Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 78–87. doi:10.2298/AADM0901078M

q-EXTENSION OF SOME SYMMETRICAL AND SEMI-CLASSICAL ORTHOGONAL POLYNOMIALS OF CLASS ONE

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We study in detail a q-extension of a symmetrical form (functional) of class one. We show that it is symmetrical and H_q-semi-classical of class one. The moments and a discrete representation are given.

1. INTRODUCTION

The monic orthogonal polynomials sequence (MOPS) $\{S_n\}_{n\geq 0}$ satisfying the recurrence relation [**1**]

$$\begin{cases} S_0(x) = 1, \quad S_1(x) = x, \\ S_{n+2}(x) = xS_{n+1}(x) - \sigma_{n+1}S_n(x), \quad n \geq 0, \end{cases}$$

where

$$\sigma_{2n+1} = -\frac{1}{4}\frac{n+\alpha}{(2n+\alpha)(2n+\alpha+1)}, \quad n \geq 0,$$
$$\sigma_{2n+2} = \frac{1}{4}\frac{n+1}{(2n+\alpha+1)(2n+\alpha+2)}, \quad n \geq 0,$$

is associated with the form $v(\alpha)$. This form is symmetrical semi-classical of class one satisfying the functional equation [**1**]

$$\big(x^3 v(\alpha)\big)' + \Big(-2(\alpha+1)x^2 - \frac{1}{2}\Big)v(\alpha) = 0.$$

2000 Mathematics Subject Classification. 42C05,33C45.
Keywords and Phrases. q-difference operator, H_q-semi-classical polynomials.

Replacing the derivative operator by the q-difference operator H_q [**4**, **6**] and -2α by $\dfrac{1-q^{-2\alpha-2}}{1-q}$ in the precedent equation, we get q-Pearson equation

$$H_q(x^3u(\alpha)) + \Big(\frac{1-q^{-2\alpha-2}}{1-q}x^2 - \frac{1}{2}\Big)u(\alpha) = 0, \quad \alpha \in \mathbb{C}. \tag{1}$$

The aim of this contribution is to determine the symmetrical quasi-definite functional $u(\alpha)$ fulfilling the last equation. This latter is considered the q-analogous of the form $v(\alpha)$. When $q \to 1$, we meet again the form $v(\alpha)$. In fact the problem of defining q-analogous of symmetrical MOPS has been the interest of some authors from different point of views [**2**, **3**, **7**, **10**, **11**, **14**].

The second section is of a preliminary and introductory character. In the third section, we determine the elements of three-term recurrence relation fulfilled by the polynomial sequence, orthogonal with respect to $u(\alpha)$. Finally, in the fourth section we give the moments and a discrete representation.

2. PRELIMINARIES

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual space. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, for any $f \in \mathcal{P}$, any $a \in \mathbb{C}\setminus\{0\}$, we let fu and $h_a u$, be the forms defined by duality

$$\langle fu, p\rangle := \langle u, fp\rangle\,;\ \langle h_a u, p\rangle := \langle u, h_a p\rangle,\ \ p \in \mathcal{P},$$

where $(h_a p)(x) = p(ax)$.

The form u is called quasi-definite functional if we can associate with it a sequence $\{P_n\}_{n\geq 0}$ of monic polynomials $\deg P_n = n,\ n \geq 0$ such that

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m},\ n, m \geq 0;\ r_n \neq 0,\ n \geq 0.$$

The sequence $\{P_n\}_{n\geq 0}$ is orthogonal with respect to u and fulfils the standard recurrence relation:

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\ n \geq 0, \end{cases} \tag{2.1}$$

with $\beta_n = \dfrac{\langle u, xp_n^2(x)\rangle}{\langle u, p_n^2\rangle}$, $n \geq 0$, $\gamma_{n+1} = \dfrac{\langle u, p_{n+1}^2\rangle}{\langle u, p_n^2\rangle}$, $n \geq 0$.

The form u is called normalized if $(u)_0 = 1$ where in general $(u)_n = \langle u, x^n\rangle,\ n \geq 0$, are the moments of u. In this paper we suppose that the forms are normalized.

Let us introduce the Hahn's operator [**6**]

$$(H_q f)(x) := \frac{f(qx) - f(x)}{(q-1)x}\ ,\ f \in \mathcal{P},\ q \in \widetilde{\mathbb{C}}, \tag{2.2}$$

where $q \neq 0$, $q^n \neq 1$, $n \geq 0$. By duality we have

$$\langle H_q u, f\rangle = -\langle u, H_q f\rangle,\ u \in \mathcal{P}',\ f \in \mathcal{P}.$$

When $q \to 1$, we meet again the derivative D.

Definition. *A form u is called H_q-semi-classical when it is regular and satisfies the equation*

$$H_q(\phi u) + \psi u = 0, \tag{2.3}$$

where (ϕ, ψ) are two polynomials, ϕ monic with $\deg \phi \geq 0$ and $\deg \psi \geq 1$. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called H_q-semi-classical.

Moreover, if u is semi-classical satisfying (2.3), the class of u, denoted s is, defined by [**9**]

$$s = \min\big(\deg(\phi) - 2, \deg(\psi) - 1\big),$$

where the minimum is taken over all pairs (ϕ, ψ) satisfying the equation (2.3).

We have the following result:

Proposition 2.1. [**9**] *Let u be a H_q-semi-classical form satisfying the equation* (2.3) *and $s = \max\big(\deg(\phi) - 2, \deg(\psi) - 1\big)$. Then the class of u is s if and only if*

$$\prod_{c\in Z(\phi)} \Big(\big|qh_q\psi(c) + (H_q\phi)(c)\big| + \big|\big\langle u, q\big(\theta_{cq}\psi\big) + \big(\theta_{cq}\circ\theta_c\phi\big)\big\rangle\big|\Big) > 0,$$

where $Z(\phi) := \{z \in \mathbb{C},\ \phi(z) = 0\}$, $(\theta_c\, p)(x) = \dfrac{p(x) - p(c)}{x - c}$, $p \in \mathcal{P}$.

When the last condition is not satisfied for $c \in Z(\phi)$ the equation (2.3) *becomes*

$$H_q\big(\theta_c(\phi)u\big) + \big(q\theta_{cq}\psi + \theta_{cq}\circ\theta_c\phi\big)u = 0.$$

Remark. If u is H_q-semi-classical of class zero, we are dealing with H_q -classical forms or classical functional [**8**, **13**].

Lemma 2.2. *Let $u \in \mathcal{P}'$ the following statements are equivalent*:

(i) *The form u satisfies*

$$H_q\big(x\phi(x)u\big) + \psi(x)u = 0. \tag{2.4}$$

(ii) *The form u satisfies*

$$h_q(\phi u) + \big((1-q)\psi - \phi\big)u = 0. \tag{2.5}$$

Proof. For $f \in \mathcal{P}$ we have

$$\begin{aligned}\langle H_q(x\phi(x)u), f\rangle &= -\langle x\phi(x)u, H_q f\rangle\\ &= -\Big\langle x\phi(x)u, \frac{h_q f - f}{(q-1)x}\Big\rangle = \Big\langle \frac{1}{1-q}\phi u, hqf\Big\rangle + \Big\langle \frac{-1}{1-q}\phi u, f\Big\rangle\\ &= \Big\langle \frac{1}{1-q}\, h_q(\phi u), f\Big\rangle + \Big\langle \frac{-1}{1-q}\phi u, f\Big\rangle.\end{aligned}$$

Therefore

$$\langle H_q(x\phi(x)u), f\rangle = \Big\langle \frac{1}{1-q}\,\big(h_q(\phi u) - \phi u\big), f\Big\rangle. \tag{2.6}$$

Indeed, from (2.6) we can deduce the desired results. □

3. THE q-EXTENSION OF THE SEQUENCE $\{S_n\}_{n\geq 0}$

We assume that $u(\alpha)$ is a symmetrical H_q-semi-classical form and $\{P_n\}_{n\geq 0}$ its orthogonal sequence satisfying the following functional equation:

$$H_q\big(x^3u(\alpha)\big) + \Big(\frac{1-q^{-2\alpha-2}}{1-q}\, x^2 - \frac{1}{2}\Big)\, u(\alpha) = 0, \quad \alpha \in \mathbb{C}, \tag{3.1}$$

we have

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x, \\ P_{n+2}(x) = xP_{n+1}(x) - \gamma_{n+1}P_n(x), \;\; n \geq 0. \end{cases} \tag{3.2}$$

Let

$$I_{n,k}(q) = \langle u(\alpha), x^k P_n(x)P_n(q^{-1}x)\rangle, \; n \geq 0, \; 0 \leq k \leq 2. \tag{3.3}$$

Lemma 3.1. *We have the following result*:

$$I_{n,2}(q^{-1}) - q^{-2\alpha-2}I_{n,2}(q) + \frac{q-1}{2}\, I_{n,0}(q) = 0, \; n \geq 0. \tag{3.4}$$

Proof. By virtue of the Lemma 2.2, the functional equation (3.1) is equivalent to

$$h_q(x^2u(\alpha) + \Big(-q^{-2\alpha-2}x^2 + \frac{q-1}{2}\Big)u(\alpha) = 0,$$

then, we obtain

$$\Big\langle h_q(x^2u(\alpha)) + \Big(-q^{-2\alpha-2}x^2 + \frac{q-1}{2}\Big)\, u(\alpha), P_n(x)P_n(q^{-1}x)\Big\rangle = 0, \; n \geq 0,$$

it is equivalent to

$$\langle x^2u(\alpha), P_n(x)P_n(qx)\rangle + \Big\langle \Big(-q^{-2\alpha-2}x^2 + \frac{q-1}{2}\Big)u(\alpha), P_n(x)P_n(q^{-1}x)\Big\rangle = 0, \; n \geq 0.$$

The previous equation can be written as the following:

$$\big\langle u(\alpha), x^2P_n(x)P_n(qx)\big\rangle - q^{-2\alpha-2}\big\langle u(\alpha), x^2P_n(x)P_n(q^{-1}x)\big\rangle + \frac{q-1}{2}\big\langle u(\alpha), P_n(x)P_n(q^{-1}x)\big\rangle = 0,\ n \geq 0.$$

Thus (3.4). □

We need the following result:

Lemma 3.2. [**12**] *Let* $\{a_n\}_{n\geq 0}$ *with* $a_n \neq 0,\ n \geq 0$, $\{b_n\}_{n\geq 0}$ *two sequences and* $\{x_n\}_{n\geq 0}$ *the sequence satisfying the recurrence relation*:

$$x_{n+1} = a_nx_n + b_n,\ n \geq 0,\ x_0 = a \in \mathbb{C}\setminus\{0\}.$$

We have

$$x_{n+1} = \prod_{k=0}^{n} a_k\bigg(a + \sum_{k=0}^{n}\Big(\prod_{\mu=0}^{k} a_\mu\Big)^{-1} b_k\bigg),\ n \geq 0.$$

Lemma 3.3. *The sequences* $\{I_{n,k}(q)\}_{n\geq 0}$ *are given by the following formulas*:

(3.5) $$I_{n,0}(q) = q^{-n}\big\langle u(\alpha), P_n^{\,2}\big\rangle,\ n \geq 0,$$

(3.6) $$I_{0,2}(q) = \gamma_1,$$

(3.7) $$I_{1,2}(q) = q^{-1}\gamma_1(\gamma_1 + \gamma_2),$$

(3.8) $$I_{n,2}(q) = q^{-n}\big\langle u(\alpha), P_n^{\,2}\big\rangle\bigg(\sum_{\nu=1}^{n+1}\gamma_\nu - q^2\sum_{\nu=1}^{n-1}\gamma_\nu\bigg),\ n \geq 2.$$

Proof. We have $I_{n,0}(q) = \big\langle u(\alpha), P_n(x)P_n(q^{-1}x)\big\rangle,\ n \geq 0$, by the orthogonality of $\{P_n\}_{n\geq 0}$ (3.5) can be deduced.

Writing $I_{0,2}(q) = \big\langle u(\alpha), x^2\big\rangle = \big\langle u(\alpha), P_2 + \gamma_1\big\rangle$, then we obtain (3.6).

Also, we have

$$\begin{aligned} I_{1,2}(q) &= \big\langle u(\alpha), x^2P_1(x)P_1(q^{-1}x)\big\rangle \\ &= \big\langle u(\alpha), x\{P_2(x) + \gamma_1\}P_1(q^{-1}x)\big\rangle \text{ (by (2.2))} \\ &= q^{-1}\big\langle u(\alpha), P_2^{\,2}\big\rangle + q^{-1}\gamma_1 I_{0,2}(q) \ \text{ (by the orthogonality of } \{P_n\}_{n\geq 0}), \end{aligned}$$

by (3.6), we get (3.7).

For $n \geq 0$, we can write

$$\begin{aligned} I_{n+1,2}(q) &= \big\langle u(\alpha), x^2P_{n+1}(x)P_{n+1}(q^{-1}x)\big\rangle \\ &= \big\langle u(\alpha), x\{P_{n+2}(x) + \gamma_{n+1}P_n(x)\}P_{n+1}(q^{-1}x)\big\rangle \ \text{ (by (3.2))} \\ &= \big\langle u(\alpha), xP_{n+2}(x)P_{n+1}(q^{-1}x)\big\rangle + \gamma_{n+1}\big\langle u(\alpha), xP_n(x)P_{n+1}(q^{-1}x)\big\rangle, \end{aligned}$$

by the orthogonality of $\{P_n\}_{n\geq 0}$, we obtain

(3.9) $$I_{n+1,2}(q) = q^{-n-1}\big\langle u(\alpha), P_{n+2}^2\big\rangle + \gamma_{n+1}\big\langle u(\alpha), xP_n(x)P_{n+1}(q^{-1}x)\big\rangle.$$

On the other hand we have

$$\begin{aligned}\left\langle u(\alpha), xP_n(x)P_{n+1}(q^{-1}x)\right\rangle &= \left\langle u(\alpha), xP_n(x)\{q^{-1}xP_n(q^{-1}x) - \gamma_n P_{n-1}(q^{-1}x)\}\right\rangle \\ &= q^{-1}\left\langle u(\alpha), x^2 P_n(x)P_n(q^{-1}x)\right\rangle - \gamma_n\left\langle u(\alpha), xP_n(x)P_{n-1}(q^{-1}x)\right\rangle,\ n \geq 1,\end{aligned}$$

on account of the orthogonality of $\{P_n\}_{n\geq 0}$, we can deduce that

$$\left\langle u(\alpha), xP_n(x)P_{n+1}(q^{-1}x)\right\rangle = q^{-1}I_{n,2}(q) - q^{-n+1}\gamma_n\left\langle u(\alpha), P_n^2\right\rangle,\ n \geq 1. \tag{3.10}$$

By virtue of (3.10), equation (3.9) becomes

$$\begin{aligned}I_{n+1,2}(q) &= q^{-1}\gamma_{n+1}I_{n,2}(q) + q^{-n-1}\left\langle u(\alpha), P_{n+2}^{\,2}\right\rangle - q^{-n+1}\gamma_n\gamma_{n+1}\left\langle u(\alpha), P_n^{\,2}\right\rangle \\ &= q^{-1}\gamma_{n+1}I_{n,2}(q) + q^{-n-1}\left\langle u(\alpha), P_{n+2}^{\,2}\right\rangle - q^{-n+1}\gamma_n\left\langle u(\alpha), P_{n+1}^2\right\rangle,\ n \geq 1.\end{aligned}$$

Using Lemma 3.2 and the relation (3.7), we get (3.8). □

Proposition 3.4. *The sequence* $\{\gamma_{n+1}\}_{n\geq 0}$ *given in* (3.2) *is defined by the following formulas*:

$$\begin{cases}\gamma_{2n+1} = \dfrac{1-q}{2}\dfrac{q^{2n+2\alpha}-1}{\left(q^{4n+2\alpha}-1\right)\left(q^{4n+2\alpha+2}-1\right)}\,q^{2n+2\alpha+2},\ n \geq 0, \\ \gamma_{2n+2} = \dfrac{q-1}{2}\dfrac{q^{2n+2}-1}{\left(q^{4n+2\alpha+2}-1\right)\left(q^{4n+2\alpha+4}-1\right)}q^{4n+4\alpha+4},\ n \geq 0.\end{cases} \tag{3.11}$$

Proof. Letting $n = 0$ and $n = 1$ in (3.4), we obtain respectively:

$$\begin{aligned}I_{0,2}(q^{-1}) - q^{-2\alpha-2}I_{0,2}(q) + \frac{q-1}{2}I_{0,0}(q) = 0, \\ I_{1,2}(q^{-1}) - q^{-2\alpha-2}I_{1,2}(q) + \frac{q-1}{2}I_{1,0}(q) = 0.\end{aligned}$$

On account of (3.5), (3.6) and (3.7), it follows that

$$\gamma_1 = \frac{1}{2}\frac{1-q}{q^{2\alpha+2}-1}\,q^{2\alpha+2}, \tag{3.12}$$

$$\gamma_1 + \gamma_2 = \frac{1}{2}\frac{1-q}{q^{2\alpha+4}-1}\,q^{2\alpha+2}. \tag{3.13}$$

Taking into account the relations (3.5)and (3.8), equation (3.4) becomes

$$(q^{2n} - q^{-2\alpha-2})\sum_{\nu=1}^{n+1}\gamma_\nu - q^2(q^{2n-4} - q^{-2\alpha-2})\sum_{\nu=1}^{n-1}\gamma_\nu + \frac{q-1}{2} = 0,\ n \geq 2. \tag{3.14}$$

Let

$$T_n = \sum_{\nu=1}^{n}\gamma_\nu,\ n \geq 1. \tag{3.15}$$

Then the system (3.12)–(3.14) can be written:

$$T_1 = \frac{1}{2}\,\frac{1-q}{q^{2\alpha+2}-1}\,q^{2\alpha+2}, \tag{3.16}$$

$$T_2 = \frac{1}{2}\,\frac{1-q}{q^{2\alpha+4}-1}\,q^{2\alpha+2}, \tag{3.17}$$

$$(q^{2n}-q^{-2\alpha-2})\,T_{n+1} - q^2(q^{2n-4}-q^{-2\alpha-2})T_{n-1} + \frac{q-1}{2} = 0,\ n \geq 2. \tag{3.18}$$

Moreover, letting $n \to 2n$ and $n \to 2n+1$ in (3.18), we get respectively:

$$(q^{4n}-q^{-2\alpha-2})T_{2n+1} - q^2(q^{4n-4}-q^{-2\alpha-2})T_{2n-1} + \frac{q-1}{2} = 0,\ n \geq 1, \tag{3.19}$$

$$(q^{4n+2}-q^{-2\alpha-2})T_{2n+2} - q^2(q^{4n-2}-q^{-2\alpha-2})T_{2n} + \frac{q-1}{2} = 0,\ n \geq 1. \tag{3.20}$$

By virtue of (3.19), (3.16) and the Lemma 3.2, we get

$$T_{2n+1} = \frac{1}{2(q+1)}\,\frac{1-q^{2n+2}}{q^{4n}-q^{-2\alpha-2}}\,,\ n \geq 0. \tag{3.21}$$

Likewise, by (3.20), (3.18) and the lemma 3.2, we obtain

$$T_{2n} = \frac{1}{2(q+1)}\,\frac{1-q^{2n}}{q^{4n-2}-q^{-2\alpha-2}}\,,\ n \geq 1. \tag{3.22}$$

From (3.15), we get respectively $\gamma_{2n+1} = T_{2n+1} - T_{2n},\ n \geq 1$ and $\gamma_{2n+2} = T_{2n+2} - T_{2n+1}$, $n \geq 0$, then by (3.21), (3.22) and (3.16), we can deduce (3.11). □

Remarks. **1.** The form $u(\alpha)$ is quasi-definite if and only if $n+\alpha \neq 0$, $n \geq 0$. $u(\alpha)$ is not positive definite.

2. When $q \to 1$ in (3.1) and(3.11), we meet again the MOPS $\{S_n\}_{n\geq 0}$.

3. Let $w(\alpha)$ be the form defined by $(w(\alpha))_n = (w(\alpha))_{2n},\ n \geq 0$.

We have

$$\big(h_{\tau^{-1}}w(\alpha)\big)_n = \frac{1}{(-aq^2;q^2)_n}\,,\ n \geq 0,\ a = -q^{2\alpha}.$$

Then, $h_{\tau^{-1}}w(\alpha)$ it is the alternative q^2-Charlier form [**8**, pp 98].

Corollary 3.5. *When $u(\alpha)$ is quasi-definite it is H_q-semi-classical of class one.*

Proof. Let $\phi(x) = x^3$ and $\psi(x) = \dfrac{1-q^{-2\alpha-2}}{1-q}\,x^2 - \dfrac{1}{2}$.

We have $qh_q\widehat{\psi}(0) + H_q\widehat{\phi}(0) = -\frac{q}{2} \neq 0$. According to the proposition 2.1 we see that the functional equation in (3.1) can not be simplified by the factor x. Therefore we get the desired result. □

4. MOMENTS AND DISCRETE REPRESENTATION

4.1 We are going to use the following notations:[**4**, **5**, **11**]

$$(a;q)_n = \begin{cases} 1, \quad n=0, \\ \prod_{k=0}^{n-1}(1-aq^k), \ n \geq 1, \end{cases} \tag{4.1}$$

$$(a;q)_\infty = \prod_{k=0}^{+\infty}(1-aq^k), \ \mid q \mid < 1. \tag{4.2}$$

We have [**5**]

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \ \mid q \mid < 1, \tag{4.3}$$

$$(z;q)_\infty = \sum_{k=0}^{+\infty} \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(q;q)_k} z^k, \ \mid q \mid < 1. \tag{4.4}$$

We need the following results:

Lemma 4.1. *Let $u \in \mathcal{P}'$ be a symmetrical form such that*

$$(u)_{2n} = \sum_{k=0}^{+\infty} a_k (c_k)^{2n}, \ n \geq 0. \tag{4.5}$$

Then

$$u = \frac{1}{2} \sum_{k=0}^{+\infty} a_k \big(\delta_{c_k} + \delta_{-c_k}\big), \tag{4.6}$$

with $\langle \delta_c, f \rangle = f(c), \ f \in \mathcal{P}$.

Proof. We have $\langle \delta_{c_k}, x^{2n} \rangle = \langle \delta_{-c_k}, x^{2n} \rangle$, and $\langle \delta_{c_k}, x^{2n} \rangle = -\langle \delta_{-c_k}, x^{2n} \rangle$. Therefore

$$(u)_n = \langle u, x^n \rangle = \Big\langle \frac{1}{2} \sum_{k=0}^{+\infty} a_k \big(\delta_{c_k} + \delta_{-c_k}\big), x^n \Big\rangle, \ n \geq 0.$$

Consequently, we get the desired result. □

4.2. Now we are able to calculate the moments and to give a discrete representation for the canonical case.

Proposition 4.2. *The moments of the form $u(\alpha)$, $\alpha \neq -n$, $n \geq 0$ defined in* (3.1) *are given by the following formulas*:

$$\big(u(\alpha)\big)_{2n} = \frac{\tau^n}{(q^{2\alpha+2};q^2)_n} \ , \ , n \geq 0; \quad \big(u(\alpha)\big)_{2n+1} = 0, \ n \geq 0, \tag{4.7}$$

where

$$\tau = \frac{1}{2} q^{2\alpha+2}(q-1). \tag{4.8}$$

Proof. Indeed, by the Lemma 2.2, the functional equation (3.1) can be written

$$h_q\big(x^2u(\alpha)\big)+\Big(-q^{-2\alpha-2}x^2+\frac{q-1}{2}\Big)u(\alpha)=0.$$

From the previous equation, we get

$$\Big\langle h_q\big(x^2u(\alpha)\big)+\Big(-q^{-2\alpha-2}x^2+\frac{q-1}{2}\Big)u(\alpha),x^{2n}\Big\rangle=0,\ n\geq 0,$$

then

$$q^{2n}\big\langle u(\alpha),x^{2n+2}\big\rangle+\Big\langle u(\alpha),\Big(-q^{-2\alpha-2}x^2+\frac{q-1}{2}\Big)x^{2n}\Big\rangle=0,\ n\geq 0.$$

Consequently, we are to the following equation:

$$\big(u(\alpha)\big)_{2n+2}=\frac{\tau}{1-q^{2n+2\alpha+2}}\,\big(u(\alpha)\big)_{2n},\ n\geq 0.$$

Therefore

$$\big(u(\alpha)\big)_{2n}=\frac{\tau^n}{(q^{2\alpha+2};q^2)_n}\ ,\ n\geq 0.$$

The form $u(\alpha)$ is symmetrical, then $\big(u(\alpha)\big)_{2n+1}=0,\ n\geq 0$. Hence the desired results. □

Proposition 4.3. *When* $0<q<1,\ \alpha=-n,\ n\geq 0$, *the form* $u(\alpha)$ *possesses the following discrete representation*:

$$u(\alpha)=\frac{1}{2(q^{2\alpha+2};q^2)_\infty}\sum_{k=0}^{+\infty}q^{2k(\alpha+1)}\,\frac{(-1)^kq^{k(k-1)}}{(q^2;q^2)_k}\,\big(\delta_{-\xi q^k}+\delta_{\xi q^k}\big),\tag{4.9}$$

with

$$\xi=\frac{i}{\sqrt{2}}\,q^{\alpha+1}\sqrt{1-q}.\tag{4.10}$$

Proof. On account of the Proposition 4.2 and the relation (4.3) we can deduce the following result:

$$\big(u(\alpha)\big)_{2n}=\tau^n\,\frac{(q^{2\alpha+2}q^{2n};q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty}\ ,\ n\geq 0.$$

By virtue of (4.4), the previous equation becomes

$$\big(\widehat{u}(\alpha)\big)_{2n}=\frac{1}{(q^{2\alpha+2};q^2)_\infty}\sum_{k=0}^{+\infty}q^{2k(\alpha+1)}\,\frac{(-1)^kq^{k(k-1)}}{(q^2;q^2)_k}\,\tau^nq^{2n},\ n\geq 0.$$

From (4.8), we get $\tau^n=\xi^{2n}$. Then, the last equation becomes

$$\big(\widehat{u}(\alpha)\big)_{2n}=\frac{1}{(q^{2\alpha+2};q^2)_\infty}\sum_{k=0}^{+\infty}q^{2k(\alpha+1)}\,\frac{(-1)^kq^{k(k-1)}}{(q^2;q^2)_k}\,(\xi q)^{2n},\ n\geq 0.$$

On account of lemma 4.1, we get (4.9). □

Acknowledgements. I would like to express my deepest gratitude to the referees for their valuable suggestions pertaining to both the formulations and bibliography.

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(Received April 28, 2008)