frac{t^{\mu+\frac{1}{2}}}{e^t-1}\,\Theta_s^{(1)}(\mu;x,t)\,\mathrm{d}t + \int_x^{+\infty}\frac{t^{\mu+\frac{1}{2}}}{e^t-1}\,\Theta_s^{(2)}(\mu;x,t)\,\mathrm{d}t\right) \qquad (x>0;\ \mu>0), \tag{3.2}$$

$$\int_0^{+\infty}\sin(xr)\,S_{\mu}^{(\alpha,0)}\big(r;\left\{k^{2/\alpha}\right\}\big)\,\mathrm{d}r = \frac{\sqrt{\pi}\,2^{\frac{3}{2}-\mu}}{\Gamma(\mu)}\left(\int_0^{x}\frac{t^{\mu-\frac{1}{2}}}{e^t-1}\,\Theta_s^{(1)}(\mu;x,t)\,\mathrm{d}t + \int_x^{+\infty}\frac{t^{\mu-\frac{1}{2}}}{e^t-1}\,\Theta_s^{(2)}(\mu;x,t)\,\mathrm{d}t\right) \qquad (x>0;\mu>1/2), \tag{3.3}$$

$$(3.4)\qquad \int_0^{+\infty} \sin(xr)\, S_\mu^{(\alpha,\beta)}\left(r; \left\{k^{2/\alpha}\right\}\right) \mathrm{d}r$$

$$= \frac{2}{\Gamma\left(2\left(\mu-\frac{\beta}{\alpha}\right)\right)} \left(\int_0^x \frac{t^{2\left(\mu-\frac{\beta}{\alpha}\right)-1}}{e^t-1}\, \Phi_s^{(1)}\left(\mu, \mu-\frac{\beta}{\alpha}, \mu-\frac{\beta}{\alpha}+\frac{1}{2}; \frac{x}{2}, \frac{t}{2}\right) \mathrm{d}t \right.$$

$$\left. + \int_x^{+\infty} \frac{t^{2\left(\mu-\frac{\beta}{\alpha}\right)-1}}{e^t-1}\, \Phi_s^{(2)}\left(\mu, \mu-\frac{\beta}{\alpha}, \mu-\frac{\beta}{\alpha}+\frac{1}{2}; \frac{x}{2}, \frac{t}{2}\right) \mathrm{d}t \right)$$

$$\left(x>0; \mu > \max\left\{\frac{2\beta}{\alpha}, \frac{\beta}{\alpha}+\frac{1}{2}\right\}\right),$$

where

$$(3.5)\qquad \Theta_s^{(1)}(\mu; x, t) = \frac{t^{\mu-\frac{1}{2}}}{2^{\mu-\frac{1}{2}}\, x\, \Gamma\left(\mu+\frac{1}{2}\right)}\, {}_2F_1\left(1, \frac{1}{2}; \mu+\frac{1}{2}; \frac{t^2}{x^2}\right)$$

$$(0<t<x; \Re(\mu)>0),$$

$$(3.6)\qquad \Theta_s^{(1)}(\mu; x, t) = \frac{x t^{\mu-\frac{5}{2}}}{2^{\mu-\frac{3}{2}}\, \Gamma\left(\mu-\frac{1}{2}\right)}\, {}_2F_1\left(1, \frac{3}{2}-\mu; \frac{3}{2}; \frac{x^2}{t^2}\right)$$

$$(0<x<t;\ \Re(\mu)>0),$$

$$(3.7)\quad \Phi_s^{(1)}\left(\mu, \mu-\frac{\beta}{\alpha}, \mu-\frac{\beta}{\alpha}+\frac{1}{2}; \frac{x}{2}, \frac{t}{2}\right) = \frac{1}{x}\, {}_3F_2\left(\frac{1}{2}, 1, \mu; \mu-\frac{\beta}{\alpha}, \mu-\frac{\beta}{\alpha}+\frac{1}{2}; \frac{t^2}{x^2}\right)$$

$$\left(0<t<x; 0<\Re(\mu)<\Re\left(2\left(\mu-\frac{\beta}{\alpha}\right)\right)\right),$$

$$(3.8)\qquad \Phi_s^{(2)}\left(\mu, \mu-\frac{\beta}{\alpha}, \mu-\frac{\beta}{\alpha}+\frac{1}{2}; \frac{x}{2}, \frac{t}{2}\right)$$

$$= \frac{2x}{t^2}\, \frac{\left(\mu-\frac{\beta}{\alpha}-1\right)\left(\mu-\frac{\beta}{\alpha}-\frac{1}{2}\right)}{\mu-1}\, {}_3F_2\left(1, 2-\mu+\frac{\beta}{\alpha}, \frac{3}{2}-\mu+\frac{\beta}{\alpha}; \frac{3}{2}, 2-\mu; \frac{x^2}{t^2}\right)$$

$$+ \frac{\sqrt{\pi}}{x}\left(\frac{x^2}{t^2}\right)^{\mu} \frac{\Gamma\left(\mu-\frac{\beta}{\alpha}\right)\Gamma\left(\mu-\frac{\beta}{\alpha}+\frac{1}{2}\right)\Gamma(1-\mu)}{\Gamma\left(-\frac{\beta}{\alpha}\right)\Gamma\left(-\frac{\beta}{\alpha}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\mu\right)}$$

$$\times\ {}_2F_1\left(1+\frac{\beta}{\alpha}, 1+\frac{\beta}{\alpha}-\frac{1}{2}; \frac{1}{2}+\mu; \frac{x^2}{t^2}\right)$$

$$\left(0<x<t; 0<\Re(\mu)<\Re\left(2\left(\mu-\frac{\beta}{\alpha}\right)\right)\right).$$

In the above, ${}_3F_2$ and ${}_2F_1$ (the GAUSS function) are of the class of so-called GAUSS-type generalized hypergeometric functions ${}_pF_q$ with $p = q+1$, see [**6**].

Applying formula (2.3) and the sine FOURIER transforms presented above, for $|x| < 1$ we get the following integral representations:

$$\sum_{k=1}^{+\infty} S\,(k)\,x^k = \frac{1}{2}\int_0^{+\infty}\left(\int_0^{+\infty}\frac{u}{e^u-1}\ln\left|\frac{t+u}{t-u}\right|\mathrm{d}u\right)\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t, \tag{3.9}$$

$$\sum_{k=1}^{+\infty} \widetilde{S}\,(k)\,x^k = \frac{1}{2}\int_0^{+\infty}\left(\int_0^{+\infty}\frac{u}{e^u+1}\ln\left|\frac{t+u}{t-u}\right|\mathrm{d}u\right)\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t, \tag{3.10}$$

$$\begin{aligned}\sum_{k=1}^{+\infty} S_{\mu+1}\,(k)\,x^k = \frac{1}{2^{\mu-1}\Gamma\,(\mu+1)}\int_0^{+\infty}\Bigg(&\int_0^{t}\frac{u^{\mu+\frac{1}{2}}}{e^u-1}\,\Theta_s^{(1)}\,(\mu;t,u)\,\mathrm{d}u\\ &+\int_t^{+\infty}\frac{u^{\mu+\frac{1}{2}}}{e^u-1}\,\Theta_s^{(2)}\,(\mu;t,u)\,\mathrm{d}u\Bigg)\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\sum_{k=1}^{+\infty} \widetilde{S}_{\mu+1}\,(k)\,x^k = \frac{1}{2^{\mu-1}\Gamma\,(\mu+1)}\int_0^{+\infty}\Bigg(&\int_0^{t}\frac{u^{\mu+\frac{1}{2}}}{e^u+1}\,\Theta_s^{(1)}\,(\mu;t,u)\,\mathrm{d}u\\ &+\int_t^{+\infty}\frac{u^{\mu+\frac{1}{2}}}{e^u+1}\,\Theta_s^{(2)}\,(\mu;t,u)\,\mathrm{d}u\Bigg)\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t,\end{aligned} \tag{3.12}$$

$$\begin{aligned}\sum_{k=1}^{+\infty} S_\mu^{(\alpha,0)}\big(k;\{n^{2/\alpha}\}\big)x^k = \frac{1}{2^{\mu-2}\Gamma\,(\mu)}\int_0^{+\infty}\Bigg(&\int_0^{t}\frac{u^{\mu-\frac{1}{2}}}{e^u-1}\,\Theta_s^{(1)}\,(\mu;t,u)\,\mathrm{d}u\\ &+\int_t^{+\infty}\frac{u^{\mu-\frac{1}{2}}}{e^u-1}\,\Theta_s^{(2)}\,(\mu;t,u)\,\mathrm{d}u\Bigg)\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\sum_{k=1}^{+\infty} \widetilde{S}_\mu^{(\alpha,0)}\big(k;\{n^{2/\alpha}\}\big)x^k = \frac{1}{2^{\mu-2}\Gamma\,(\mu)}\int_0^{+\infty}\Bigg(&\int_0^{t}\frac{u^{\mu-\frac{1}{2}}}{e^u+1}\,\Theta_s^{(1)}\,(\mu;t,u)\,\mathrm{d}u\\ &+\int_t^{+\infty}\frac{u^{\mu-\frac{1}{2}}}{e^u+1}\,\Theta_s^{(2)}\,(\mu;t,u)\,\mathrm{d}u\Bigg)\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\sum_{k=1}^{+\infty} S_\mu^{(\alpha,\beta)}\big(k;\{n^{2/\alpha}\}\big)x^k &= \frac{2\sqrt{2}}{\sqrt{\pi}\,\Gamma\left(2\left(\mu-\frac{\beta}{\alpha}\right)\right)}\int_0^{+\infty}\Bigg(\int_0^{t}\frac{u^{2\left(\mu-\frac{\beta}{\alpha}\right)-1}}{e^u-1}\,\Phi_s^{(1)}\,(\mu;t,u)\,\mathrm{d}u\\ &+\int_t^{+\infty}\frac{u^{2\left(\mu-\frac{\beta}{\alpha}\right)-1}}{e^u-1}\,\Phi_s^{(2)}\,(\mu;t,u)\,\mathrm{d}u\Bigg)\frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t,\end{aligned} \tag{3.15}$$

$$(3.16)\qquad \sum_{k=1}^{+\infty} \widetilde{S}_{\mu}^{(\alpha,\beta)}\big(k;\{n^{2/\alpha}\}\big)x^k$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}\,\Gamma\left(2\left(\mu-\frac{\beta}{\alpha}\right)\right)} \int_0^{+\infty} \left(\int_0^t \frac{u^{2\left(\mu-\frac{\beta}{\alpha}\right)-1}}{e^u+1}\,\Phi_s^{(1)}(\mu;t,u)\,\mathrm{d}u \right.$$

$$\left. + \int_t^{+\infty} \frac{u^{2\left(\mu-\frac{\beta}{\alpha}\right)-1}}{e^u+1}\,\Phi_s^{(2)}(\mu;t,u)\,\mathrm{d}u \right) \frac{1-x\sin t}{1-2x\cos t+x^2}\,\mathrm{d}t.$$

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