Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **2** (2008), 241–254. doi:10.2298/AADM0802241B

THE CURL OF A WEIGHTED NETWORK

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In this work we introduce an accurate definition of the curl operator on weighted networks that completes the discrete vector calculus developed by the authors. This allows us to define the circulation of a vector field along a curve and to characterize the conservative fields. In addition, we obtain an adequate discrete version of the De Rham cohomology of a compact manifold, giving in particular discrete analogues of the Poincaré and Hodge's decomposition theorems.

1. INTRODUCTION

The discrete vector calculus theory is a very fruitful area of work in many mathematical branches not only for its intrinsic interest but also for its applications, [**1**, **2**, **5**, **8**, **10**, **12**, **14**]. There exist different approaches and we want to mention mainly three of them. The first one is based on simplicial complexes that approximate locally a smooth manifold [**8**], the second one consists in the so-called mimetic methods see [**2**] and the references therein. The last approach deals with the mesh as the only existent space and then the discrete vector calculus is described through Algebraic Topology tools since the geometric realization of the mesh is a unidimensional CW-complex, [**1**, **4**, **9**, **10**, **14**]. Our work falls within the last ambit but, instead of importing the tools from Algebraic Topology, we construct the discrete vector calculus from the graph structure itself following the guidelines of Differential Geometry, see [**2**, **3**]. In these works we developed a discrete calculus by defining the tangent space at each vertex and hence the concepts of discrete vector fields and bilinear forms are a likely result of the definition of tangent space. Moreover, we obtained discrete versions of the derivative, gradient, divergence and Laplace-Beltrami operators that satisfy the same properties that its differential analogues.

In this work we complete our discrete vector calculus on weighted networks by introducing the curl operator. This allows us to define the circulation of a vector

2000 Mathematics Subject Classification. 39A12.
Keywords and Phrases. Networks, Laplacian, curl, Hodge's decomposition, Helmholtz's decomposition.

field along a curve and to characterize the conservative fields. In addition, we obtain an adequate discrete version of the De Rham cohomology of a compact manifold, giving in particular discrete analogues of the Poincaré and Hodge decomposition theorems. We must remark that unlike other works, here it is not necessary to provide the weighted network with an orientation to develop a satisfactory discrete vector calculus. Moreover, our approach is based on the consideration of general vector fields whereas other authors consider only symmetric or antisymmetric fields, see [**1**, **4**, **6**]. Therefore, they can not consider the curl operator and consequently the Hodge Laplacian and Hodge's decomposition do not contain all the possible information about the network. In fact, the version of Hodge's decomposition theorem here obtained represents not only a mimetic version of its continuous counterpart but a satisfactory answer to the main question raised by K. Gustafson and F. Harary in [**7**]. We end with the application of our developments to purely resistive networks and to the study of difference schemes on n-dimensional uniform grids.

2. PRELIMINARIES

Throughout the paper, $\Gamma = (V, E)$ denotes a simple and finite connected graph without loops, with vertex set V and edge set E. Two different vertices, $x, y \in V$, are called *adjacent*, which is represented by $x \sim y$, if $\{x, y\} \in E$ and then the edge $\{x, y\}$ is denoted by e_{xy}. We denote by $\mathcal{C}(V)$, $\mathcal{C}(E)$, $\mathcal{C}(V \times V)$ and $\mathcal{C}(V \times V \times V)$, the vector spaces of real functions defined on the sets that appear between brackets. A function $\nu \in \mathcal{C}(V)$ is called a *weight on* V if $\nu(x) > 0$ for all $x \in V$. We also consider the subsets:

$$\mathcal{C}(\Gamma) := \{f \in \mathcal{C}(V \times V) : f(x, y) = 0, \quad \text{if } x \not\sim y\},$$

$$\mathcal{C}(\Gamma \times \Gamma) := \{f \in \mathcal{C}(V \times V \times V) : f(x, y, z) = 0, \quad \text{if either } y \not\sim x \text{ or } z \not\sim x\}.$$

A function $f \in \mathcal{C}(\Gamma)$ is called *symmetric* or *antisymmetric* iff for each $x, y \in V$, $f(x, y) = f(y, x)$ or $f(x, y) = -f(y, x)$ respectively. In addition, for each $f \in \mathcal{C}(\Gamma)$ it is verified that $f = f^s + f^a$, where $f^s(x, y) = \frac{1}{2}\,(f(x, y) + f(y, x))$ and $f^a(x, y) = \frac{1}{2}\,(f(x, y) - f(y, x))$. On the other hand, it is clear that the space $\mathcal{C}(E)$ is naturally identified with the subspace of $\mathcal{C}(\Gamma)$ formed by the symmetric functions. It is easy to conclude that the same identification occurs for the subspace of antisymmetric functions, provided that an orientation is given on Γ, see for instance [**1**, **4**, **13**].

For given $x \in V$, the real vector space of formal linear combinations of the edges incident with x is called *tangent space at* x and it is denoted by $T_x(\Gamma)$, see [**2**, **3**]. The set of edges incident with x is a basis of $T_x(\Gamma)$ called *coordinate basis of* $T_x(\Gamma)$. Therefore, $\dim T_x(\Gamma)$ equals the number of adjacent vertices to x.

Any function $\mathsf{f} : V \longrightarrow \bigcup_{x \in V} T_x(\Gamma)$ such that $\mathsf{f}(x) \in T_x(\Gamma)$ for each $x \in V$ is called *vector field on* Γ. A vector field f is uniquely determined by its components in the coordinate basis and hence we can consider the function $f \in \mathcal{C}(\Gamma)$ such that for each $x \in V$, $\mathsf{f}(x) = \sum\limits_{y\sim x} f(x,y)\, e_{xy}$. A vector field is called *symmetric or antisymmetric* when its component function has the same property. If f is a vector field and $f \in \mathcal{C}(\Gamma)$ is its component function, the vector fields f^s and f^a whose component functions are f^s and f^a are called *symmetric and antisymmetric part of* f. An antisymmetric field is also called a *flow*.

The space of vector fields on Γ is denoted by $\mathcal{X}(\Gamma)$, whereas the subspaces of symmetric or antisymmetric vector fields are denoted by $\mathcal{X}^s(\Gamma)$ and $\mathcal{X}^a(\Gamma)$, respectively. Clearly $\mathcal{X}(\Gamma) = \mathcal{X}^s(\Gamma) \oplus \mathcal{X}^a(\Gamma)$, $\dim \mathcal{X}^s(\Gamma) = \dim \mathcal{X}^a(\Gamma) = |E|$ and moreover, $\mathcal{X}^s(\Gamma)$ is naturally isomorphic to $\mathcal{C}(E)$.

For each $x \in V$, let $\mathcal{T}^1_x(\Gamma)$ and $\mathcal{T}^2_x(\Gamma)$ be the vector spaces of endomorphisms and bilinear forms on $T_x(\Gamma)$, respectively. A function $\mathsf{G}\colon V \longrightarrow \bigcup\limits_{x\in V} \mathcal{T}^1_x(\Gamma)$ such that for any $x \in V$, $\mathsf{G}(x) \in \mathcal{T}^1_x(\Gamma)$ is called *field of endomorphisms on* Γ, whereas a function $\mathsf{B}\colon V \longrightarrow \bigcup\limits_{x\in V} \mathcal{T}^2_x(\Gamma)$ such that for any $x \in V$, $\mathsf{B}(x) \in \mathcal{T}^2_x(\Gamma)$ is called *field of bilinear forms on* Γ. The space of fields of endomorphisms on Γ is denoted by $\mathcal{T}^1(\Gamma)$, whereas the space of fields of bilinear forms on Γ is denoted by $\mathcal{T}^2(\Gamma)$.

If $\mathsf{G} \in \mathcal{T}^1(\Gamma)$ and $\mathsf{G}(x)$ is an automorphism for each $x \in V$, we denote by G^{-1} the field of endomorphisms determined by $\mathsf{G}(x)^{-1}$ for each $x \in V$.

We say that the field B of bilinear forms is *symmetric* or *orthogonal* if for any $x \in V$, $\mathsf{B}(x)$ is symmetric or the coordinate basis of $T_x(\Gamma)$ is orthogonal with respect to $\mathsf{B}(x)$. If for any $x \in V$, $\mathsf{B}(x)$ is an inner product on $T_x(\Gamma)$, then B is called a *metric on* Γ. In particular, the only metric on Γ such that for any $x \in V$, the coordinate basis of $T_x(\Gamma)$ is orthonormal is named *canonical metric on* Γ, and it is represented by $\langle \cdot, \cdot \rangle$. So, if $f, h \in \mathcal{C}(\Gamma)$ are the component functions of the vector fields $\mathsf{f}, \mathsf{h} \in \mathcal{X}(\Gamma)$, then $\langle \mathsf{f}, \mathsf{h}\rangle(x) = \sum\limits_{y\in V} f(x,y)h(x,y)$, for any $x \in V$.

If $\mathsf{B} \in \mathcal{T}^2(\Gamma)$, we can consider the function $m \in \mathcal{C}(\Gamma \times \Gamma)$ such that if $x \in V$ and y, z are adjacent to x then $m(x,y,z) = \mathsf{B}(x)(e_{xy}, e_{xz})$. Clearly, B is uniquely determined by m, which is called *component function of* B and so we can identify the vector space $\mathcal{T}^2(\Gamma)$ with $\mathcal{C}(\Gamma \times \Gamma)$. Then, the subspace of orthogonal bilinear forms is identified with $\mathcal{C}(\Gamma)$. The existence of a coordinate basis on $T_x(\Gamma)$ allows us to identify naturally $\mathcal{T}^1(\Gamma)$ with $\mathcal{T}^2(\Gamma)$. If $\mathsf{B} \in \mathcal{T}^2(\Gamma)$ and $\mathsf{G} \in \mathcal{T}^1(\Gamma)$ are identified, the component function of B is also called the *component function of* G. Therefore, a field of endomorphisms is called *symmetric* or *diagonal* if its associated field of bilinear forms is symmetric or orthogonal, respectively.

If $\mathsf{G} \in \mathcal{T}^1(\Gamma)$, then for any $\mathsf{f} \in \mathcal{X}(\Gamma)$, Gf is the vector field defined as $\mathsf{Gf}(x) := \mathsf{G}(x)(\mathsf{f}(x))$ for any $x \in V$. Therefore, if $g \in \mathcal{C}(\Gamma \times \Gamma)$ and $f \in \mathcal{C}(\Gamma)$ are the component functions of G and f respectively, then the component function of Gf is given by $h(x,y) = \sum\limits_{z\in V} g(x,y,z)f(x,z)$, for any $x, y \in V$. Analogously, if $\mathsf{B} \in \mathcal{T}^2(\Gamma)$, G is its associated field of endomorphisms and $\mathsf{f}, \mathsf{h} \in \mathcal{X}(\Gamma)$, we can consider $\mathsf{B}(\mathsf{f}, \mathsf{h}) \in \mathcal{C}(V)$, the function given by $\mathsf{B}(\mathsf{f}, \mathsf{h})(x) := \langle \mathsf{Gf}, \mathsf{h}\rangle(x)$, for each $x \in V$.

Next we recall the notion of weighted network and the basic difference operators defined on it. These operators are the fundamental tools to obtain an accurate discrete vector calculus. In [**2**, **3**] the interested reader can find a deeper analysis of these notions together with the development of the integration techniques as well as their applications to the study of boundary value problems.

For a given graph Γ, a metric B and weights μ and ν the ordered quadruple $(\Gamma, \mathsf{B}, \mu, \nu)$ will be called *weighted network*. The following expression determines an inner product on $\mathcal{C}(V)$

$$\int_V u\, v\, \nu \,\mathrm{d}x := \sum_{x\in V} u(x)\, v(x)\, \nu(x), \qquad u, v \in \mathcal{C}(V),$$

Moreover, for any $\mathsf{f}, \mathsf{h} \in \mathcal{X}(\Gamma)$ the expression

$$\frac{1}{2}\int_V \mathsf{B}(\mathsf{f},\mathsf{h})\, \mu \,\mathrm{d}x := \frac{1}{2}\sum_{x,y,z\in V} \mu(x) g(x,y,z) f(x,z) h(x,y),$$

determines an inner product on $\mathcal{X}(\Gamma)$. From now on we suppose fixed the weighted network, $(\Gamma, \mathsf{B}, \mu, \nu)$ and also G the field of automorphisms identified with B.

The linear function $\mathsf{d}\colon \mathcal{C}(V) \longrightarrow \mathcal{X}(\Gamma)$ that assigns to any $u \in \mathcal{C}(V)$ the flow

$$(\mathsf{d}u)(x) := \sum_{y\sim x} \big(u(y) - u(x)\big)\, e_{xy} \tag{1}$$

is called *derivative operator*, whereas the flow $\mathsf{d}u$ is called *derivative of* u. The operator $\nabla := \mathsf{G}^{-1} \circ \mathsf{d}$ is called the *gradient operator*. A vector field f is named *gradient field* if there exists $u \in \mathcal{C}(V)$ such that $\mathsf{f} = \nabla u$. Clearly, it is verified that $\nabla u = \mathsf{0}$ iff u is a constant function.

We define the *divergence operator* as $\mathsf{div} := -\nabla^*$, where * stands for the adjoint operator. Clearly, for any vector field $\mathsf{f} \in \mathcal{X}(\Gamma)$, $\mathsf{div}\,\mathsf{f}$ is determined by the identity

$$\int_V u\,\mathsf{div}\,\mathsf{f}\, \nu \,\mathrm{d}x = -\frac{1}{2}\int_V \mathsf{B}(\nabla u, \mathsf{f})\, \mu \, dx, \quad \text{for any } u \in \mathcal{C}(V). \tag{2}$$

Observe that the operator div does not depend on the metric B since $\mathsf{B}(\nabla u, \mathsf{f}) = \langle \mathsf{d}u, \mathsf{f}\rangle$. In addition, if $f \in \mathcal{C}(\Gamma)$ is the component function of f, then

$$\mathsf{div}\,\mathsf{f}(x) = \frac{1}{\nu(x)} \sum_{y\in V} (\mu f)^{\mathrm{a}}(x,y). \tag{3}$$

Given a function $h \in \mathcal{C}(V)$ there exists $\mathsf{f} \in \mathcal{X}(\Gamma)$ such that $\mathsf{div}\,\mathsf{f} = h$ iff $\displaystyle\int_V h\, \nu \,\mathrm{d}x = 0$ since $\nabla u = 0$ iff u is a constant.

A divergence-free vector field is called a *solenoidal field*.

3. INTEGRATION ALONG CURVES AND THE CURL OPERATOR

One of the first questions raised in continuous vector calculus is to define the circulation of a field along a curve, whose physical meaning is the work necessary to carry a single particle from the curve origin to its end. The question of if the realized work depends or does not depend on the followed trajectory, leads naturally to the notion of conservative fields and raises the problem of how to characterize this property. As far as the authors know, the discrete counterpart of these concepts has not been accurately introduced in the literature. Therefore, in this section we attempt to define them and to show their fundamental properties.

For given $x, y \in V$ an ordered sequence of $n+1$ vertices, $\alpha = \{x_0, \dots, x_n\}$, such that $x_0 = x$, $x_n = y$ and $x_j \sim x_{j+1}$, $j = 0, \dots, n-1$ is called a *curve of length* n *from* x *to* y. The vertices x and y are called the *ends of the curve* and a curve whose ends coincide is called a *closed curve*. If $\alpha = \{x_j\}_{j=0}^n$ is a curve of length n on Γ, the vector field $\mathsf{t}_\alpha := \dfrac{1}{\mu}\, \mathsf{1}_\alpha^a$, where

$$\mathsf{1}_\alpha(x) := \begin{cases} \displaystyle\sum_{\substack{x_j = x \\ j=0,\dots,n-1}} e_{x_j x_{j+1}}, & \text{if there exists } i = 0, \dots, n-1 \text{ such that } x = x_i \\ \mathsf{0}, & \text{otherwise} \end{cases}$$

is called the *tangent field to the curve*. It is easy to prove that $\operatorname{\mathsf{div}} \mathsf{t}_\alpha = \dfrac{1}{\nu}\,(\varepsilon_{x_0} - \varepsilon_{x_n})$, where ε_x stands for the Dirac function at $x \in V$, and hence t_α is a solenoidal field when α is a closed curve. Given $\mathsf{f} \in \mathcal{X}(\Gamma)$ and α a curve, we define the *circulation of* f *along* α as the value

$$\oint_\alpha \mathsf{f} := \frac{1}{2} \int_V \mathsf{B}(\mathsf{f}, \mathsf{t}_\alpha)\, \mu \, dx. \tag{4}$$

Applying Identity (2), if α is a curve with ends x_0 and x_n, the circulation of any gradient field along α is

$$\oint_\alpha \nabla u = u(x_n) - u(x_0), \qquad u \in \mathcal{C}(V)$$

and, in particular, the circulation of a gradient field along any closed curve equals 0. A vector field whose circulation along any closed curve is zero is called *conservative*. So, any gradient field is conservative. Next, we characterize this kind of vector fields.

Proposition 3.1. *A vector field* f *is conservative iff there exists* $u \in \mathcal{C}(V)$ *satisfying that* $\mathsf{d}u = (\mathsf{Gf})^a$.

Proof. If $\mathsf{d}u = (\mathsf{Gf})^a$ then, keeping in mind that $\mu \mathsf{t}_\alpha$ is a flow, for any closed curve α we obtain that

$$\oint_\alpha \mathsf{f} = \frac{1}{2} \int_V \langle \mathsf{Gf}, \mu \mathsf{t}_\alpha \rangle \, dx = \frac{1}{2} \int_V \langle (\mathsf{Gf})^a, \mu \mathsf{t}_\alpha \rangle \, dx = \frac{1}{2} \int_V \mathsf{B}(\nabla u, \mathsf{t}_\alpha)\, \mu \, dx = \oint_\alpha \nabla u = 0$$

and hence f is conservative. Conversely, let $\mathsf{f} \in \mathcal{X}(\Gamma)$ be a conservative field and consider $x, y \in V$ and α_1, α_2 two curves from x to y, that is, $\alpha_1 = \{x_j\}_{j=0}^{n}$ and $\alpha_2 = \{y_j\}_{j=0}^{m}$, where $x_0 = y_0 = x$ and $x_n = y_m = y$. Then, the ordered sequence $\alpha_1 * \alpha_2 = \{z_j\}_{j=0}^{n+m}$ where $z_j = x_j$, $j = 0, \dots, n$ and $z_{n+j} = y_{m-j}$, $j = 1, \dots, m$ is a closed curve and satisfies that $\mathsf{t}_{\alpha_1 * \alpha_2} = \mathsf{t}_{\alpha_1} - \mathsf{t}_{\alpha_2}$ and hence $\oint_{\alpha_1} \mathsf{f} = \oint_{\alpha_2} \mathsf{f}$, since f is a conservative vector field.

For a fixed $x \in V$ and for any $y \in V$ there exists a curve from x to y, since Γ is connected. So, we can define unambiguously the function $u \in \mathcal{C}(V)$ as $u(x) = 0$ and $u(y) = \oint_{\alpha_y} \mathsf{f}$ if $y \neq x$, where α_y is any curve from x to y. Moreover, if $z \sim y$ and α_{zy} denotes the curve $\{z, y\}$, then $\alpha_y * \alpha_{zy}$ is a curve from x to z, which implies that

$$u(z) - u(y) = \oint_{\alpha_y * \alpha_{zy}} \mathsf{f} - \oint_{\alpha_y} \mathsf{f} = -\oint_{\alpha_{zy}} \mathsf{f}.$$

If $h \in \mathcal{C}(\Gamma)$ is the component function of the field Gf, then $\mathsf{t}_{\alpha_{zy}}(z) = \dfrac{1}{\mu(z)}\, e_{yz}$, $\mathsf{t}_{\alpha_{zy}}(y) = -\dfrac{1}{\mu(y)}\, e_{yz}$, $\mathsf{t}_{\alpha_{zy}}(w) = 0$ otherwise and hence we obtain that $\mathsf{d}u = (\mathsf{Gf})^a$, since

$$\begin{aligned}\oint_{\alpha_{zy}} \mathsf{f} &= \frac{1}{2}\int_V \langle \mathsf{Gf}, \mu \mathsf{t}_{\alpha_{zy}} \rangle \, dx = \frac{1}{2}\langle \mathsf{Gf}, \mu \mathsf{t}_{\alpha_{zy}} \rangle(z) + \frac{1}{2}\langle \mathsf{Gf}, \mu \mathsf{t}_{\alpha_{zy}} \rangle(y) \\ &= \frac{1}{2}\big(h(z, y) - h(y, z)\big).\end{aligned}$$ □

In the continuous setting, there exists a close relation between conservative fields and irrotational fields. To study this relation in the discrete realm we introduce the discrete counterpart of the curl operator. Note that the above characterization of conservative vector fields can be re-written as follows: f is a conservative field iff there exists $u \in \mathcal{C}(V)$ such that $\mathsf{f} = \nabla u + \mathsf{G}^{-1}(\mathsf{Gf})^s$. So, if $\mathsf{f} \in \mathcal{X}(\Gamma)$, the vector field

$$\mathsf{curl}\, \mathsf{f} := \frac{1}{\mu}\, (\mathsf{Gf})^s \tag{5}$$

is called *curl* of f. Clearly curl defines an endomorphism of $\mathcal{X}(\Gamma)$. A vector field f is called *irrotational* if $\mathsf{curl}\, \mathsf{f} = 0$. Therefore, f is irrotational iff $\mathsf{Gf} \in \mathcal{X}^a(\Gamma)$ and hence $\mathsf{G}^{-1}(\mathcal{X}^a(\Gamma))$ is the space of irrotational vector fields which implies that its dimension equals $|E|$. Now, we can re-write the claim of the above proposition.

Corollary 3.2. *An irrotational vector field is conservative iff it is a gradient field.*

The following result shows that the definitions of the first order operators given here lead to a coherent vector calculus on weighted networks that is mimetic to its continuous counterpart.

Proposition 3.3. $\mathsf{curl}^* = \mathsf{curl}$, $\mathsf{div} \circ \mathsf{curl} = 0$ *and* $\mathsf{curl} \circ \nabla = 0$.

Proof. For given $\mathsf{f}, \mathsf{h} \in \mathcal{X}(\Gamma)$, it is verified that

$$\int_V \mathsf{B}(\mathsf{curl}\,\mathsf{f}, \mathsf{h})\mu\,\mathrm{d}x = \int_V \langle \mathsf{G}\,\mathsf{curl}\,\mathsf{f}, \mathsf{h}\rangle \mu\,\mathrm{d}x = \int_V \langle (\mathsf{Gf})^s, \mathsf{Gh}\rangle\,\mathrm{d}x = \int_V \langle (\mathsf{Gf})^s, (\mathsf{Gh})^s\rangle\,\mathrm{d}x$$
$$= \int_V \langle \mathsf{Gf}, \mathsf{curl}\,\mathsf{h}\rangle \mu\,\mathrm{d}x = \int_V \mathsf{B}(\mathsf{f}, \mathsf{curl}\,\mathsf{h})\mu\,\mathrm{d}x,$$

which means that $\mathsf{curl}^* = \mathsf{curl}$. The remaining claims follow straightforwardly from the definition of the involved operators. □

We remark that all the above properties and also those given in the following section are still in force if we consider $\dfrac{a}{\mu}\,(\mathsf{Gf})^s$ for any non zero $a \in \mathbb{R}$ instead of $\dfrac{1}{\mu}\,(\mathsf{Gf})^s$ as the definition of the curl operator.

4. THE COHOMOLOGY OF A WEIGHTED NETWORK

We aim here at obtaining some results about weighted networks that classically fall within the ambit of the Algebraic Topology whose terminology and techniques have provided a rigorous treatment of infinite networks, see [**14**]. However, this is not the case of finite network and the only comments about this issue can be found in [**1**, **4**]. Our main goal is to show that the status of weighted networks is mimetic to the one of compact differentiable Riemannian manifolds: The discrete analogue of the *De Rham cohomology* gives the fundamental properties of the ordinary cohomology in the ambit of Algebraic Topology. On the other hand, this development will confirm the appropriateness of the definitions of the difference operators on a weighted network given in the preceding sections. These definitions allow us to consider the Hodge Laplacian on the space of vector fields of the manifold. Although this definition is not new, see for instance [**6**, **8**, **12**], the Hodge Laplacian is obtained in the above-mentioned works by considering the network as the 1-skeleton of a simplicial complex which dimension equals the one of the ambient space. In contrast, the results presented here do not need to consider the underlying graph as a part of a higher dimensional CW-complex.

Taking into account Proposition 3.3, we define the *De Rham complex* as

$$0 \overset{0}{\hookrightarrow} \mathcal{C}(V) \overset{\nabla}{\longrightarrow} \mathcal{X}(\Gamma) \overset{\mathsf{curl}}{\longrightarrow} \mathcal{X}(\Gamma) \overset{\mathsf{div}}{\longrightarrow} \mathcal{C}(V) \overset{0}{\longrightarrow} 0. \tag{6}$$

Moreover $H^0(\Gamma) := \ker \nabla$ and $H^1(\Gamma) := \ker \mathsf{curl}/\mathrm{Img}\,\nabla$ are called the *De Rham cohomology groups of* Γ, whereas $\beta_n := \dim H^n(\Gamma)$, $n = 0, 1$ are named the *Betti numbers* of Γ. The *Euler characteristic* of Γ is the number $\chi(\Gamma) = |V| - |E|$ and it is well-known that $\chi(\Gamma) \leq 1$ and the equality holds iff Γ is a tree.

Proposition 4.1. (Euler-Poincaré Formula.) *It is verified that* $\beta_0 = 1$, $\beta_1 = |E| - |V| + 1$ *and hence that* $\chi(\Gamma) = \beta_0 - \beta_1$.

We remark that a tree is the unique connected graph whose geometric realization is a simply connected unidimensional CW-complex, in fact it is contractible.

So, the next result can be interpreted as the discrete analogue of the Poincaré Lemma.

Proposition 4.2. (Poincaré Lemma.) *The following statements are equivalent*:

(i) Γ *is a tree.*

(ii) $H^1(\Gamma)$ *is trivial.*

(iii) *Each solenoidal field is the curl of another field.*

(iv) *Each irrotational field is a gradient field.*

Proof. The equivalence between the two first claims is due to the fact that $\beta_1 = 1 - \chi(\Gamma)$. On the other hand, $\ker \mathsf{div} = [\operatorname{Img} \nabla]^\perp$ since $\mathsf{div} = -\nabla^*$, and hence $\dim\ker\mathsf{div} = |E| + 1 - \chi(\Gamma) \geq |E|$. So, Γ is a tree iff $\dim\ker\mathsf{div} = |E|$, or equivalently, iff $\dim\operatorname{Img}\nabla = |E|$. From Proposition , we get that

$$\operatorname{Img}\mathsf{curl} = [\ker\mathsf{curl}]^\perp, \qquad \operatorname{Img}\mathsf{curl} \subset \ker\mathsf{div}, \qquad \text{and} \quad \operatorname{Img}\nabla \subset \ker\mathsf{curl}.$$

The first identity implies that $\dim\operatorname{Img}\mathsf{curl} = |E|$. Now, the second inclusion implies that each solenoidal field is the curl of another field iff $\dim\ker\mathsf{div} = |E|$; that is, iff Γ is a tree. Finally, the last inclusion establishes that each irrotational field is a gradient field iff $\dim\operatorname{Img}\nabla = |E|$; that is, iff Γ is a tree. □

Following the guidelines of the differentiable framework, the endomorphisms of $\mathcal{C}(\Gamma)$ and $\mathcal{X}(\Gamma)$ defined respectively as

$$\Delta = -\mathsf{div} \circ \nabla \qquad \text{and} \qquad \boldsymbol{\Delta} = \mathsf{curl} \circ \mathsf{curl} - \nabla \circ \mathsf{div}$$

are called *Laplace-Beltrami operator* or *Laplacian* and *Hodge Laplacian operator*, respectively. In addition a function $u \in \mathcal{C}(V)$ such that $\Delta u = 0$ is called *harmonic function*, whereas a field $\mathsf{f} \in \mathcal{X}(\Gamma)$ such that $\boldsymbol{\Delta}\mathsf{f} = 0$ is called *harmonic field.*

If we denote $d_{-1} = 0$, $d_0 = \nabla$, $d_1 = \mathsf{curl}$, $\Delta_0 = \Delta$ and $\Delta_1 = \boldsymbol{\Delta}$, then $\Delta_n = d_n^* \circ d_n + d_{n-1} \circ d_{n-1}^*$, $n = 0, 1$ and hence the Laplacian and the Hodge Laplacian of a network become the discrete counterpart of the well-known differentiable operators. In addition, the following results represent the discrete version of the properties satisfied in the differentiable setting.

Proposition 4.3. *The Laplacian and the Hodge Laplacian are self-adjoint positive semidefinite operators. Moreover* $u \in \mathcal{C}(V)$ *is harmonic iff it is constant, whereas* $\mathsf{f} \in \mathcal{X}(\Gamma)$ *is a harmonic field iff it is both irrotational and solenoidal.*

Proof. Clearly $\Delta^* = -\nabla^* \circ \mathsf{div}^* = \Delta$, whereas $\boldsymbol{\Delta}^* = \mathsf{curl}^* \circ \mathsf{curl}^* - \mathsf{div}^* \circ \nabla^* = \boldsymbol{\Delta}$, since curl is a self-adjoint operator whereas ∇ and div are mutually adjoint. On the other hand, given $u \in \mathcal{C}(V)$ and $\mathsf{f} \in \mathcal{X}(\Gamma)$, the last claims follow taking into account the following identities

$$\int_V u\Delta u\,\nu\,\mathrm{d}x = \frac{1}{2}\int_V \mathsf{B}(\nabla u, \nabla u)\,\mu\,\mathrm{d}x,$$
$$\int_V \mathsf{B}(\boldsymbol{\Delta}\mathsf{f}, \mathsf{f})\,\mu\,\mathrm{d}x = \int_V \mathsf{B}(\mathsf{curl}\,\mathsf{f}, \mathsf{curl}\,\mathsf{f})\,\mu\,\mathrm{d}x + 2\int_V (\mathsf{div}\,\mathsf{f})^2\nu\,\mathrm{d}x.$$ □

Proposition 4.4. (HODGE's decomposition theorem.) *The following orthogonal decompositions hold*

$$\mathcal{C}(V) = \ker \Delta \oplus \mathrm{Img}\ \mathsf{div} \quad \textit{and} \quad \mathcal{X}(\Gamma) = \ker \boldsymbol{\Delta} \oplus \mathrm{Img}\ \nabla \oplus \mathrm{Img}\ \mathsf{curl}.$$

Proof. The first identity is a direct consequence of the facts $\mathrm{Img}\,\mathsf{div} = [\ker \nabla]^{\perp}$ and $\ker \Delta = \ker \nabla$. On the other hand, it can be easily proved that the three subspaces in the second decomposition are mutually orthogonal. In particular, this property implies that

$$\mathrm{Img}\,\nabla \oplus \mathrm{Img}\,\mathsf{curl} \subset \big(\ker \boldsymbol{\Delta}\big)^{\perp} = \mathrm{Img}\,\boldsymbol{\Delta} \subset \mathrm{Img}\,\nabla \oplus \mathrm{Img}\,\mathsf{curl},$$

since $\boldsymbol{\Delta}$ is self-adjoint. Therefore it is verified that $\big(\ker \boldsymbol{\Delta}\big)^{\perp} = \mathrm{Img}\,\nabla \oplus \mathrm{Img}\,\mathsf{curl}$ and definitely

$$\mathcal{X}(\Gamma) = \ker \boldsymbol{\Delta} \oplus \big(\ker \boldsymbol{\Delta}\big)^{\perp} = \ker \boldsymbol{\Delta} \oplus \mathrm{Img}\,\nabla \oplus \mathrm{Img}\,\mathsf{curl}. \qquad \square$$

Corollary 4.5. $H^0(\Gamma) \simeq \ker \Delta$ *and* $H^1(\Gamma) \simeq \ker \boldsymbol{\Delta}$*. In particular,* $\boldsymbol{\Delta}$ *is an automorphism iff* Γ *is a tree.*

Keeping in mind Proposition 4.3, we can re-interpret HODGE's decomposition in terms of the discrete version of *Helmholtz's Theorem*: *For each vector field* f *there exist a function* u *and two vector fields* g, h*, such that* $\mathsf{f} = \nabla u + \mathsf{curl}\,\mathsf{g} + \mathsf{h}$, where $\mathsf{div}\,\mathsf{h} = 0$ and $\mathsf{curl}\,\mathsf{h} = 0$. The triple $(u, \mathsf{g}, \mathsf{h})$ form the so-called *Helmholtz's components* of f, where u is the *scalar potential*, g is the *vector potential* and h is the *harmonic component* of f. Clearly, the harmonic component is uniquely determined whereas the scalar potential is unique up to a constant and the vector potential is unique up to an element of $G^{-1}(\mathcal{X}^a(\Gamma))$. So HODGE's decomposition theorem, or equivalently HELMHOLTZ's theorem, becomes not only an effective answer but also the accurate framework to the main question formulated in [**7**] where it was raised the decomposition of a graph into a curl-free part, a divergence-free part and a simultaneously curl-free and divergence-free part.

5. PURELY RESISTIVE NETWORKS

Consider now a weighted network $(\Gamma, \mathsf{B}, \nu, \mu)$ where B is an orthogonal metric and $\mu = 1$. Then given $\mathsf{f}, \mathsf{h} \in \mathcal{X}(\Gamma)$, if $g, f, h \in \mathcal{C}(\Gamma)$ are the component functions of B, f and h respectively, we obtain that

$$\int_V \mathsf{B}(\mathsf{f}, \mathsf{h})\, \mu \,\mathrm{d}x = \sum_{x,y \in V} g(x,y) f(x,y) h(x,y).$$

In particular, if $\mathsf{f}, \mathsf{h} \in \mathcal{X}^s(\Gamma)$ then defining $r = g^s$ and considering r, f, h as functions in $\mathcal{C}(E)$ we get

$$\frac{1}{2} \int_V \mathsf{B}(\mathsf{f}, \mathsf{h})\, dx = \sum_{x,y \in V} g^s(x,y) f(x,y) h(x,y) = \sum_{e \in E} r(e)\, f(e)\, h(e).$$

So, the inner product induced on $\mathcal{X}^s(\Gamma)$ as subspace of $\mathcal{X}(\Gamma)$, corresponds to the standard inner product on $\mathcal{C}(E)$, with respect to the weight r which is usually called *resistance*. Conversely, if r is a resistance; that is, a weight on E, then the inner product on $\mathcal{C}(E)$ with respect to r coincides with the induced by the orthogonal metric B whose component function is the symmetric function identified with r.

For a given graph Γ, an orthogonal metric B with symmetric component function and weights $\mu = 1$ and ν the ordered quadruple $(\Gamma, \mathsf{B}, \mu, \nu)$ will be called *purely resistive network*. Therefore a purely resistive network is nothing else than a triple (Γ, r, ν) where ν is a weight and $r\colon V \times V \longrightarrow [0, +\infty)$ is a symmetric function, called *resistance of the network*, such that $r(x, y) > 0$ iff $x \sim y$. Moreover, the function $c\colon V \times V \longrightarrow [0, +\infty)$ given by $c(x, y) = \dfrac{1}{r(x, y)}$ when $x \sim y$ and $c(x, y) = 0$ otherwise, is usually called the *conductance of the network*. So, the concept of purely resistive network introduced here coincides with the one established in the literature as *finite network*, see for instance [**14**]. In addition, the inner product on $\mathcal{X}(\Gamma)$ defined in this work includes those inner products on $\mathcal{C}(E)$ that have been widely used in the context of networks, see for instance [**9**, **11**, **14**].

If (Γ, r, ν) is a purely resistive network, then $\mathcal{X}(\Gamma) = \mathcal{X}^s(\Gamma) \oplus \mathcal{X}^a(\Gamma)$ is an orthogonal decomposition. In addition, given $u \in \mathcal{C}(V)$, we obtain that

$$\nabla u(x) = \sum_{y \sim x} c(x, y)\big(u(y) - u(x)\big) e_{xy} \quad \text{and} \quad \Delta u(x) = \frac{1}{\nu(x)} \sum_{y \in V} c(x, y)\big(u(x) - u(y)\big).$$

In particular, when $\nu = 1$ then Δ equals the standard *combinatorial Laplacian*; whereas when $\nu(x) = \sum\limits_{y \in V} c(x, y)$ for any $x \in V$, Δ is nothing else than the so-called *probabilistic Laplacian*. In addition, if $\mathsf{f} \in \mathcal{X}(\Gamma)$ and $f \in \mathcal{C}(\Gamma)$ is its component function, then for any $x \in V$ we get that

$$\mathsf{curl}\,\mathsf{f}(x) = \mathsf{curl}\,\mathsf{f}^s(x) = \sum_{y \sim x} r(x, y)\, f^s(x, y)\, e_{xy}.$$

In particular $\mathsf{f} \in \mathcal{X}(\Gamma)$ is an irrotational field iff it is a flow; that is, iff $\mathsf{f}^s = 0$. Moreover,

$$\mathsf{curl}\,\mathsf{curl}\,\mathsf{f}(x) = \sum_{y \sim x} r(x, y)^2 f^s(x, y)\, e_{xy}.$$

On the other hand, as $\mathsf{div}\,\mathsf{f}(x) = \dfrac{1}{\nu(x)} \sum\limits_{z \in V} f^a(x, z)$ we obtain that

$$\nabla \mathsf{div}\,\mathsf{f}(x) = \sum_{y \sim x} \left(\frac{c(x, y)}{\nu(x)\nu(y)} \sum_{z \in V} \Big[\nu(x) f^a(y, z) - \nu(y) f^a(x, z) \Big] \right) e_{xy}$$

and hence

$$\boldsymbol{\Delta}\mathsf{f}(x) = \sum_{y \sim x} \left(r(x, y)^2 f^s(x, y) + \frac{c(x, y)}{\nu(x)\nu(y)} \sum_{z \in V} \Big[\nu(y) f^a(x, z) + \nu(x) f^a(z, y) \Big] \right) e_{xy}.$$

In particular, $\boldsymbol{\Delta}\mathsf{f} = \mathsf{curl}\,\mathsf{curl}\,\mathsf{f}$ when $\mathsf{f} \in \mathcal{X}^s(\Gamma)$ and in this case

$$\boldsymbol{\Delta}\mathsf{f}(x) = \sum_{y\sim x} r(x,y)^2 f(x,y)\, e_{xy},$$

whereas $\boldsymbol{\Delta}\mathsf{f} = \nabla\,\mathsf{div}\,\mathsf{f}$ when $\mathsf{f} \in \mathcal{X}^a(\Gamma)$ and in this case

$$\boldsymbol{\Delta}\mathsf{f}(x) = \sum_{y\sim x} \frac{c(x,y)}{\nu(x)\nu(y)}\Big(\sum_{z\in V}\Big[\nu(y)f(x,z) + \nu(x)f(z,y)\Big]\Big)\, e_{xy}.$$

If $\nu = 1$, these expressions coincide with those obtained in [**1**] since in this case $\nabla\colon \mathcal{C}(V) \longrightarrow \mathcal{X}^a(\Gamma)$ and hence $\Delta = \nabla^* \circ \nabla$, whereas $\boldsymbol{\Delta} = \nabla \circ \nabla^*$ on $\mathcal{X}^a(\Gamma)$.

6. APPLICATION TO UNIFORM GRIDS

Our aim here is to obtain the expression of the operators defined in the above sections when we consider a metric on the n-dimensional lattice. Specifically, for each $h > 0$ we consider Γ_h the graph whose vertex set is $V_h = h\mathbb{Z}^n$ and where two vertices x, y are adjacent iff their euclidean distance $|x-y|$ equals h. Therefore, if $\{\mathsf{e}_j\}_{j=1}^n$ denotes the standard basis of $\mathbb{R}^n$ and we define $\mathsf{e}_{n+j} = -\mathsf{e}_j$, $j = 1,\dots,n$, then for any $x \in V_h$, the adjacent vertices to x are $x_j = x + h\mathsf{e}_j$, $j = 1,\dots,2n$. If ν and μ are the weights on V_h defined as $\mu(x) = h^{n-1}$ and $\nu(x) = h^n$, for any $x \in V_h$, the weighted graph (Γ_h, μ, ν) is called *n-dimensional uniform grid of size h*. For any $x \in V_h$ we also consider the vertices $x_{ij} = x + h\,(\mathsf{e}_i + \mathsf{e}_j)$, $1 \le i \le j \le 2n$, $j \ne n+i$ and the set $S(x) = \Big\{\{x\} \cup \{x_j\}_{j=1}^{2n} \cup \{x_{ij}\}_{\substack{1\le i\le j\le 2n\\ j\ne i+n}}\Big\}$ is called *stencil at* x, see Figure 1. Fixed a symmetric and positive definite $2n$-order matrix G, we consider the field of endomorphisms G that assigns to each $x \in V$ the matrix $\mathsf{G}(x) = hG$. This field is called *homogeneous field determined by* G, see [**2, 3**]. Clearly the field of bilinear forms identified with G is a metric on Γ_h. Therefore for each $h > 0$ we can consider the weighted network $(\Gamma_h, \mathsf{G}, \nu, \mu)$. In [**2, 3**] it was proved that the LAPLACE-BELTRAMI operator Δ_h determines a difference scheme with constant coefficients on $\{\Gamma_h\}_{h>0}$. One of the fundamental questions in Numerical Analysis is to characterize all difference schemes with constant coefficients on $\{\Gamma_h\}_{h>0}$ that are consistent with a second order differential operator with constant coefficients. Recall that given $r > 0$, the difference scheme Δ_h is called *r-consistent* with the differential operator L on $\{\Gamma_h\}_{h>0}$ if for any $x \in V_h$ and for any u smooth enough it is verified that $L(u)(x) - \Delta_h(u)(x) = O(h^r)$. The following result was established in [**2**, Proposition 4.8] in a more general setting, see also [**3**, Proposition 6.1] and it refers to the consistency of Δ_h with the LAPLACE-BELTRAMI operator associated with an uniform metric in $\mathbb{R}^n$.

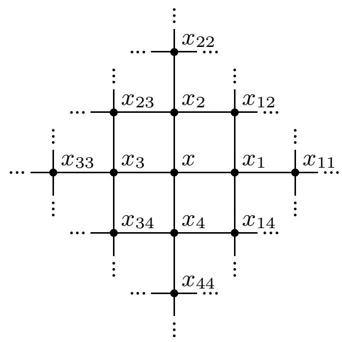

Figure 1. Bidimensional stencil

Proposition 6.1. *Let $K = (k_{ij})$ a symmetric and positive definite n-matrix and consider the elliptic partial differential operator with constant coefficients $L(u) = -\sum_{i,j=1}^{n} k_{ij} u_{x_i x_j}$. Then the operator Δ_h is a difference scheme 2-consistent with L iff there exists a symmetric n-matrix $M = (m_{ij})$ such that $G^{-1} = \begin{bmatrix} K+M & M \\ M & K+M \end{bmatrix}$. In this case, for any $u \in \mathcal{C}(V_h)$ and any $x \in V_h$ we get that*

$$\nabla u(x) = \frac{1}{h}\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\big((k_{ij}+m_{ij})u(x_i) + m_{ij}u(x_{n+i}) - (k_{ij}+2m_{ij})u(x)\big)\right) e_{xx_j}$$
$$+ \frac{1}{h}\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\big((k_{ij}+m_{ij})u(x_{n+i}) + m_{ij}u(x_i) - (k_{ij}+2m_{ij})u(x)\big)\right) e_{xx_{n+j}}$$

and

$$\begin{aligned}\Delta_h(u)(x) &= \sum_{j=1}^{n}\Big(\sum_{i=1}^{n}\frac{(k_{ij}+2m_{ij})}{h^2}\Big)\big(2u(x) - u(x_j) - u(x_{n+j})\big)\\ &- \sum_{1\le i<j\le n}\frac{m_{ij}}{h^2}\big(2u(x) - u(x_{ij}) - u(x_{n+in+j})\big)\\ &- \sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{(k_{ij}+m_{ij})}{h^2}\big(2u(x) - u(x_{jn+i}))\big)\\ &- \sum_{j=1}^{n}\frac{m_{jj}}{2h^2}\big(2u(x) - u(x_{jj}) - u(x_{n+jn+j})\big).\end{aligned}$$

In view of the above result in what follows we only consider homogeneous metrics determined by matrices of the form $G = \begin{bmatrix} K+M & M \\ M & K+M \end{bmatrix}^{-1}$.

Lemma 6.2. *Consider the symmetric n-matrices K and M and let $\lambda_{_K}$ and $\lambda_{_M}$ their lowest eigenvalue, respectively. If the matrix $A = \begin{bmatrix} K+M & M \\ M & K+M \end{bmatrix}$ is positive definite then $\lambda_{_K} > 0$. Conversely, if $\lambda_{_K} > 0$ and $2\lambda_{_M} > -\lambda_{_K}$, then A is positive definite and moreover, $A^{-1} = \begin{bmatrix} K^{-1}-Q & -Q \\ -Q & K^{-1}-Q \end{bmatrix}$, where $Q = K^{-1}M(K+2M)^{-1}$.*

Proof. Given $x, y \in \mathbb{R}^n$ and $z = \begin{bmatrix} x \\ y \end{bmatrix}$ then

$$z^tAz = x^tKx + y^tKy + (x+y)^tM(x+y).$$

Therefore, taking $y = -x$, we obtain that $z^tAz = 2x^tKx$ which implies that K is positive definite when A is. On the other hand, we get that

$$\begin{aligned} z^tAz &= x^tKx + y^tKy + (x+y)^tM(x+y)\\ &\ge \lambda_{_K}(x^tx + y^ty) + \lambda_{_M}x^tx + \lambda_{_M}y^ty + 2\lambda_{_M}y^tx\\ &= (\lambda_{_K} + \lambda_{_M})z^tz + 2\lambda_{_M}y^tx.\end{aligned}$$

The CAUCHY-SCHWARTZ inequality implies that

$$|2\lambda_{_M} y^t x| \le 2|\lambda_{_M}|\,|y^t x| \le 2|\lambda_{_M}|\,\sqrt{y^t y}\,\sqrt{x^t x} \le |\lambda_{_M}|\,z^t z,$$

and hence $2\lambda_{_M}\, y^t x \ge -|\lambda_{_M}|\, z^t z$ and so $z^t A z \ge (\lambda_{_K} + \lambda_{_M} - |\lambda_{_M}|) z^t z$. In conclusion, if $\lambda_{_K} > 0$, then A is positive definite when $2\lambda_{_M} > -\lambda_{_K}$. In this case, $K + 2M$ is positive definite and hence invertible. □

Consider K a symmetric and positive definite matrix, M a symmetric matrix such that $2\lambda_{_M} > -\lambda_{_K}$, G the homogeneous metric determined by

$$G = \begin{bmatrix} K^{-1} - Q & -Q \\ -Q & K^{-1} - Q \end{bmatrix},$$

where $Q = K^{-1}M(K+2M)^{-1}$ and let the weighted network $(\Gamma_h, \mathsf{G}, \nu, \mu)$. As usual, we denote $K = (k_{ij})$, $K^{-1} = (k^{ij})$, $M = (m_{ij})$ and $Q = (q^{ij})$.

As we are considering only homogeneous metrics, we mainly focus on vector fields f such that for any $x \in V_h$, $f(x, x_j) = b_j$, $j = 1, \ldots, 2n$. In this case f is called the *homogeneous vector field determined by* $\mathsf{b} = (b_j) \in \mathbb{R}^{2n}$. Moreover, $\mathsf{f} \in \mathcal{X}^s(\Gamma_h)$ iff $b_{n+i} = b_i$, $i = 1, \ldots, n$, whereas f is a flow iff $b_{n+i} = -b_i$, $i = 1, \ldots, n$.

Next we obtain the expression of the fundamental operators div, curl and $\boldsymbol{\Delta}$ on homogenous vector fields. So, if $\mathsf{f} \in \mathcal{C}(V_h)$ is a homogeneous vector field, then $\boldsymbol{\Delta}\mathsf{f} = \mathsf{curl}\,\mathsf{curl}\,\mathsf{f}$ since $\mathsf{div}\mathsf{f} = 0$.

In the case of uniform n-dimensional grids of size h it is useful to define the curl as the operator $\mathsf{curl}\,\mathsf{f} = \dfrac{h^{n-3}}{\mu}\,(\mathsf{Gf})^s = \dfrac{1}{h^2}\,(\mathsf{Gf})^s$ for any $\mathsf{f} \in \mathcal{X}(\Gamma)$. Therefore, if f is the homogeneous vector field determined by $\mathsf{b} = (b_i) \in \mathbb{R}^{2n}$, then

$$\mathsf{curl}\,\mathsf{f} = \frac{1}{2h}\sum_{j=1}^{n}\left[\sum_{i=1}^{n}(k^{ji} - 2q^{ji})(b_i + b_{n+i})\right](e_{xx_j} + e_{xx_{n+j}}).$$

Consequently $\mathsf{curl}\,\mathsf{f} = \mathsf{curl}\,\mathsf{f}^s$ and moreover

$$\boldsymbol{\Delta}\mathsf{f} = \mathsf{curl}\,\mathsf{curl}\,\mathsf{f} = \frac{1}{2h^2}\sum_{j=1}^{n}\left[\sum_{i=1}^{n}(b_i + b_{n+i})\sum_{\ell=1}^{n}(k^{j\ell} - 2q^{j\ell})(k^{\ell i} - 2q^{\ell i})\right](e_{xx_j} + e_{xx_{n+j}}).$$

Acknowledgments. This work has been partly supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología,) under project MTM2007-62551.

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(Received March 8, 2008)
(Revised July 7, 2008)