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SHARP UPPER BOUNDS FOR THE NUMBER OF SPANNING TREES OF A GRAPH

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This note presents two new upper bounds for the number of spanning trees of a graph in terms of the order, edge number and maximum degree of a graph.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with n vertices and e edges. Suppose the vertex set is $V = \{v_1, v_2, \dots, v_n\}$ with non increasing degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, where d_i is the degree of vertex v_i for $i = 1, 2, \dots, n$. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of graph G, where $D(G) = \operatorname{diag}\big(d(u), u \in V\big)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the adjacency matrix of G. The eigenvalues of $L(G)$ are called the *Laplacian eigenvalues* and denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. It is well known that $\lambda_1 \leq n$.

We denote the number of spanning trees (also known as complexity) of G by $\kappa(G)$. The following formula in terms of the Laplacian eigenvalues of G is well known (see, for example, [**2**], p. 39):

$$\kappa(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$

Next, we present some known upper bounds for $\kappa(G)$.

(1) Grimmett [**4**].

$$\kappa(G) \leq \frac{1}{n} \left(\frac{2e}{n-1} \right)^{n-1}. \tag{1}$$

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(2) Grone and Merris [**5**].

$$\kappa(G) \leq \left(\frac{n}{n-1}\right)^{n-1} \left(\frac{\prod_{i=1}^{n} d_i}{\sum_{i=1}^{n} d_i}\right). \tag{2}$$

(3) Nosal [**8**]. For r-regular graph,

$$\kappa(G) \leq n^{n-2}\left(\frac{r}{n-1}\right)^{n-1}. \tag{3}$$

(4) Kelmanns ([**2**], p. 222).

$$\kappa(G) \leq n^{n-2}\left(1-\frac{2}{n}\right)^{e}. \tag{4}$$

(5) Das [**3**].

$$\kappa(G) \leq \left(\frac{2e-d_1-1}{n-2}\right)^{n-2}. \tag{5}$$

(6) Zhang [**9**]. (Always better than (1).) For $a = \left(\frac{n(n-1)-2e}{2en(n-2)}\right)^{1/2}$,

$$\kappa(G) \leq \big(1+(n-2)a\big)(1-a)^{n-2}\,\frac{1}{n}\left(\frac{2e}{n-1}\right)^{n-1}. \tag{6}$$

In this note, we establish the following two new upper bounds for the complexity of a connected graph.

Theorem 1.1. *For a connected graph G, we have*

$$\kappa(G) \leq \left(\frac{d_1+1}{n}\right)\left(\frac{2e-d_1-1}{n-2}\right)^{n-2}. \tag{7}$$

The equality in (7) *holds if and only if G is a complete graph or a star graph.*

Theorem 1.2. *For a connected graph G, we have*

$$\kappa(G) \leq \left(\frac{\sum_{i=1}^{n} d_i^2 + 2e - (d_1+1)^2}{n-2}\right)^{\frac{n-2}{2}}. \tag{8}$$

The equality in (8) *holds if and only if G is a complete graph or a star graph.*

From Theorem 1.2 and Lemma 2.3 in the next section, we have

Corollary 1.3. *For a connected graph G, we have*

$$\kappa(G) \leq \left(\frac{e\Big(\frac{2e}{n-1}+n\Big)-(d_1+1)^2}{n-2} \right)^{\frac{n-2}{2}}. \tag{9}$$

The equality in (9) *holds if and only if G is a complete graph or a star graph.*

2. LEMMAS AND PROOFS

Lemma 2.1. [**6**] *If G is a graph of order n with at least one edge, and the maximum degree of G is d_1, then $\lambda_1(G) \geq d_1+1$. Moreover, if G is connected, then the equality holds if and only if $d_1 = n-1$.*

Lemma 2.2. [**3**] *Let G be a connected graph of order $n \geq 3$. Then $\lambda_2 = \cdots = \lambda_{n-1}$ if and only if G is a complete graph or a star graph or a (d_1, d_1) complete bipartite graph.*

Lemma 2.3. [**1, 7**] *Let G be a connected graph with n vertices and e edges and let $\pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of G. Then*

$$\sum_{i=1}^{n} d_i^2 \leq e\Big(\frac{2e}{n-1}+n-2\Big).$$

The equality holds if and only if G is a complete graph or a star graph.

Next, we present the proof of the main results of this note.

Proof. (Theorem 1.1) We have

$$\kappa(G) = \frac{1}{n}\prod_{i=1}^{n-1}\lambda_i = \frac{1}{n}\lambda_1\prod_{i=2}^{n-1}\lambda_i \leq \frac{\lambda_1}{n}\left(\frac{\sum\limits_{i=2}^{n-1}\lambda_i}{n-2}\right)^{n-2} = \frac{\lambda_1}{n}\left(\frac{\sum\limits_{i=1}^{n-1}\lambda_i-\lambda_1}{n-2}\right)^{n-2}$$
$$= \frac{\lambda_1}{n}\left(\frac{2e-\lambda_1}{n-2}\right)^{n-2}.$$

For $d_1+1 \leq x \leq n$, let

$$f(x) = x\Big(2e-x\Big)^{n-2}.$$

Taking derivative with respect to x, we have

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = f(x)\,\frac{2e-(n-1)x}{x\,(2e-x)}.$$

Since $d_1+1 \geq \frac{2e}{n-1}$, so $\frac{\mathrm{d}f(x)}{\mathrm{d}x} \leq 0$ for $d_1+1 \leq x \leq n$. Hence $f(x)$ takes the maximum value at $x = d_1+1$ and we can get the result directly.

If the equality holds in (7), then all the inequalities in the above proof must be equalities. Hence we have $\lambda_1 = d_1+1$, $\lambda_2 = \cdots = \lambda_{n-1}$. By Lemma 2.1, we have $d_1 = n-1$. By Lemma 2.2, we have that G is a complete graph or a star graph or a (d_1, d_1) complete bipartite graph. Combining the above cases, we conclude that G must be a complete graph or a star graph. □

Proof. (Theorem 1.2) We have

$$\kappa^2(G) = \frac{1}{n^2}\prod_{i=1}^{n-1}\lambda_i^2 = \frac{1}{n^2}\lambda_1^2\prod_{i=2}^{n-1}\lambda_i^2 \le \frac{\lambda_1^2}{n^2}\left(\frac{\sum_{i=2}^{n-1}\lambda_i^2}{n-2}\right)^{n-2} \le \left(\frac{\sum_{i=1}^{n-1}\lambda_i^2-\lambda_1^2}{n-2}\right)^{n-2}$$
$$= \left(\frac{\sum_{i=1}^{n} d_i^2+2e-(d_1+1)^2}{n-2}\right)^{n-2}.$$

If the equality holds in (8), then all the inequalities in the above proof must be equalities. Hence we have $\lambda_1 = n$, $\lambda_2 = \cdots = \lambda_{n-1}$. By Lemma 2.1, we have $d_1 = n-1$. By Lemma 2.2, we have that G is a complete graph or a star graph or a (d_1, d_1) complete bipartite graph. Combining the above cases, we conclude that G must be a complete graph or a star graph. □

Proof. (Corollary 1.3) By the proof of Theorem 1.2 and Lemma 2.3, we can get the result. □

Remark. It is easy to see that (7) is never worse than (5).

At last, we will show that (7) is never worse than (8), and hence by Lemma (2.3), (7) is never worse than (9). Denote the right hand side of (7) and (8) by M_1 and M_2, respectively. From the proof of Theorems 1.1 and 1.2, we have

$$M_1 = \frac{\lambda_1}{n}\left(\frac{\sum_{i=2}^{n-1}\lambda_i}{n-2}\right)^{n-2} \le \left(\frac{\sum_{i=2}^{n-1}\lambda_i}{n-2}\right)^{n-2}, \qquad M_2^2 = \left(\frac{\sum_{i=2}^{n-1}\lambda_i^2}{n-2}\right)^{n-2}.$$

Since $\sum_{i=2}^{n-1}\lambda_i^2 \ge \frac{\left(\sum_{i=2}^{n-1}\lambda_i\right)^2}{n-2}$, we have

$$M_1^2 - M_2^2 \le \left(\frac{\left(\sum_{i=2}^{n-1}\lambda_i\right)^2}{(n-2)^2}\right)^{n-2} - \left(\frac{\sum_{i=2}^{n-1}\lambda_i^2}{n-2}\right)^{n-2}$$
$$\le \left(\frac{\left(\sum_{i=2}^{n-1}\lambda_i\right)^2}{(n-2)^2}\right)^{n-2} - \left(\frac{\left(\sum_{i=2}^{n-1}\lambda_i\right)^2}{(n-2)^2}\right)^{n-2}$$
$$= 0.$$

Hence, we get $M_1 \leq M_2$.

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