Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **2** (2008), 260–275. doi:10.2298/AADM0802260F

SPECTRAL PROPERTY OF CERTAIN CLASS OF GRAPHS ASSOCIATED WITH GENERALIZED BETHE TREES AND TRANSITIVE GRAPHS

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A generalized Bethe tree is a rooted tree for which the vertices in each level having equal degree. Let $\mathcal{B}_k$ be a generalized Bethe tree of k level, and let $\mathcal{T}^r$ be a connected transitive graph on r vertices. Then we obtain a graph $\mathcal{B}_k \circ \mathcal{T}^r$ from r copies of $\mathcal{B}_k$ and $\mathcal{T}^r$ by appending r roots to the vertices of $\mathcal{T}^r$ respectively. In this paper, we give a simple way to characterize the eigenvalues of the adjacency matrix $A(\mathcal{B}_k \circ \mathcal{T}^r)$ and the Laplacian matrix $L(\mathcal{B}_k \circ \mathcal{T}^r)$ of $\mathcal{B}_k \circ \mathcal{T}^r$ by means of symmetric tridiagonal matrices of order k. We also present some structure properties of the Perron vectors of $A(\mathcal{B}_k \circ \mathcal{T}^r)$ and the Fiedler vectors of $L(\mathcal{B}_k \circ \mathcal{T}^r)$. In addition, we obtain some results on transitive graphs.

1. INTRODUCTION

Let G be a simple graph on n vertices. The *Laplacian matrix* of G is defined to an $n \times n$ matrix $L(G) = D(G) - A(G)$, where $A(G)$ is the *adjacency matrix* and $D(G)$ is the diagonal matrix of vertex degrees. It is well known that $L(G)$ is a positive semidefinite matrix, and 0 is the least eigenvalue with an all ones vector being the corresponding eigenvector. Fiedler [**1**] proved that G is a connected graph if and only if the second smallest eigenvalue of $L(G)$ is positive. This eigenvalue is called *algebraic connectivity* of G, and the corresponding eigenvectors are usually called *Fiedler vectors.*

2000 Mathematics Subject Classification. 05C50, 15A18.
Keywords and Phrases. Bethe tree, Laplacian matrix, adjacency matrix, Perron vector, Fiedler vector.
Supported by National Natural Science Foundation of China (10601001, 60772121), Anhui Provincial Natural Science Foundation (050460102, 070412065), NSF of Department of Education of Anhui Province (2005kj005zd), Project of Anhui University on Leading Researchers Construction, Foundation of Innovation Team on Basic Mathematics of Anhui University.

A *tree* is a connected acyclic graph, and a *Bethe tree* [**5**] of k levels is a rooted tree such that the vertex root has degree d, the vertices in the intermediate levels have degree $(d+1)$ and the vertices in level k are the pendant vertices. A *generalized Bethe tree* $\mathcal{B}_k$ of k levels is introduced by Rojo and Soto [**7**] and is defined to a rooted tree with the vertices in each level having equal degree. They found the eigenvalues of the adjacency matrix and Laplacian matrix of $\mathcal{B}_k$, which are respectively the eigenvalues of leading principal submatrices of two nonnegative symmetric tridiagonal matrices of order k whose entries are given in terms of the vertex degrees.

In 2006 Rojo [**8**] discusses the trees $\mathcal{B}_k^{(2)}$ obtained from two copies of $\mathcal{B}_k$ by joining an edge between two roots. Recently, Rojo [**9**] discusses the graphs $\mathcal{B}_k^{(r)}$ obtained from r copies of $\mathcal{B}_k$ and a cycle $\mathcal{C}_r$ on r vertices by appending r roots to the vertices of $\mathcal{C}_r$ respectively. For the trivial case of $r=1$, then $\mathcal{B}_k^{(r)}$ is a generalized Bethe tree.

Motivated by the work of Rojo et.al., we consider more general graph $\mathcal{B}_k \circ \mathcal{T}^r$, which is obtained from r copies of B_k and a connected transitive graph $\mathcal{T}^r$ on r vertices by appending r roots to the vertices of $\mathcal{T}^r$ respectively. Note that a graph is called *transitive* if for any two vertices u, w of the graph, there exists an automorphism of the graph that maps u to w. Obviously, the cycle $\mathcal{C}_r$ is transitive. So the graphs $\mathcal{B}_k^{(r)}$ are special cases of our graphs $\mathcal{B}_k \circ \mathcal{T}^r$.

In this paper, we give a simple way to characterize the eigenvalues of the adjacency matrix $A(\mathcal{B}_k \circ \mathcal{T}^r)$ and the Laplacian matrix $L(\mathcal{B}_k \circ \mathcal{T}^r)$ of $\mathcal{B}_k \circ \mathcal{T}^r$ by means of symmetric tridiagonal matrices of order k. We also present some structure properties of the Perron vectors of $A(\mathcal{B}_k \circ \mathcal{T}^r)$ and the Fiedler vectors of $L(\mathcal{B}_k \circ \mathcal{T}^r)$. In addition, we obtain some results on transitive graphs.

2. PRELIMINARIES

Let $\mathcal{B}_k \circ \mathcal{T}^r$ be on n vertices, where $k \geq 2, r \geq 3$. In general, $\mathcal{B}_k \circ \mathcal{T}^r$ can be considered as a graph of k levels such that in each level the vertices have equal degree, where the vertices of $\mathcal{T}^r$ are at level 1. Obviously, $\mathcal{T}^r$ is regular whose degree is denoted by d_0. We assume that $d_0 \geq 2$. (Note that if $d_0 = 1$ then $\mathcal{T}^r$ is an edge and in this case $\mathcal{B}_k \circ \mathcal{T}^r = \mathcal{B}_k^{(2)}$.) For $j = 1, 2, \ldots, k$, let d_{k-j+1} and n_{k-j+1} be the degree of the vertices and number of them in level j. Thus, we get that $d_k > d_0, n_k = r, d_1 = 1$, and $n = \sum\limits_{j=1}^{k} n_{k-j+1}$ is the total number of vertices. Moreover,

$$n_{k-1} = (d_k - d_0) n_k, \tag{2.1}$$

$$n_{k-j} = (d_{k-j+1} - 1) n_{k-j+1}, \; 2 \leq j \leq k-1. \tag{2.2}$$

Denote by

$$m_j = \frac{n_j}{n_{j+1}}, \; j = 1, 2, \ldots, k-1. \tag{2.3}$$

We give some notations that we will use in our paper. $\mathbf{0}_m, \mathbf{I}_m$ are respectively the zero matrix and the identity matrix of order $m \times m$. $\mathbf{e}_m$ is the all ones column vector of dimension m. In some places we simply write $\mathbf{0}, \mathbf{I}, \mathbf{e}$ if there exists no confusion. For $j = 1, 2, \ldots, k-1$, denote

$$C_j = \mathrm{diag}\{\mathbf{e}_{m_j}, \mathbf{e}_{m_j}, \ldots, \mathbf{e}_{m_j}\},$$

a block diagonal matrix with n_{j+1} diagonal blocks. Thus, the size of C_j is $n_j \times n_{j+1}$ by Eq. (2.3). Denote

$$E_r(\alpha) = \alpha \mathbf{I}_r + A(\mathcal{T}^r).$$

Using the labels $1, 2, 3, \ldots, n$, in this order, our labeling for the vertices of $\mathcal{B}_k \circ \mathcal{T}^r$ is: *Label the vertices of* $\mathcal{B}_k \circ \mathcal{T}^r$ *from the level* k *to the level* 1 *and, in each level, in a counterclockwise sense*; see an example below.

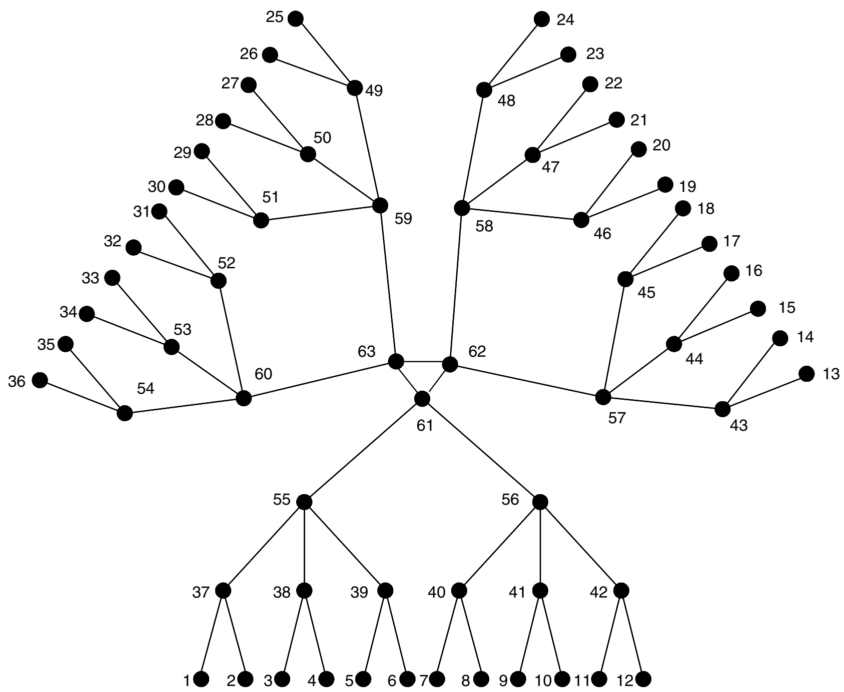

Fig. 2.1. An example of labeling for vertices of $\mathcal{B}_4 \circ \mathcal{C}^3$ (c.f. [**9**]).

Our labeling for the vertices of $\mathcal{B}_k \circ \mathcal{T}^r$ yields to the adjacency matrix and

Laplacian matrix as follows, which are both tridiagonal block matrices.

$$A(\mathcal{B}_k \circ \mathcal{T}^r) = \begin{bmatrix} \mathbf{0}_{n_1} & C_1 & & & \\ C_1^T & \mathbf{0}_{n_2} & C_2 & & \\ & C_2^T & \ddots & \ddots & \\ & & \ddots & \mathbf{0}_{n_{k-1}} & C_{k-1} \\ & & & C_{k-1}^T & E_r(0) \end{bmatrix}, \tag{2.4}$$

$$L(\mathcal{B}_k \circ \mathcal{T}^r) = \mathrm{diag}\{\mathbf{I}_{n_1}, d_2\mathbf{I}_{n_2}, \ldots, d_k\mathbf{I}_k\} - A(\mathcal{B}_k \circ \mathcal{T}^r). \tag{2.5}$$

As $\mathcal{T}^r$ is connected, $A(\mathcal{T}^r)$ is symmetric and irreducible, the spectral radius of $A(\mathcal{T}^r)$, also called *Perron value* of $A(\mathcal{T}^r)$, is exactly the largest eigenvalue; and by Perron-Frobenius Theorem, this eigenvalue is simple. They exists a unique (up to multiples) corresponding positive eigenvector, usually referred to the *Perron vector* of $A(\mathcal{T}^r)$. As $\mathcal{T}^r$ is regular of degree d_0, $\mathbf{e}_r$ is a Perron vector corresponding the Perron value d_0. Suppose $A(\mathcal{T}^r)$ has p ($p \leq r$) distinct eigenvalues arranged as

$$d_0 = \lambda_1(\mathcal{T}^r) > \lambda_2(\mathcal{T}^r) > \cdots > \lambda_p(\mathcal{T}^r).$$

and denote the multiplicity of $\lambda_\ell(\mathcal{T}^r)$ by $m_\ell(\mathcal{T}^r)$ for each $\ell = 1, 2, \ldots, p$. By above discussion, $m_1(\mathcal{T}^r) = 1$, and $\sum_{\ell=1}^{p} m_\ell(\mathcal{T}^r) = r$ (the order of $A(\mathcal{T}^r)$). Obviously, $\det A(\mathcal{T}^r) = \prod_{\ell=1}^{p} \lambda_\ell(\mathcal{T}^r)^{m_\ell(\mathcal{T}^r)}$, and

$$\det E_r(\alpha) = \prod_{\ell=1}^{p} \big(\lambda_\ell(\mathcal{T}^r) + \alpha\big)^{m_\ell(\mathcal{T}^r)}. \tag{2.6}$$

Lemma 2.1. *Let*

$$M = \begin{bmatrix} \alpha_1 \boldsymbol{I}_{n_1} & C_1 & & & \\ C_1^T & \alpha_2 \boldsymbol{I}_{n_2} & C_2 & & \\ & C_2^T & \ddots & \ddots & \\ & & \ddots & \alpha_{k-1}\boldsymbol{I}_{n_{k-1}} & C_{k-1} \\ & & & C_{k-1}^T & E_r(\alpha_k) \end{bmatrix},$$

and let

$$\beta_1 = \alpha_1, \beta_j = \alpha_j - \frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}}, \quad j = 2, 3, \ldots, k, \ \beta_{j-1} \neq 0.$$

Suppose $\beta_j \neq 0$ for all $j = 1, 2, \ldots, k-1$. Then

$$\det M = \beta_1^{n_1}\beta_2^{n_2} \ldots \beta_{k-1}^{n_{k-1}} \prod_{\ell=1}^{p} \big(\lambda_\ell(\mathcal{T}^r) + \beta_k\big)^{m_\ell(\mathcal{T}^r)}.$$

Proof. Applying the Gaussian elimination procedure to M without row interchanges, we obtain the block upper triangular matrix

$$\begin{bmatrix} \beta_1\mathbf{I}_{n_1} & C_1 & & & \\ & \beta_2\mathbf{I}_{n_2} & C_2 & & \\ & & \ddots & \ddots & \\ & & & \beta_{k-1}\mathbf{I}_{n_{k-1}} & C_{k-1} \\ & & & & E_r(\beta_k) \end{bmatrix}.$$

Hence,

$$\det M = \beta_1^{n_1}\beta_2^{n_2}\ldots\beta_{k-1}^{n_{k-1}}\det E_r(\beta_k).$$

The result follows by Eq. (2.6). □

3. SPECTRUM OF THE LAPLACIAN MATRIX

Denote

$$\Omega = \{j : n_j > n_{j+1}, j = 1, 2, \ldots, k-1\}.$$

Definition 3.1 *Let*

$$P_0(\lambda) = 1$$
$$P_1(\lambda) = \lambda - 1,$$
$$P_j(\lambda) = (\lambda - d_j)P_{j-1}(\lambda) - \frac{n_{j-1}}{n_j}P_{j-2}(\lambda), \text{ for } j = 2, 3, \ldots, k;$$

and let

$$P_{k,\ell}(\lambda) = \big(\lambda - (d_k - \lambda_\ell(\mathcal{T}^r))\big)P_{k-1}(\lambda) - \frac{n_{k-1}}{n_k}P_{k-2}(\lambda), \text{ for } \ell = 1, 2, \ldots, p.$$

Theorem 3.2. *The characteristic polynomial of $L(\mathcal{B}_k \circ \mathcal{T}^r)$ is*

$$\det\big(\lambda\boldsymbol{I} - L(\mathcal{B}_k \circ \mathcal{T}^r)\big) = \prod_{j\in\Omega} P_j(\lambda)^{n_j - n_{j+1}} \prod_{\ell=1}^{p} P_{k,\ell}(\lambda)^{m_\ell(\mathcal{T}^r)}. \tag{3.1}$$

Proof. Suppose that $\lambda \in \mathbb{R}$ is such that $P_j(\lambda) \neq 0$ for all $j = 1, 2, \ldots, k-1$. We apply Lemma 2.1 to the matrix $\lambda\mathbf{I} - L(\mathcal{B}_k \circ \mathcal{T}^r)$ by substituting $\alpha_1 = \lambda - 1$, $\alpha_\ell = \lambda - d_\ell$ for $\ell = 2, 3, \ldots, k$. Then

$$\beta_1 = \lambda - 1 = P_1(\lambda),$$
$$\beta_2 = (\lambda - d_2) - \frac{n_1}{n_2}\frac{1}{\beta_1} = (\lambda - d_2) - \frac{n_1}{n_2}\frac{1}{P_1(\lambda)}$$
$$= \frac{(\lambda - d_2)P_1(\lambda) - \frac{n_1}{n_2}P_0(\lambda)}{P_1(\lambda)} = \frac{P_2(\lambda)}{P_1(\lambda)}.$$

Similarly for $j = 3, \ldots, k$, we get

$$\beta_j = \frac{P_j(\lambda)}{P_{j-1}(\lambda)} .$$

In addition,

$$\begin{aligned}
\beta_k + \lambda_\ell(\mathcal{T}^r) &= \frac{P_k(\lambda)}{P_{k-1}(\lambda)} + \lambda_\ell(\mathcal{T}^r) \\
&= \frac{(\lambda - d_k)P_{k-1}(\lambda) - \frac{n_{k-1}}{n_k} P_{k-2}(\lambda) + \lambda_\ell(\mathcal{T}^r)P_{k-1}(\lambda)}{P_{k-1}(\lambda)} \\
&= \frac{\big(\lambda - (d_k - \lambda_\ell(\mathcal{T}^r))\big) P_{k-1}(\lambda) - \frac{n_{k-1}}{n_k} P_{k-2}(\lambda)}{P_{k-1}(\lambda)} = \frac{P_{k,\ell}(\lambda)}{P_{k-1}(\lambda)}.
\end{aligned}$$

Then by Lemma 2.1,

$$\begin{aligned}
\det\big(\lambda\mathbf{I} - L(\mathcal{B}_k \circ \mathcal{T}^r)\big) &= \beta_1^{n_1}\beta_2^{n_2}\ldots\beta_{k-1}^{n_{k-1}} \det E_r(\beta_k) \\
&= P_1^{n_1}\frac{P_2^{n_2}}{P_1^{n_2}}\frac{P_3^{n_3}}{P_2^{n_3}}\cdots\frac{P_{k-1}^{n_{k-1}}}{P_{k-2}^{n_{k-1}}}\prod_{\ell=1}^{p}\big(\lambda_\ell(\mathcal{T}^r) + \beta_k\big)^{m_\ell(\mathcal{T}^r)} \\
&= P_1^{n_1}\frac{P_2^{n_2}}{P_1^{n_2}}\frac{P_3^{n_3}}{P_2^{n_3}}\cdots\frac{P_{k-1}^{n_{k-1}}}{P_{k-2}^{n_{k-1}}}\prod_{\ell=1}^{p}\frac{P_{k,\ell}^{m_\ell(\mathcal{T}^r)}}{P_{k-1}^{m_\ell(\mathcal{T}^r)}} \\
&= \prod_{j\in\Omega} P_j^{n_j - n_{j+1}} \prod_{\ell=1}^{p} P_{k,\ell}^{m_\ell(\mathcal{T}^r)},
\end{aligned}$$

where, in above steps the variable λ is omitted for brevity, and the last equality follows as $\sum_{\ell=1}^{p} m_\ell(\mathcal{T}^r) = r = n_k$.

Thus (3.1) is proved for all $\lambda \in \mathbb{R}$ such that $P_j(\lambda) \neq 0$ for $j = 1, 2, \ldots, k-1$. By the fact that $\det(\lambda\mathbf{I} - L(\mathcal{B}_k \circ \mathcal{T}^r))$ is a polynomial of finite degree and there exist infinite λ's such that (3.1) holds, so (3.1) holds for all $\lambda \in \mathbb{R}$. The result follows. □

Corollary 3.3. *Up to multiplicities, the spectrum of $L(\mathcal{B}_k \circ \mathcal{T}^r)$ is*

$$\sigma\big(L(\mathcal{B}_k \circ \mathcal{T}^r)\big) = \Big(\underset{j\in\Omega}{\cup}\{\lambda : P_j(\lambda) = 0\}\Big) \bigcup \Big(\overset{p}{\underset{\ell=1}{\cup}}\{\lambda : P_{k,\ell}(\lambda) = 0\}\Big). \tag{3.2}$$

Definition 3.4. *Let $T_{k,\ell}$ be the $k \times k$ symmetric tridiagonal matrix given below*

$$T_{k,\ell} = \begin{bmatrix}
1 & \sqrt{d_2 - 1} & & & \\
\sqrt{d_2 - 1} & d_2 & \sqrt{d_3 - 1} & & \\
& \sqrt{d_3 - 1} & \ddots & \ddots & \\
& & \ddots & d_{k-1} & \sqrt{d_k - d_0} \\
& & & \sqrt{d_k - d_0} & d_k - \lambda_\ell(\mathcal{T}^r)
\end{bmatrix}$$

for $\ell = 1, 2, \ldots, p$.

Lemma 3.5. *For* $j = 1, 2, \ldots, k-1$, *let* T_j *be the* j*th leading principal submatrix of* $T_{k,1}$. *Then*

$$\det(\lambda \boldsymbol{I} - T_j) = P_j(\lambda), \text{ for } j = 1, 2, \ldots, k-1, \tag{3.3}$$

$$\det(\lambda \boldsymbol{I} - T_{k,\ell}) = P_{k,\ell}(\lambda), \text{ for } \ell = 1, 2, \ldots, p. \tag{3.4}$$

Proof. It is easily seen that $\det(\lambda \mathbf{I} - T_1) = \lambda - 1 = P_1(\lambda)$ and

$$\det(\lambda \mathbf{I} - T_2) = (\lambda - d_2)(\lambda - 1) - \left(\sqrt{d_2 - 1}\right)^2 = (\lambda - d_2)P_1(\lambda) - \frac{n_1}{n_2} P_0(\lambda) = P_2(\lambda),$$

where $P_0(\lambda) = 1$ and $d_2 - 1 = \frac{n_1}{n_2}$ by (2.2). Generally for $j = 3, \ldots, k-1$, by the fact $d_j - 1 = \frac{n_{j-1}}{n_j}$,

$$\det(\lambda \mathbf{I} - T_j) = (\lambda - d_j)P_{j-1}(\lambda) - \left(\sqrt{d_j - 1}\right)^2 P_{j-2}(\lambda) = P_j(\lambda).$$

Similarly, as $d_k - d_0 = \frac{n_{k-1}}{n_k}$ by (2.1),

$$\det(\lambda \mathbf{I} - T_{k,\ell}) = \big(\lambda - (d_k - \lambda_\ell(\mathcal{T}^r))\big)P_{k-1}(\lambda) - \left(\sqrt{d_k - d_0}\right)^2 P_{k-2}(\lambda) = P_{k,\ell}(\lambda).$$

□

By Corollary 3.3 and Lemma 3.5, we have following result.

Theorem 3.6. *Let* T_j, $j = 1, 2, \ldots, k-1$, *and* $T_{k,\ell}$, $\ell = 1, 2, \ldots, p$ *be as above. Then, up to multiplicities,*

$$\sigma\big(L(\mathcal{B}_k \circ \mathcal{T}^r)\big) = \Big(\underset{j \in \Omega}{\cup} \sigma(T_j) \Big) \bigcup \Big(\overset{p}{\underset{\ell=1}{\cup}} \sigma(T_{k,\ell}) \Big).$$

Denote by $\rho(A)$ the spectral radius of a square matrix A. Now we recall some well known facts in the following Lemma.

Lemma 3.7. Fact 1. *The eigenvalues of a Hermitian matrix do not decrease if a positive semidefinite matrix is added to it* [**6**].

Fact 2. *If* A *is an* $m \times m$ *symmetric tridiagonal matrix with nonzero codiagonal entries, then the eigenvalues of any* $(m-1) \times (m-1)$ *principal submatrix strictly interlace the eigenvalues of* A [**4**].

Fact 3. *If* A *is an irreducible nonnegative matrix , then* $\rho(A)$ *strictly increases when any entry of* A *strictly increases* [**11**].

Fact 4. *If* A *is an irreducible nonnegative matrix , then* $\rho(A)$ *is a simple eigenvalue of* A [**11**].

Theorem 3.8.

(a) *For* $j = 2, 3, \ldots, k$, $\sigma(T_{j-1}) \cap \sigma(T_j) = \emptyset$; *and for* $\ell = 1, 2, \ldots, p$, $\sigma(T_{k-1}) \cap \sigma(T_{k,\ell}) = \emptyset$.

(b) *If* $1 \le i \ne j \le p$, *then* $\sigma(T_{k,i}) \cap \sigma(T_{k,j}) = \emptyset$.

(c) *For* $j = 1, 2, \ldots, k-1$, $\det T_j = 1$; *and for* $\ell = 1, 2, \ldots, p$, $\det T_{k,\ell} = d_0 - \lambda_\ell^G$, *in particular*, $\det T_{k,1} = 0$.

(d) *For* $j = 1, 2, \ldots, k-1$, *each eigenvalue of* T_j *is simple*; *and for* $\ell = 1, 2, \ldots, p$, *each eigenvalue of* $T_{k,\ell}$ *is also simple, and*

$$\rho(T_{k,1}) < \rho(T_{k,2}) < \cdots < \rho(T_{k,p}).$$

(e) *The largest eigenvalue of* $T_{k,p}$ *is the spectral radius of* $L(\mathcal{B}_k \circ \mathcal{T}^r)$.

(f) *The smallest eigenvalue of* $T_{k,2}$ *is the algebraic connectivity of* $\mathcal{B}_k \circ \mathcal{T}^r$.

Proof. (a) It follows from Fact 2 of Lemma 3.7.

(b) Suppose that there exists $\bar\lambda \in \sigma(T_{k,i}) \cap \sigma(T_{k,j})$. From Eq. (3.4),

$$P_{k,i}(\bar\lambda) = P_{k,j}(\bar\lambda).$$

That is

$$\begin{aligned}\big(\bar\lambda - (d_k - \lambda_i(\mathcal{T}^r))\big)P_{k-1}(\bar\lambda) - \frac{n_{k-1}}{n_k}\, P_{k-2}(\bar\lambda)\\ = \big(\bar\lambda - (d_k - \lambda_j(\mathcal{T}^r))\big)P_{k-1}(\bar\lambda) - \frac{n_{k-1}}{n_k}\, P_{k-2}(\bar\lambda),\end{aligned}$$

and hence $\lambda_i(\mathcal{T}^r) = \lambda_j(\mathcal{T}^r)$, a contradiction.

(c) For $1 \le j \le k-1$, by the Gaussian elimination procedure without row interchanges, we reduce T_j to the upper triangular matrix

$$\begin{bmatrix} 1 & \sqrt{d_2 - 1} & & & & \\ & 1 & \sqrt{d_3 - 1} & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \sqrt{d_{j-1} - 1} & \\ & & & & 1 & \sqrt{d_j - 1} \\ & & & & & 1 \end{bmatrix}.$$

Then $\det T_j = 1$. By a similar procedure, we get $\det T_{k,\ell} = d_0 - \lambda_\ell(\mathcal{T}^r)$, for $\ell = 1, 2, \ldots, p$. As $\lambda_1(\mathcal{T}^r) = d_0$, $\det T_{k,1} = 0$.

(d) Clearly, the result holds for T_1 as it is of order 1. Assume that T_j ($j \ge 2$) has an eigenvalue $\bar\lambda$ of multiplicity greater than 1. Then by interlacing property, T_{j-1} also has $\bar\lambda$ as one of its eigenvalues. Hence T_j and T_{j-1} have a common eigenvalue, a contradiction to Fact 2 of Lemma 3.7. So each eigenvalue of T_j is simple for $j = 1, 2, \ldots, k-1$. By a similar discussion, we get that each eigenvalue of $T_{k,\ell}$ is also simple for $\ell = 1, 2, \ldots, p$. Moreover, $d_k - \lambda_p(\mathcal{T}^r) > d_k - \lambda_{p-1}(\mathcal{T}^r) > \cdots > d_k - \lambda_1(\mathcal{T}^r)$, so we get the strict inequalities by Fact 3 of Lemma 3.7.

(e) From Fact 2 of Lemma 3.7, any eigenvalue in the set $\bigcup_{j=1}^{k-1} \sigma(T_j)$ is bounded above by $\rho(T_{k-1})$. This fact also implies that $\rho(T_{k-1})$ is strictly less than $\rho(T_{k,\ell})$ for all ℓ. Finally we use Theorem 3.6 and the inequalities in (d) to obtain the desired result.

(f) For $\ell = 1, 2, \ldots, p-1$.

$$T_{k,\ell+1} = T_{k,\ell} + \mathrm{diag}\{0, 0, \ldots, 0, \lambda_\ell(\mathcal{T}^r) - \lambda_{\ell+1}(\mathcal{T}^r)\}.$$

As $\lambda_\ell(\mathcal{T}^r) - \lambda_{\ell+1}(\mathcal{T}^r) > 0$, by Fact 1 of Lemma 3.7 each eigenvalue of $T_{k,\ell+1}$ is greater or equal to the corresponding eigenvalue of $T_{k,\ell}$. In particular, if $\lambda_k(T_{k,\ell})$ denotes the smallest eigenvalue of $T_{k,\ell}$, then

$$\lambda_k(T_{k,1}) < \lambda_k(T_{k,2}) < \cdots < \lambda_k(T_{k,t}),$$

where, the strict inequalities hold since the matrices $T_{k,\ell}, \ell = 1, 2, \ldots, p$ have no common eigenvalues by the result of (b). We already proved that $\det T_{k,1} = 0$, hence $\lambda_k(T_{k,1}) = 0$. In addition, since the eigenvalues of each matrix T_j interlace the eigenvalues of $T_{k,2}$, it follows that $\lambda_k(T_{k,2})$ is the second smallest positive eigenvalue of $L(\mathcal{B}_k \circ \mathcal{T}^r)$, i.e., the algebraic connectivity of $\mathcal{B}_k \circ \mathcal{T}^r$. □

Next we give some results on the multiplicity of the eigenvalues of $L(\mathcal{B}_k \circ \mathcal{T}^r)$.

Theorem 3.9. *Let T_j, $j = 1, 2, \ldots, k-1$, and $T_{k,\ell}$, $\ell = 1, 2, \ldots, p$, be as above. Then*

(a) *For $j \in \Omega$, each eigenvalue of T_j, as an eigenvalue of $L(\mathcal{B}_k \circ \mathcal{T}^r)$ has a multiplicity greater than or equal to $(n_j - n_{j+1})$.*

(b) *For $\ell = 1, 2, \ldots, p$, each eigenvalue of $T_{k,\ell}$, as an eigenvalue of $L(\mathcal{B}_k \circ \mathcal{T}^r)$ has a multiplicity greater than or equal to $m_\ell(\mathcal{T}^r)$.*

(c) *The spectral radius of $L(\mathcal{B}_k \circ \mathcal{T}^r)$ has a multiplicity $m_p(\mathcal{T}^r)$.*

(d) *The algebraic connectivity of $\mathcal{B}_k \circ \mathcal{T}^r$ has a multiplicity $m_2(\mathcal{T}^r)$.*

Proof. The results (a) and (b) are immediate consequence of Theorem 3.2 and Lemma 3.5. Now we prove (c). Note that $\rho\big(L(\mathcal{B}_k \circ \mathcal{T}^r) = \rho(T_{k,p})\big)$, $\rho(T_{k,p})$ is a simple eigenvalue of $T_{k,p}$ and is strictly greater than the largest eigenvalue of T_j for $j = 1, 2, \ldots, k-1$, and is also strictly greater than the largest eigenvalue of $T_{k,\ell}$ for $\ell = 1, 2, \ldots, p-1$. Hence the multiplicity of $\rho(T_{k,p})$ as an eigenvalue of $L(\mathcal{B}_k \circ \mathcal{T}^r)$ is given by the exponent of the polynomial $P_{k,p}$ as in (3.1) of Theorem 3.2.

For the last result (d), we have proved that $\lambda_k(T_{k,2})$ is the algebraic connectivity of $\mathcal{B}_k \circ \mathcal{T}^r$. We assert that $\lambda_k(T_{k,2}) \notin \sigma(T_j)$ for $j = 1, 2, \ldots, k-1$; otherwise, by the interlacing property, $\lambda_k(T_{k,2}) \leq \lambda_{k-1}(T_{k-1}) \leq \cdots \leq \lambda_j(T_j)$, where $\lambda_\ell(T_\ell)$ denotes the smallest eigenvalue of T_ℓ, then $\lambda_k(T_{k,2}) = \lambda_j(T_j)$, and hence $\lambda_k(T_{k,2}) = \lambda_{k-1}(T_{k-1})$, a contradiction to Fact 2 of Lemma 3.7. In addition, $\lambda_k(T_{k,2}) \notin \sigma(T_{k,\ell})$ for $\ell \neq 2$ by Theorem 3.8(b), and $\lambda_k(T_{k,2})$ is a simple eigenvalue of $T_{k,2}$ by Theorem 3.8(d). Therefore, the multiplicity of $\lambda_k(T_{k,2})$ as an eigenvalue of $L(\mathcal{B}_k \circ \mathcal{T}^r)$ is given by the exponent of the polynomial $P_{k,2}$ in (3.1) of Theorem 3.2. □

4. SPECTRUM OF THE ADJACENCY MATRIX

Let

$$D = \mathrm{diag}\{-\mathbf{I}_{n_1}, \mathbf{I}_{n_2}, -\mathbf{I}_{n_3}, \cdots, (-1)^{k-1}\mathbf{I}_{n_{k-1}}, (-1)^k \mathbf{I}_{n_k}\}.$$

From the matrix form [2.4], we can easily get that

$$D(\lambda \mathbf{I} + A(\mathcal{B}_k \circ \mathcal{T}^r) D^{-1} = \lambda \mathbf{I} - A(\mathcal{B}_k \circ \mathcal{T}^r). \tag{4.1}$$

Definition 4.1 *Let*

$$\begin{aligned} S_0(\lambda) &= 1, \\ S_1(\lambda) &= \lambda, \\ S_j(\lambda) &= \lambda S_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} S_{j-2}(\lambda), \ \textit{for } j = 2, \dots, k. \end{aligned}$$

and

$$S_{k,\ell}(\lambda) = \big(\lambda + \lambda_\ell(\mathcal{T}^r)\big) S_{k-1}(\lambda) - \frac{n_{k-1}}{n_k} S_{k-2}(\lambda), \ \textit{for } \ell = 1, 2, \dots, p.$$

Theorem 4.2. *The characteristic polynomial of $A(\mathcal{B}_k \circ \mathcal{T}^r)$ is*

$$\det(\lambda \boldsymbol{I} - A(\mathcal{B}_k \circ \mathcal{T}^r) = \prod_{j \in \Omega} S_j(\lambda)^{n_j - n_{j+1}} \prod_{\ell=1}^{p} S_{k,\ell}(\lambda)^{m_\ell(\mathcal{T}^r)}. \tag{14}$$

Proof. The proof is similar to that of Theorem 3.2. Apply Lemma 2.1 to the matrix $\lambda \mathbf{I} + A(\mathcal{B}_k \circ \mathcal{T}^r)$ by substituting α_j with λ for $j = 1, 2, \dots, k$. The result follows from (4.1). □

Corollary 4.3. *Up to multiplicities, the spectrum of $A(\mathcal{B}_k \circ \mathcal{T}^r)$ is*

$$\sigma\big(A(\mathcal{B}_k \circ \mathcal{T}^r)\big) = \Big(\underset{j \in \Omega}{\cup} \{\lambda : S_j(\lambda) = 0\}\Big) \bigcup \Big(\overset{p}{\underset{\ell=1}{\cup}} \{\lambda : S_{k,\ell}(\lambda) = 0\}\Big). \tag{4.2}$$

Definition 4.4. *Let $R_{k,\ell}$ be the $k \times k$ symmetric tridiagonal matrix given below*

$$R_{k,\ell} = \begin{bmatrix} 0 & \sqrt{d_2 - 1} & & & \\ \sqrt{d_2 - 1} & 0 & \sqrt{d_3 - 1} & & \\ & \sqrt{d_3 - 1} & \ddots & \ddots & \\ & & \ddots & 0 & \sqrt{d_k - d_0} \\ & & & \sqrt{d_k - d_0} & \lambda_\ell(\mathcal{T}^r) \end{bmatrix}$$

for $\ell = 1, 2, \ldots, p$.

Lema 4.5. *For* $j = 1, 2, \ldots, k-1$, *let* R_j *be the* j*th leading principal submatrix of* $R_{k,1}$. *Then*

$$\det(\lambda \boldsymbol{I} - R_j) = S_j(\lambda), \text{ for } j = 1, 2, \ldots, k-1.$$
$$\det(\lambda \boldsymbol{I} - R_{k,\ell}) = S_{k,\ell}(\lambda), \text{ for } \ell = 1, 2, \ldots, p.$$

Proof. The proof is similar to that of Lemma 3.5. □

By Corollary 4.3 and Lemma 4.5, we have following result.

Theorem 4.6. *Let* $R_j, j = 1, 2, \ldots, k-1$ *and* $R_{k,\ell}$, $\ell = 1, 2, \ldots, p$, *be as above. Then, up to multiplicities,*

$$\sigma\big(A(\mathcal{B}_k \circ \mathcal{T}^r)\big) = \Big(\bigcup_{j=1}^{k-1} \sigma(R_j)\Big) \bigcup \Big(\bigcup_{\ell=1}^{p} \sigma(R_{k,\ell})\Big).$$

Parallel to Theorem 3.8 and Theorem 3.9, we get the results below, where the proofs are similar and are omitted here.

Theorem 4.7.

(a) $\sigma(R_{j-1}) \cap \sigma(R_j) = \emptyset$, *for* $j = 2, 3, \ldots, k-1$; $\sigma(R_{k-1}) \cap \sigma(R_{k,\ell}) = \emptyset$, *for* $\ell = 1, 2, \ldots, p$.

(b) *If* $1 \leq i \neq j \leq p$, *then* $\sigma(R_{k,i}) \cap \sigma(R_{k,j}) = \emptyset$.

(c) *For* $j = 1, 2, \ldots, k-1$, *each eigenvalue of* R_j *is simple. For* $\ell = 1, 2, \ldots, p$, *each eigenvalue of* $R_{k,\ell}$ *is simple, and*

$$\rho(R_{k,1}) > \rho(R_{k,2}) > \cdots > \rho(R_{k,p}).$$

(d) *The largest eigenvalue of* $R_{k,1}$ *is the spectral radius of* $A(\mathcal{B}_k \circ \mathcal{T}^r)$.

Theorem 4.8. *Let* $R_j, j = 1, 2, \ldots, k-1$, *and* $R_{k,\ell}, \ell = 1, 2, \ldots, p$, *be as above. Then*

(a) *For* $j \in \Omega$, *each eigenvalue of* R_j, *as an eigenvalue of* $A(\mathcal{B}_k \circ \mathcal{T}^r)$ *has a multiplicity greater than or equal to* $(n_j - n_{j+1})$.

(b) *For* $\ell = 1, 2, \ldots, p$, *each eigenvalue of* $R_{k,\ell}$, *as an eigenvalue of* $A(\mathcal{B}_k \circ \mathcal{T}^r)$ *has a multiplicity greater than or equal to* $m_\ell(\mathcal{T}^r)$.

(c) *The spectral radius of* $A(\mathcal{B}_k \circ \mathcal{T}^r)$ *is a simple eigenvalue.*

Remark. The results in Section 4 and Section 5 on the graph $\mathcal{B}_k \circ \mathcal{T}^r$ also hold in the case of $\mathcal{T}^r$ being regular of degree d_0 (not necessarily being transitive), as we just use the regularity of transitive graphs in above discussions.

5. PROPERTY OF EXTREME EIGENVECTORS

For more precisely, denote $\mathcal{B}(v_i)$ to the generalized BETHE tree of $\mathcal{B}_k \circ \mathcal{T}^r$ appending at the vertex v_i of $\mathcal{T}^r$ for each $i = 1, 2, \dots, r$, where $v_1, v_2, \dots, v_r$ are the vertices of $\mathcal{T}^r$. Let $x = (x_1, x_2, \dots, x_n)$ be an arbitrary given eigenvector of $A(\mathcal{B}_k \circ \mathcal{T}^r)$ (or $L(\mathcal{B}_k \circ \mathcal{T}^r)$) of a graph $\mathcal{B}_k \circ \mathcal{T}^r$ on vertices $v_1, v_2, \dots, v_n$. Then x can be considered as a function defined on the vertex set of $\mathcal{B}_k \circ \mathcal{T}^r$, that is, for any vertex v_i, we map it to $x_i = x(v_i)$. We often say x_i is a value of the vertex v_i given by x. A vertex v is called positive (respectively, negative, zero, nonzero) if $x(v) > 0$ (respectively, $x(v) < 0$, $x(v) = 0$, $x(v) \neq 0$). For each $\mathcal{B}(v_i)$ we use $x(\mathcal{B}(v_i))$ to denote the subvector of x arranged in the original order.

Lemma 5.1. *Let u, w be two vertices of $\mathcal{B}_k \circ \mathcal{T}^r$ at same level. Then the following results hold.*

(a) *If u, w belong to the same generalized Bethe tree $\mathcal{B}(v_i)$ for some i, then there exists an automorphism φ of $\mathcal{B}_k \circ \mathcal{T}^r$ such that $\varphi(u) = w, \varphi(v_i) = v_i, \varphi\big(\mathcal{B}(v_i)\big) = \mathcal{B}(v_i)$ and φ preserves the vertices of $\mathcal{B}_k \circ \mathcal{T}^r - \mathcal{B}(v_i)$, where $\varphi\big(\mathcal{B}(v_i)\big) = \{\varphi(w) : w \in \mathcal{B}(v_i)\}$.*

(b) *If u, w belong to $\mathcal{B}(v_i)$ and $\mathcal{B}(v_j)$ respectively for $i \neq j$, then there exists an automorphism φ such that $\varphi(u) = w, \varphi(v_i) = v_j, \varphi(\mathcal{B}(v_i)) = \mathcal{B}(v_j)$.*

Proof. (a) Clearly there exists an automorphism of $\mathcal{B}(v_i)$ that maps u to w and preserves v_i. Extend this automorphism to $\mathcal{B}_k \circ \mathcal{T}^r$ by preserving the vertices of $\mathcal{B}_k \circ \mathcal{T}^r - \mathcal{B}(v_i)$, and the result follows.

(b) As $\mathcal{T}^r$ is transitive, there exists an automorphism φ of $\mathcal{T}^r$ mapping v_i to v_j. We can extend this automorphism to $\mathcal{B}_k \circ \mathcal{T}^r$ which maps u to w and maps $\mathcal{B}_i$ to $\mathcal{B}_j$. □

First we consider the PERRON vectors of $A(\mathcal{B}_k \circ \mathcal{T}^r)$, i.e., the positive eigenvectors of $A(\mathcal{B}_k \circ \mathcal{T}^r)$ corresponding the spectral radius (or the largest eigenvalue) of $A(\mathcal{B}_k \circ \mathcal{T}^r)$.

Theorem 5.2. *Let x be a Perron vector of $A(\mathcal{B}_k \circ \mathcal{T}^r)$ which gives values of the vertices of $\mathcal{B}_k \circ \mathcal{T}^r$. Then the vertices at the same level of $\mathcal{B}_k \circ \mathcal{T}^r$ have equal values, i.e.,*

$$x\big(\mathcal{B}(v_1)\big) = x\big(\mathcal{B}(v_2)\big) = \cdots = x\big(\mathcal{B}(v_r)\big) = (a_1 \boldsymbol{e}_{n_1/r}, a_2 \boldsymbol{e}_{n_2/r}, \dots, a_k \boldsymbol{e}_1),$$

for some positive real numbers $a_1, a_2, \dots, a_k$.

Proof. Let x be a PERRON vector of $A(\mathcal{B}_k \circ \mathcal{T}^r)$. For arbitrary given two different vertices u, w at the same level of $\mathcal{B}_k \circ \mathcal{T}^r$, by Lemma 5.1, there exists an automorphism φ such that $\varphi(u) = w$. Define a vector x_φ by $x_\varphi(v) = x\big(\varphi(v)\big)$ for each vertex v of $\mathcal{B}_k \circ \mathcal{T}^r$. Then x_φ is also a PERRON vector of $A(\mathcal{B}_k \circ \mathcal{T}^r)$. Note that the spectral radius (as an eigenvalue) of $A(\mathcal{B}_k \circ \mathcal{T}^r)$ is simple. Hence $x_\varphi = kx$ for some nonzero k. Since x_φ and x are both positive and have same length, $x_\varphi = x$, and hence $x(w) = x_\varphi(u) = x(u)$. The result follows. □

Recall that the algebraic connectivity of $\mathcal{B}_k \circ \mathcal{T}^r$ is the second smallest eigenvalue of $L(\mathcal{B}_k \circ \mathcal{T}^r)$, and the corresponding eigenvectors are called FIEDLER vectors of $\mathcal{B}_k \circ \mathcal{T}^r$. Let x be a FIEDLER vectors of $\mathcal{B}_k \circ \mathcal{T}^r$ which gives values of the vertices of $\mathcal{B}_k \circ \mathcal{T}^r$. By FIEDLER result [**2**, Theorem 3.12], $\mathcal{B}_k \circ \mathcal{T}^r$ has either a block containing both positive vertices and negative vertices, called *characteristic block*, or a zero cut-vertex which is adjacent to a nonzero vertex, called *characteristic vertex*. Both characteristic block and characteristic vertex (if one exists) are unique. If $\mathcal{B}_k \circ \mathcal{T}^r$ contains the unique characteristic block B, then each other block has either positive vertices only, or negative vertices only, or zero vertices only, and every path P which contains at most two cut-vertices in each block, which starts in B and contains just one vertex v in B has the property that the values at cut-vertices contained in P form either an increasing, or decreasing, or a zero sequence along this path according to whether $y(v) > 0$, $y(v) < 0$, or $y(v) = 0$; in the last case all vertices in P have value zeros.

Lemma 5.3. [**3**] *A transitive graph of degree d has vertex connectivity at least $\frac{2}{3}(d+1)$.*

Throughout this paper we assume the degree of $\mathcal{T}^r$ is $d_0 \geq 2$. So by Lemma 5.3, $\mathcal{T}^r$ has vertex connectivity at least 2, i.e., $\mathcal{T}^r$ has no cut-vertices and is a block itself. By Theorem 3.9, the algebraic connectivity of $\mathcal{B}_k \circ \mathcal{T}^r$ has multiplicity $m_2(\mathcal{T}^r)$, i.e., the multiplicity of second largest eigenvalue of $A(\mathcal{T}^r)$.

Theorem 5.4. *Suppose that the second largest eigenvalue of $A(\mathcal{T}^r)$ is simple, and x is a Fiedler vector of $\mathcal{B}_k \circ \mathcal{T}^r$ which gives values of the vertices of $\mathcal{B}_k \circ \mathcal{T}^r$. Then the following results hold:*

(a) *For $1 \leq i \neq j \leq r$, $x\big(\mathcal{B}(v_i)\big) = x\big(\mathcal{B}(v_j)\big)$ or $x\big(\mathcal{B}(v_i)\big) = -x\big(\mathcal{B}(v_j)\big)$.*

(b) *$\mathcal{T}^r$ is the unique block of $\mathcal{B}_k \circ \mathcal{T}^r$ which contains both positive and negative vertices and contains no zero vertices.*

(c) *x contains no zero vertices.*

(d) *r is an even number, i.e., $\mathcal{T}^r$ has an even number of vertices.*

(e) *There are exactly $r/2$ generalized Bethe trees of $\mathcal{B}_k \circ \mathcal{T}^r$ whose values given by a positive vector Y and $r/2$ remaining generalized Bethe trees of $\mathcal{B}_k \circ \mathcal{T}^r$ whose values given $-Y$, where*

$$Y = (a_1 \boldsymbol{e}_{n_1/r}, a_2 \boldsymbol{e}_{n_2/r}, \ldots, a_k \boldsymbol{e}_1),$$

for some positive real numbers $a_1, a_2, \ldots, a_k$ with $a_1 > a_2 > \cdots > a_k$. In particular, for the graph $\mathcal{T}^r$, there are $r/2$ vertices with equal value $a_k > 0$ and $r/2$ remaining vertices with equal value $-a_k$.

Proof. (a) The proof is similar to Theorem 5.2. There exists an automorphism φ of $\mathcal{B}_k \circ \mathcal{T}^r$ such that $\varphi\big(\mathcal{B}(v_i)\big) = \mathcal{B}(v_j)$ and φ preserves the order of vertices. Observe that x_φ is also a FIEDLER vector of $\mathcal{B}_k \circ \mathcal{T}^r$. Hence $x_\varphi = x$ or $x_\varphi = -x$ as they have same length, which implies that $x\big(\mathcal{B}(v_i)\big) = x\big(\mathcal{B}(v_j)\big)$ or $x\big(\mathcal{B}(v_i)\big) = -x\big(\mathcal{B}(v_j)\big)$.

(b) If the unique characteristic block or characteristic vertex (except the vertex of $\mathcal{T}^r$) of $\mathcal{B}_k \circ \mathcal{T}^r$ is contained in some $\mathcal{B}(v_i)$, then by the result (a) of this theorem, it is contained in all other generalized BETHE trees, a contradiction to the uniqueness. So the characteristic block or characteristic vertex must be contained in $\mathcal{T}^r$. If $\mathcal{T}^r$ contains the characteristic vertex u of $\mathcal{B}_k \circ \mathcal{T}^r$, then $x(u) = 0$ and u is adjacent to a nonzero vertex. Also by the result (a), $x(\mathcal{T}^r) = 0$, and hence all vertices of $\mathcal{T}^r$ are characteristic vertices, also a contradiction. So $\mathcal{T}^r$ contains the characteristic block of $\mathcal{B}_k \circ \mathcal{T}^r$, and therefore contains both positive and negative valued vertices. By (a), $\mathcal{T}^r$ contains no zero vertices.

(c) The result directly follows from (b) and Case A of Theorem (3.12) of [**2**].

(d) By (a) and (b), for each vertex v_i of $\mathcal{T}^r$, $x(v_i) =: a > 0$ for some positive number a or $x(v_i) = -a$. If $x(v_i) > 0$ (< 0), then $x(\mathcal{B}(v_i)) > 0$ (< 0) by FIEDLER's result [**1**, Theorem (3.12),Case A] Assume that $\mathcal{B}_k \circ \mathcal{T}^r$ has exactly t ($1 \le t \le r-1$) generalized BETHE trees with positive value, and $r - t$ generalized BETHE trees with negative value. By (a), without loss of generality, let $x(\mathcal{B}_1) = x(\mathcal{B}_2) = \cdots = x(B_t) =: Y > 0$, and hence $x(\mathcal{B}_{t+1}) = x(\mathcal{B}_{t+2}) = \cdots = x(B_r) = -Y$. As x is orthogonal to $\mathbf{e}_n$ (an eigenvector corresponding to the zero eigenvalue),

$$tY + (r-t)(-Y) = (r-2t)Y = 0,$$

which implies $r = 2t$, an even number.

(e) From the proof of (d), There are exactly $r/2$ generalized BETHE trees whose values given by a positive vector Y and $r/2$ remaining generalized BETHE trees whose values given $-Y$. For any arbitrary given two different vertices u, w at the same level of $\mathcal{B}(v_i)$, by Lemma 5.1, there exists an automorphism φ such that $\varphi(u) = w, \varphi(v_i) = v_i$ and $\varphi(\mathcal{B}_i) = \mathcal{B}_i$. As $x\big(\mathcal{B}(v_i)\big)$ is either positive or negative, $x_\sigma = x$ and hence $x(u) = x(w)$. Hence $Y = (a_1 \mathbf{e}_{n_1/r}, a_2 \mathbf{e}_{n_2/r}, \ldots, a_k \mathbf{e}_1)$, for some positive real numbers $a_1, a_2, \ldots, a_k$. Also by Case A of Theorem (3.12) of [**2**], we have $a_1 > a_2 > \cdots > a_k$. □

For the transitive graph $\mathcal{T}^r$ of degree d_0,

$$L(\mathcal{T}^r) = d_0 \mathbf{I} - A(\mathcal{T}^r).$$

Then the algebraic connectivity of $\mathcal{T}^r$ is $d_0 - \lambda_2(\mathcal{T}^r)$, and the FIEDLER vectors of $\mathcal{T}^r$ is exactly the eigenvector of $A(\mathcal{T}^r)$ corresponding to $\lambda_2(\mathcal{T}^r)$. Hence for transitive graphs, we only consider the adjacency matrices. At final, we give some results on transitive graph $\mathcal{T}^r$, whose proof is very similar to that of Theorem 5.4. [One can also consider $\mathcal{T}^r$ as a trivial $\mathcal{B}_k \circ \mathcal{T}^r$ by taking every generalized BETHE tree as a vertex.]

Corollary 5.5. *Let $\mathcal{T}^r$ be a transitive graph on r vertices, and let x be an eigenvector of $A(\mathcal{T}^r)$ corresponding the second largest eigenvalue $\lambda_2(\mathcal{T}^r)$. If the eigenvalue $\lambda_2(\mathcal{T}^r)$ is simple, then the following results hold.*

(1) *x has no zero entries*;

(2) *$\mathcal{T}^r$ has an even number of vertices, i.e. $r = 2t$ for some positive integer t;*

(3) *Up to multiples, x contains exactly t entries with value 1 and t entries with value -1.*

Corollary 5.6. *Let $\mathcal{T}^r$ be a transitive graph on r vertices, and let $\lambda_2(\mathcal{T}^r)$ be the second largest eigenvalue of $A(\mathcal{T}^r)$. Then the eigenvalue $\lambda_2(\mathcal{T}^r)$ is simple if and only if each corresponding eigenvector has no zero entries.*

Corollary 5.7. *For a regular graph G on an odd number of vertices, if $A(G)$ has a simple second largest eigenvalue, then it is not transitive.*

EXAMPLE. Let $\mathcal{T}^6$ be obtained by two $\mathcal{C}_3$ by joining 3 independent edges between vertices of them. Then $\lambda_2(A(\mathcal{T}^6)) = 1$ is a simple second largest eigenvalue of $A(\mathcal{T}^6)$. The corresponding eigenvector is shown on the graph in Fig. 5.1 below.

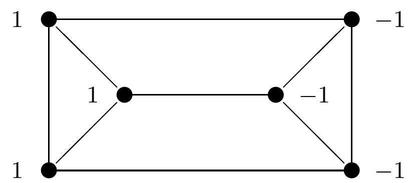

Fig. 5.1. A transitive graph with a simple 2nd largest eigenvalue.

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(Received February 8, 2008)

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