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SOME PROPERTIES OF THE SEQUENCE OF PRIME NUMBERS

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Let p_n be the n-th prime number and $x_n = p_{n+1}^{n+1}/p_n^n$. We show that the sequence $(x_n)_{n\geq N}$ is not monotonic for any integer $N > 1$ and that the series $\sum\limits_{n=1}^{+\infty} 1/x_n$ is divergent. Related series are studied as well.

1. INTRODUCTION

We use the well-known notation

- $\pi(x)$ – the number of prime numbers $\leq x$,
- p_n – the n-th prime number,
- $d_n = p_{n+1} - p_n$, for $n \geq 1$,
- $f(n) \asymp g(n)$ if there exist $0 < c_1 < c_2$ such that $c_1 f(n) < g(n) < c_2 f(n)$ for n large enough,
- $f(n) \sim g(n)$ if $\lim\limits_{n\to+\infty} \dfrac{f(n)}{g(n)} = 1$.

The following results are also well known:

$$p_n \sim n \log n, \tag{1}$$

$$\sum_{k=1}^{n} \frac{1}{p_k} = \log\log n + O(1). \tag{2}$$

Moreover, we need the following results.

I. *We have*

$$\limsup_{n\to+\infty} \frac{p_{n+1} - p_n}{\log n} = +\infty. \tag{3}$$

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This result can be found in [**6**], but [**4**] contains sharper results, which were later proved.

Erdős and Prachar proved in [**1**] the following theorem:

II. *Let $A(x)$ be the number of indices k such that $x/2 < p_k \le x$ and $p_{k+1} - p_k < (1-\delta)\log x$, then*

$$A(x) > c_1 \frac{x}{\log x} \tag{4}$$

for some $\delta \in (0,1)$ and $c_1 > 0$, and for all $x > 0$ large enough.

Erdős shows in [**3**] the following fact:

III. *There exists $c > 1$ such that the inequality*

$$d_n > cd_{n+1} \tag{5}$$

holds for infinitely many values of n, and the inequality

$$d_{n+1} > cd_n \tag{6}$$

holds for infinitely many values of n as well.

The following result is proved in [**5**].

IV. *If the sequence $(u_n)_{n\ge1}$ is decreasing and consists only of positive numbers, and the sequence $(\alpha_n)_{n\ge1}$ has the property that there exist $M \ge m > 0$ such that $M \ge \dfrac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} \ge m$ for every n, then*

$$M \sum_{k=1}^{n} u_k \ge \sum_{k=1}^{n} \alpha_k u_k \ge m \sum_{k=1}^{n} u_k,$$

and thus the series $\sum\limits_{n=1}^{+\infty} u_n$ and $\sum\limits_{n=1}^{+\infty} \alpha_n u_n$ are equiconvergent.

We shall denote $x_n = \dfrac{p_{n+1}^{n+1}}{p_n^n}$ and we are going to point out some properties of the sequence $(x_n)_{n\ge1}$.

2. THE MONOTONICITY OF THE SEQUENCE $(x_n)_{n\ge1}$

It immediately follows from Theorem III that the sequence $(d_n)_{n\ge1}$ is not monotonic. It is also known that the sequence $(p_{n+1}/p_n)_{n\ge1}$ is not monotonic. Thus the monotonicity problem for the sequence $(x_n)_{n\ge1}$ arises in a natural way. Since $x_n > p_{n+1}$, it follows that $\lim\limits_{n\to+\infty} x_n = +\infty$, hence the sequence $(x_n)_{n\ge1}$ cannot be decreasing. The complete result in this connection is given by

Theorem 1. *The sequence $(x_n)_{n\ge N}$ is not monotonic for any integer $N \ge 1$.*

Proof. It suffices to show that the sequence is nonincreasing. To this end, we show that $x_{n+1} < x_n$ for infinitely many values of n.

We consider only the indices n such that $d_{n-1} > cd_n$ with $c > 1$ (see the theorem III above) and moreover $n > \dfrac{c+1}{c-1}$. We have

$$x_n < x_{n-1} \iff p_{n+1}^{n+1}\, p_{n-1}^{n-1} < p_n^{2n}. \tag{7}$$

Since $d_{n-1} > cd_n$ we deduce $p_n > \dfrac{cp_{n+1} + p_{n-1}}{c+1}$. To prove (7), it suffices to show that $\left(\dfrac{cp_{n+1} + p_{n-1}}{c+1}\right)^{2n} > p_{n+1}^{n+1}\, p_{n-1}^{n-1}$. If we denote $\dfrac{p_{n+1}}{p_{n-1}} = x > 1$, then it remains to show that $\left(\dfrac{cx+1}{c+1}\right)^{2n} > x^{n+1}$, that is,

$$cx - (c+1)\, x^{\frac{n+1}{2n}} + 1 > 0. \tag{8}$$

For $x > 1$ let $f(x) = cx - (c+1)x^{\frac{n+1}{2n}} + 1$. Then $f'(x) = c - \dfrac{n+1}{2n}\,(c+1)x^{\frac{1-n}{2n}} > 0$ because $x > 1$ implies $x^{\frac{1-n}{2n}} \le 1$ while $n > \dfrac{c+1}{c-1}$ implies $\dfrac{(n+1)(c+1)}{2n} < c$.

Consequently, the function f is increasing for $x > 1$. Since $\lim\limits_{x\to 1} f(x) = 0$, the desired inequality (8) follows. □

3. THE SERIES $\sum\limits_{n=1}^{+\infty} 1/x_n$

The series $\sum\limits_{n=1}^{+\infty} \dfrac{1}{p_n}$ is divergent, but (2) shows that the sequence $\sum\limits_{k=1}^{n} \dfrac{1}{p_k}$ tends to infinity fairly slowly. Since $\dfrac{1}{x_n} < \dfrac{1}{p_{n+1}}$, the series $\sum\limits_{n=1}^{+\infty} \dfrac{1}{x_n}$ could be convergent. Moreover we have

$$\frac{1}{x_n} = \frac{1}{p_{n+1}} \cdot \left(\frac{p_n}{p_{n+1}}\right)^n = \frac{1}{p_{n+1}} \cdot \frac{1}{\left((1 + d_n/p_n)^{p_n/d_n}\right)^{nd_n/p_n}}. \tag{9}$$

It now follows by (1) and the result in I that $\limsup\limits_{n\to+\infty} \dfrac{nd_n}{p_n} = +\infty$.

Since $\lim\limits_{n\to\infty} \dfrac{d_n}{p_n} = 0$, we have $\lim\limits_{n\to+\infty} \left(1 + \dfrac{d_n}{p_n}\right)^{\frac{p_n}{d_n}} = e$, so $\liminf\limits_{n\to+\infty} \dfrac{1/x_n}{1/p_{n+1}} = 0$. This could mislead us to conclude that the series $\sum\limits_{n=1}^{+\infty} \dfrac{1}{x_n}$ is convergent. But, we prove the opposite

Theorem 2. *The series* $\sum\limits_{n=1}^{+\infty} \dfrac{1}{x_n}$ *is divergent.*

We first need the following auxiliary result.

Lemma 1. *We have* $\sum\limits_{k=1}^{n} e^{-kd_k/p_k} \asymp n$.

Proof. Let $s_n = \sum\limits_{k=1}^{n} e^{-kd_k/p_k}$. Since $-kd_k/p_k < 0$, we have $s_n < n$.

We put $x = p_n$ in the theorem II, and it follows that there exist $A(p_n)$ indices k such that $p_n/2 < p_k \le p_n$ and $d_k < (1-\delta)\log p_n$. We have $A(p_n) > c_1 \dfrac{p_n}{\log p_n}$ and (1) implies that there exists $c_2 > 0$ such that $A(p_n) > c_2 n$. For these indices k we have

$$e^{-kd_k/p_k} > e^{-(1-\delta)k\log p_n/p_k}. \tag{10}$$

We have that $p_k \sim k\log k \sim k\log p_k$ and, since $p_n/2 < p_k \le p_n$, it follows that $\log p_k \sim \log p_n$, hence $\dfrac{k\log p_n}{p_k} \sim 1$. Consequently $\dfrac{k\log p_n}{p_k} < c_3$ and then by (10) we have $e^{-kd_k/p_k} > e^{-c_3(1-\delta)} = c_4$. This implies $s_n \ge A(p_n)\cdot c_4 > c_2c_4 n$ and, since $s_n < n$, we get $s_n \asymp n$. □

Proof of Theorem 2. Let $\alpha_k = e^{-kd_k/p_k}$ and $u_k = 1/p_{k+1}$. Since $\sum\limits_{k=1}^{n} \alpha_k \asymp n$ and the series $\sum\limits_{n=1}^{+\infty} \dfrac{1}{p_{n+1}}$ is divergent, the property IV implies that the series $\sum\limits_{n=1}^{+\infty} \dfrac{e^{-nd_n/p_n}}{p_{n+1}}$ is divergent.

Since for $x > 0$ we have $(1+x)^{1/x} < e$, we get from (9)

$$\frac{1}{x_n} > \frac{1}{p_{n+1}} \cdot \frac{1}{e^{nd_n/p_n}} \tag{11}$$

and the divergence of the series $\sum\limits_{n=1}^{+\infty} \dfrac{1}{x_n}$ follows. □

Remark 1. The above result can be stated in a more precise form. With the above notation we have $s_n/n > c_2c_4$. It follows by IV that

$$\sum_{k=1}^{n} \frac{e^{-kd_k/p_k}}{p_{k+1}} > c_2c_4 \sum_{k=1}^{n} \frac{1}{p_{k+1}}$$

hence by (2) and (10) we have $S_n = \sum\limits_{k=1}^{n} \dfrac{1}{x_k} > c_5 \log\log n$ with $c_5 > 0$.

Since $\dfrac{1}{x_k} < \dfrac{1}{p_{k+1}}$, it follows that $S_n < \sum\limits_{k=1}^{n} \dfrac{1}{p_k} < c_6 \log\log n$. Thus we have $S_n \asymp \log\log n$.

Remark 2. Since $\dfrac{p_{n+1}^n}{p_n^{n+1}} > \dfrac{1}{x_n}$, we conclude that the series $\sum\limits_{n=1}^{+\infty} \dfrac{p_{n+1}^n}{p_n^{n+1}}$ is divergent. We denote $\sigma_n = \sum\limits_{k=1}^{n} \dfrac{p_{k+1}^k}{p_k^{k+1}}$ and it follows that $\sigma_n > S_n > c_5 \log\log n$. In this this regard we may raise the following

Open Problem. Is it true that $\sigma_n \asymp \log\log n$?

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