CHROMATIC ZEROS AND THE GOLDEN RATIO

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In this note, we investigate τ^n, where $\tau = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio as chromatic roots. Using some properties of FIBONACCI numbers, we prove that $\tau^n (n \in \mathbb{N})$, cannot be roots of any chromatic polynomial.

1. INTRODUCTION

Let $G = (V, E)$ be a simple graph and $\lambda \in \mathbb{N}$. A mapping $f \colon V \to \{1, 2, \ldots, \lambda\}$ is called a λ-*colouring* of G if $f(u) \neq f(v)$ whenever the vertices u and v are adjacent in G. The number of distinct λ-colourings of G, denoted by $P(G, \lambda)$ is called the *chromatic polynomial* of G. A root of $P(G, \lambda)$ is called a *chromatic root* of G. An interval is called a *root-free interval* for a chromatic polynomial $P(G, \lambda)$ if G has no chromatic root in this interval. It is well-known that $(-\infty, 0)$ and $(0, 1)$ are two maximal root-free intervals for the family of all graphs (see [**2**]). JACKSON [**2**] showed that $(1, 32/27]$ is another maximal root-free interval for the family of all graphs and the value $32/27$ is best possible.

A *plane graph* is a graph drawn in the plane in such a way that any pair of edges meet only once at their end vertices. A *plane triangulation* is a loopless plane graph in which all faces have size three. A *near-triangulation* graph, is a plane graph with at most one non-triangular face.

We recall that a complex number ζ is called an *algebraic number* (resp., an *algebraic integer*) if it is a root of some monic polynomial with rational (resp., integer) coefficients (see [**6**]). Since the chromatic polynomial $P(G, \lambda)$ is a monic polynomial in λ with integer coefficients, its roots are, by definition, algebraic integers. This naturally raises the question: *Which* algebraic integers can occur as zeros of chromatic polynomials? Clearly those lying in $(-\infty, 0) \cup (0, 1) \cup (1, 32/27]$ are forbidden.

Using this reasoning, TUTTE [**7**] proved that $B_5 = \dfrac{3+\sqrt{5}}{2} = 1 + \tau = \tau^2$ ($B_n = 2 + 2\cos(2\pi/n)$ is the n-th *Beraha number* [**1**]) cannot be a chromatic root,

for otherwise $B_5^* = \dfrac{3-\sqrt{5}}{2}$ would also be a chromatic root, which is impossible, since $B_5^* \in (0,1)$. SALAS and SOKAL in [**5**] extended this result to show that the generalized BERAHA numbers $B_n^{(k)} = 4\cos^2(k\pi/n)$ for $n = 5, 7, 8, 9$ and $n \geq 11$, with k coprime to n, are never chromatic roots. For $n = 10$ they showed the weaker result that $B_{10} = \dfrac{5+\sqrt{5}}{2}$ and $B_{10}^* = \dfrac{5-\sqrt{5}}{2}$ are not chromatic roots of any plane near-triangulation. The results and conjectures in [**3**, **8**, **9**, **10**] show that τ and τ^2 has a special significance for chromatic polynomials of plane triangulations.

In this note we investigate τ^n as chromatic zeros. We use some properties of FIBONACCI numbers and the golden ratio to prove that $\tau^n (n \in \mathbb{N})$, where $\tau = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio, cannot be roots of any chromatic polynomials.

2. FIBONACCI NUMBERS AND THE GOLDEN RATIO

FIBONACCI numbers are terms of the sequence defined in a quite simple recursive fashion. However, despite their simplicity, they have some curious properties which are worth attention. In this section, we look at some of the important features of these numbers which are needed to prove the main result in next section.

Definition 1. *The sequence F_n of natural numbers defined by the equations $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ is called the Fibonacci sequence or the sequence of the Fibonacci numbers. The n-th term in the sequence is called the n-th Fibonacci number.*

We recall the following theorem:

Theorem 1 ([**4**], p. 78) $\tau^n = F_n\tau + F_{n-1}$ $(n \in \mathbb{N})$.

Corollary 1. *For every $n \in \mathbb{N}$, $\big| -\tau^{-1}F_n + F_{n-1} \big| < 1$.*

Proof. Similar to theorem 1, by induction on n, we have $(1-\tau)F_n + F_{n-1} = (1-\tau)^n$. Therefore, we have the result.

3. CHROMATIC ZEROS AND THE GOLDEN RATIO

In this section, we prove that τ^n can not be a chromatic root for any natural number n. We state and prove following theorem:

Theorem 2. *Suppose that a, b are rational numbers, and $r \geq 2$ is an integer that is not a perfect square, and $a - |b|\sqrt{r} < 32/27$. Then $a + b\sqrt{r}$ is not the root of any chromatic polynomial.*

Proof. If $\lambda = a + b\sqrt{r}$ is a root of some polynomial with integer coefficients (e.g. a chromatic polynomial), then so is $\lambda^* = a - b\sqrt{r}$. But λ or λ^* can not belong to $(-\infty, 0) \cup (0, 1) \cup (1, 32/27]$, a contradiction. $\square$

The following theorem is our main result:

Theorem 3. $\tau^n (n \in \mathbb{N})$ *is not a root of any chromatic polynomials.*

Proof. By Theorem 1, we can consider τ^n of the form $\lambda = a + b\sqrt{r}$ with $a = \dfrac{F_n}{2} + F_{n-1}, b = \dfrac{F_n}{2}$, and $r = 5$. Since $a - |b|\sqrt{r} = -\tau^{-1}F_n + F_{n-1}$, by Corollary 1, $a - |b|\sqrt{r} < 1 < 32/27$. Therefore we have the result by Theorem 2. $\qquad\square$

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