Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 97–119. doi:10.2298/AADM0901097M

ON CONFORMALLY INVARIANT EXTREMAL PROBLEMS

Vesna Manojlović

Abstract This paper deals with conformal invariants in the euclidean space $\mathbb{R}^n$, $n \geq 2$, and their interrelation. In particular, conformally invariant metrics and balls of the respective metric spaces are studied.

1. INTRODUCTION

Conformal invariance has played a predominant role in the study of geometric function theory during the past century. Some of the landmarks are the pioneering contributions of Grötzsch and Teichmüller prior to the Second World War, and the paper of Ahlfors and Beurling **[1]** in 1950. These results lead to farreaching applications and have stimulated many later studies **[12]**. For instance, Gehring and Väisälä **[7]**, **[16]** have built the theory of quasiconformal mappings in $\mathbb{R}^n$ based on the notion of the modulus of a curve family introduced in **[1]**.

Our goal here is to study two kinds of conformally invariant extremal problems, which in special cases reduce to problems due to Grötzsch and Teichmüller, respectively. These two classical extremal problems are extremal problems for moduli of ring domains. The Grötzsch and Teichmüller rings are the extremal rings for extremal problems of the following type, which were first posed for the case of the plane. Among all ring domains which separate two given closed sets E_1 and E_2, $E_1 \cap E_2 = \emptyset$, find one whose module has the greatest value.

In the general case these extremal problems lead to conformal invariants $\lambda_G(x, y)$ and $\mu_G(x, y)$ defined for a domain $G \subset \mathbb{R}^n$ and $x, y \in G$. A basic fact is that $\lambda_G(x, y)^{1/(1-n)}$ and $\mu_G(x, y)$ are metrics. In the recent survey of Vuorinen **[21]** an extensive research program was suggested for the study of metric spaces in the context of geometric function theory. Motivated by **[21]** and following closely

2000 Mathematics Subject Classification. 30C75.
Keywords and Phrases. Conformal invariants, hyperbolic-type metrics, moduli of continuity.

the ideas developed in **[18]** and **[19]** we study two topics: (a) the geometry of the metric spaces (G, d) when d is $\lambda_G(x,y)^{1/(1-n)}$ or $\mu_G(x,y)$ and (b) the relations of these two metrics to several other metrics. The main result is a revised version of Chart 1 on p. 86 of **[18]**, which takes into account some later developments, such as **[9]**, **[10]**, **[19]**.

Then we present an application to the geometry of balls in these metrics. As a special case we investigate λ_G metric in $B^2 \setminus \{0\}$, continuing work of **[9]**.

The main method in this paper is the extremal length method of Ahlfors and Beurling and various inequalities for the moduli of curve families. More applications of the same method to quasiconformal mappings will be given in the next paper of the author **[22]**.

2. THE EXTREMAL PROBLEMS OF GRÖTZSCH AND TEICHMÜLLER

In what follows, we adopt the standard notions related to quasiconformal mappings from **[16]**.

We use notation $B^n(x,r) = \{y \in \mathbb{R}^n \colon |x-y| < r\}$, $S^{n-1}(x,r) = \{y \in \mathbb{R}^n \colon |x-y| = r\}$, $H^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \colon x_n > 0\}$ and abbreviations $B^n(r) = B^n(0,r)$, $B^n = B^n(1)$, $S^{n-1}(r) = S^{n-1}(0,r)$ and $S^{n-1} = S^{n-1}(1)$. The $(n-1)$-dimensional surface area of S^{n-1} is denoted by ω_{n-1}.

For the modulus $M(\Gamma)$ of a curve family Γ and its basic properties we refer the reader to **[16]**. Its basic property is conformal invariance.

For $E, F, G \subset \overline{\mathbb{R}^n}$ let $\Delta(E,F,G)$ be the family of all closed curves joining E to F within G. More precisely, a path $\gamma : [a,b] \to \overline{\mathbb{R}^n}$ belongs to $\Delta(E,F,G)$ iff $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$.

If G is a proper subdomain of $\overline{\mathbb{R}^n}$, then for $x, y \in G$ with $x \neq y$ we define

$$\lambda_G(x,y) = \inf_{C_x, C_y} M\big(\Delta(C_x, C_y; G)\big) \tag{1}$$

where $C_z = \gamma_z[0,1)$ and $\gamma_z : [0,1) \longrightarrow G$ is a curve such that $\gamma_z(0) = z$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x, y$. This conformal invariant was introduced by J. Ferrand (see **[19]**).

For $x \in \mathbb{R}^n \setminus \{0, e_1\}$, $n \geq 2$, we define

$$p(x) = \inf_{E,F} M\big(\Delta(E, F; \mathbb{R}^n)\big), \tag{2}$$

where the infimum is taken over all pairs of continua E and F in $\overline{\mathbb{R}^n}$ with $0, e_1 \in E$, $x, \infty \in F$. This extremal quantity was introduced by O. Teichmüller (see **[19]**, **[10]**). For a connection between $p(x)$ and λ_G, $G = \mathbb{R}^n \setminus \{0\}$, see **[19, (8.23)]**.

For a proper subdomain G of $\overline{\mathbb{R}^n}$ and for all $x, y \in G$ define

$$\mu_G(x,y) = \inf_{C_{xy}} M\big(\Delta(C_{xy}, \partial G; G)\big) \tag{3}$$

where the infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0,1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$. For the case $G = B^n$ the function $\mu_{B^n}(x, y)$ is the extremal quantity of H. GRÖTZSCH (see **[19]**).

Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \to Y$ be a continuous mapping. Then we say that f is uniformly continuous if there exists an increasing continuous function $\omega : [0, r_1) \to [0, r_2)$ with $r_1, r_2 > 0$, $\omega(0) = 0$ and $d_2(f(x), f(y)) \le \omega(d_1(x, y))$ for all $x, y \in X$ with $d_1(x, y) < r_1$. We call the function ω the modulus of continuity of f. If there exist $C, \alpha > 0$ such that $\omega(t) \le Ct^\alpha$ for all $t \in (0, t_0)$, $t_0 \in (0, r_1)$, we say that f is HÖLDER-continuous with HÖLDER exponent α. If $\alpha = 1$, we say that f is LIPSCHITZ with the LIPSCHITZ constant C or simply C-LIPSCHITZ. If f is a homeomorphism and both f and f^{-1} are C-LIPSCHITZ, then f is C-bilipschitz or C-quasiisometry and if $C = 1$ we say that f is an isometry. These conditions are said to hold locally, if they hold for each compact subset of X.

In this section we introduce five metrics:

1) Spherical (chordal) metric q.

2) Quasihyperbolic metric k_G of a domain $G \subset \mathbb{R}^n$.

3) Distance ratio metric j_G.

4) Seittenranta's metric δ_G.

5) Apollonian metric α_G.

The first one is defined on $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. The second and the third ones are defined in any proper subdomain $G \subset \mathbb{R}^n$, both of them generalize hyperbolic metric (on B^n or H^n) to arbitrary proper subdomain $G \subset \mathbb{R}^n$. SEITTENRANTA's metric is a natural, Möbius invariant analogue of the j_G-metric.

Hyperbolic-type metrics

The metric q in $\overline{\mathbb{R}^n}$ is defined by

$$q(x,y) = \begin{cases} \dfrac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & x \ne \infty \ne y, \\[2ex] \dfrac{1}{\sqrt{1+|x|^2}}, & y = \infty. \end{cases} \tag{4}$$

Absolute (cross) ratio of an ordered quadruple a, b, c, d of distinct points in $\overline{\mathbb{R}^n}$ is defined by

$$|a, b, c, d| = \frac{q(a,c)\,q(b,d)}{q(a,b)\,q(c,d)} = \frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}. \tag{5}$$

The basic property of the absolute ratio is invariance under MÖBIUS transformations. For these facts see [**19**]. Now we introduce distance ratio metric or j_G-*metric*. For an open set $G \subset \mathbb{R}^n$, $G \neq \mathbb{R}^n$ we define $d(z) = d(z, \partial G)$ for $z \in G$ and

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right) \tag{6}$$

for $x, y \in G$.

For a nonempty $A \subset G$ we define the m-diameter of A by

$$m(A) = \sup\{m(x,y) \mid x, y \in A\},$$

where m is any metric. The Euclidean diameter we denote by $\operatorname{diam}(A)$.

For an open set $G \subset \mathbb{R}^n$, $G \neq \mathbb{R}^n$, and a nonempty $A \subset G$ such that $d(A, \partial G) > 0$ we define

$$r_G(A) = \frac{d(A)}{d(A, \partial G)}.$$

If $\rho(x) > 0$ for $x \in G$, ρ is continuous and if γ is a rectifiable curve in G, then we define

$$\ell_\rho(\gamma) = \int_\gamma \rho \, \mathrm{d}s.$$

The Euclidean length of a curve γ is denoted by $\ell(\gamma)$.

Also, for $x, y \in G$ we define

$$d_\rho(x,y) = \inf \ell_\rho(\gamma), \tag{7}$$

where the infimum is taken over all rectifiable curves from x to y.

It is easy to show that d_ρ is a metric in G.

Now we take any proper domain $G \subset \mathbb{R}^n$ and set $\rho(x) = 1/d(x, \partial G)$.

The corresponding metric, denoted by k_G, is called the *quasihyperbolic metric* in G. Observing that

$$\rho\big(\varphi(x)\big) = \frac{1}{d\big(\varphi(x), \partial\,(\varphi G)\big)} = \frac{1}{d(x, \partial G)} = \rho(x),$$

for a Euclidean isometry φ, we see that

$$k_{G'}(x', y') = k_G(x,y), \qquad \text{where } G' = \varphi(G),\ \ x' = \varphi(x),\ \ y' = \varphi(y).$$

Now we introduce *Seittenranta's metric* δ_G [**14**]. For more details on MÖBIUS transformations in $\mathbb{R}^n$ see [**3**]. For an open set $G \subset \mathbb{R}^n$ with $\operatorname{card} \partial G \geq 2$ we set

$$m_G(x,y) = \sup_{a,b \in \partial G} |a, x, b, y|$$

and

$$\delta_G(x,y) = \log\big(1 + m_G(x,y)\big) \tag{8}$$

for all $x, y \in G$.

Consider now the case of an unbounded domain $G \subset \mathbb{R}^n, \infty \in \partial G$. Note that if a or b in the supremum equals infinity, then we get exactly j_G metric. This implies that we always have $j_G(x,y) \le \delta_G(x,y)$.

We will also use *Apollonian metric* studied by Beardon **[4]**, (also see **[2]**, 7.28 (2)) defined in open proper subsets $G \subset \mathbb{R}^n$ by

$$\alpha_G(x,y) = \sup_{a,b\in\partial G} \log|a,x,y,b| \quad \text{for all } x,y\in G.$$

This formula defines a metric iff $\mathbb{R}^n \setminus G$ is not contained in an $(n-1)$-dimensional sphere in $\mathbb{R}^n$.

In general, the hyperbolic-type metrics can be divided into length-metrics, defined by means of integrating a weight function and point-distance metrics.

Another group may again be classified by the number of boundary points used in the definition. So for instance, the j metric is one-point metric, while the Apollonian metric is two-point metric.

Definition 1. *A domain $A \subset \overline{\mathbb{R}^n}$ is a ring if $C(A)$ has exactly two components, where $C(A)$ denotes the complement of $A \subset \mathbb{R}^n$.*

If the components of $C(A)$ are C_0 and C_1, we denote $A = R(C_0, C_1)$, $B_0 = C_0 \cap \overline{A}$ and $B_1 = C_1 \cap \overline{A}$. To each ring $A = R(C_0, C_1)$, we associate the curve family $\Gamma_A = \Delta(B_0, B_1, A)$ and the modulus of A is defined by $\mod(A) = M(\Gamma_A)$. Next, the capacity of A is by definition $\mathrm{cap}A = \omega_{n-1}(\mod A)^{1-n}$.

The complementary components of the Grötzsch ring $R_{G,n}(s)$ in $\mathbb{R}^n$ are $\overline{B}^n$ and $[s\cdot e_1, \infty]$, $s>1$, while those of the Teichmüller ring $R_{T,n}(t)$ are $[-e_1, 0]$ and $[t\,e_1, \infty]$, $t>0$. We shall need two special functions $\gamma_n(s)$, $s>1$, and $\tau_n(t)$, $t>0$, to designate the moduli of the families of all those curves which connect the complementary components of the Grötzsch and Teichmüller rings in $\mathbb{R}^n$, respectively:

$$\gamma_n(s) = M(\Gamma_s) = \gamma(s), \quad \Gamma_s = \Gamma_{R_{G,n}}(s),$$
$$\tau_n(t) = M(\Delta_t) = \tau(t), \quad \Delta_t = \Gamma_{R_{T,n}}(t).$$

These functions are related by a functional identity **[5]**, Lemma 6

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2-1). \tag{9}$$

Definition 2. *Given $r>0$, we let $R\Psi_n(r)$ be the set of all rings $A = R(C_0, C_1)$ in $\overline{\mathbb{R}^n}$ with the following properties*:

1) *C_0 contains the origin and a point a such that $|a| = 1$.*

2) *C_1 contains ∞ and a point b such that $|b| = r$.*

Teichmüller first considered the following quantity in the planar case $(n=2)$:

$$\tau_n(r) = \inf M(\Gamma_A) = \inf\{p(x) \,\big|\, |x| = r\},$$

where the infimum is taken over all rings $A \in R\Psi_n(r)$ *and* $p(x)$ *is as in* (2). *For* $n \geq 3$ *the function* τ_n *was studied in* **[5]** *and in* **[10]**.

Theorem 1. [**16**, Theorem 11.7] *The function* $\tau_n : (0,\infty) \to (0,\infty)$ *has the following properties*:

1) τ_n *is decreasing,*
2) $\lim_{r\to\infty} \tau_n(r) = 0$,
3) $\lim_{r\to 0} \tau_n(r) = \infty$,
4) $\tau_n(r) > 0$ *for every* $r > 0$.

Moreover, $\tau_n : (0,\infty) \to (0,\infty)$ *and* $\gamma_n : (1,\infty) \to (0,\infty)$ *are homeomorphisms.*

From the definition of τ_n and from the conformal invariance of the modulus, we obtain the following estimate:

Theorem 2. *Suppose that* $A = R(C_0, C_1)$ *is a ring and that* $a, b \in C_0$ *and* $c, \infty \in C_1$. *Then*

$$M(\Gamma_A) \geq \tau_n\left(\frac{|c-a|}{|b-a|}\right).$$

Here equality holds for the Teichmüller ring $R_{T,n}(t)$, *when* $a = 0, b = -e_1, c = te_1, t > 0$ *and* $C_0 = [-e_1, 0], C_1 = [te_1, \infty)$.

Theorem 3. *Let* $C \subset B^n$ *be a connected compact set containing* 0 *and* x, *where* $|x| < 1$. *Then the capacity of a ring domain with components* $C_0 = C$, $C_1 = \{x : |x| \geq 1\}$ *is at least* $\gamma_n(1/|x|)$. *Here equality holds for the ring with the complementary components* $[0, |x|e_1]$ *and* $\mathbb{R}^n \setminus B^n$ *the image of the Grötzsch ring* $R_{G,n}(1/|x|)$ *under the inversion in* S^{n-1}.

These theorems state the extremal properties of the Teichmüller and Grötzsch rings and their proofs are based on the symmetrization theorem in [**5**, Theorem 1].

3. MODULI OF CONTINUITY

In this section we investigate the moduli of continuity of the identity mappings $id_G : (G,\rho) \to (G,d)$ where ρ and d are chosen from the set of interesting metrics defined on G (like quasihyperbolic metric k_G, modulus metric μ_G etc.).

Hence, we are interested in results of type

$$d(x,y) \leq \zeta\big(\rho(x,y)\big) = \zeta_\rho^d\big(\rho(x,y)\big), \quad x,y \in G. \tag{10}$$

We give several estimates of this type, and then we collect these results in charts at the end of this section.

Note that in our charts we have λ_G^{-1}, as well as in the inequalities of type (10); however reader should be aware that in general λ_G^{-1} is not a metric. In fact $\lambda_G^{1/(1-n)}$ is always a metric. For more details on this matter see **[21]**.

It is well known [**8**] that $j_G(x,y) \le k_G(x,y)$, so $\zeta_k^j(t) = t$. We shall next prove this inequality.

Lemma 1. *Let G be a proper subdomain of $\mathbb{R}^n$.*

1) *For $x, y \in G$*

$$k_G(x,y) \ge \log\left(1 + \frac{m(x,y)}{\min\{d(x), d(y)\}}\right) \ge j_G(x,y),$$

where $m(x,y) = \inf\{\ell(\gamma) \,|\, \gamma$ is a curve joining x and y in $G\}$.

2) *If there is $M \in (0,\infty)$ such that $d(x) \le M$ for all $x \in G$, then for all $x, y \in G$*

$$k_G(x,y) \ge |x-y|/M.$$

Proof. 1) We may assume $0 < d(x) \le d(y)$. Choose a rectifiable arc $\gamma : [0,s] \to G$ from x to y, parametrized by arc length:

$$\gamma(0) = x, \qquad \gamma(s) = y;$$

obviously $s \ge |x-y|$. For any $0 \le t \le s$ we have

$$d(\gamma(t)) \le d(x) + t, \qquad \text{(a key observation)},$$

so,

$$l_\rho(\gamma) \ge \int_0^s \frac{\mathrm{d}t}{d(x)+t} = \log\frac{d(x)+s}{d(x)} \ge \log\frac{d(x)+|x-y|}{d(x)} = j_G(x,y).$$

2) Let γ be a rectifiable curve joining x with y. Then

$$\int_\gamma \frac{|\mathrm{d}x|}{d(x,\partial G)} \ge \int_\gamma |\mathrm{d}x|/M \ge |x-y|/M$$

and hence the assertion follows. □

The inequality reverse to Lemma 16 (1) is not true in general; a domain G such that there is a constant $c > 0$ with $k_G(x,y) \le c\, j_G(x,y)$ for all $x, y \in G$ is called a uniform domain, so in that case $\zeta_j^k(t) = ct$.

Lemma 2. [**18**, Lemma 2.21] *Let G be a proper subdomain of $\mathbb{R}^n$. If $x \in G$, $d(x) = d(x,\partial G)$ and $y \in B^n(x, d(x)) = B_x$, $x \ne y$, then*

$$\lambda_G(x,y) \ge \lambda_{B_x}(x,y) \ge c_n \log\left(\frac{d(x)}{|x-y|}\right) \tag{11}$$

where c_n is the positive number in [**16**, (10.11)]. *There exists a strictly increasing function $h_1 : (0,+\infty) \longrightarrow (0,+\infty)$ with $\lim_{t\to 0_+} h_1(t) = 0$ and $\lim_{t\to+\infty} h_1(t) = +\infty$, depending only on n, such that*

$$\lambda_G(x,y) \le h_1\left(\frac{\min\{d(x), d(y)\}}{|x-y|}\right) \tag{12}$$

for $x, y \in G$, $x \neq y$. *If* $x \in G$ *and* $y \in B^n\big(x, d(x)\big) = B_x$, $x \neq y$, *then*

$$\mu_G(x,y) \le \mu_{B_x}(x,y) = \mathrm{cap} R_G\left(\frac{d(x)}{|x-y|}\right) \le \omega_{n-1}\left(\log\left(\frac{d(x)}{|x-y|}\right)\right)^{1-n}. \tag{13}$$

From (13) we get $\mu_G(x,y) \le \gamma_n(d(x)/|x-y|)$ for $x \in G$ and $y \in B_x$. It is equivalent with $\mu_G(x,y) \le \gamma_n(1/r)$ where $r = |x-y|/d(x)$.

We can express $j_G(x,y)$ in terms of r: $r = e^{j_G(x,y)} - 1$ and obtain

$$\mu_G(x,y) \le \gamma_n\left(\frac{1}{e^{j_G(x,y)} - 1}\right).$$

This gives $\zeta_j^\mu(t) = \gamma_n\big(1/(e^t - 1)\big)$ locally.

Lemma 3. [**18**, Lemma 2.39] *For* $n \ge 2$ *there exists a strictly increasing function* $h_2 : [0, +\infty) \to [0, +\infty)$ *with* $h_2(0) = 0$ *and* $\lim_{t \to +\infty} h_2(t) = +\infty$ *with the following properties.*

If E *is closed and* F *is compact in* $\mathbb{R}^n$ *then*

$$M\big(\Delta(E,F)\big) \le h_2(T); \quad T = \min\{j_{\mathbb{R}^n \setminus E}(F), j_{\mathbb{R}^n \setminus F}(E)\}. \tag{14}$$

In particular, if G *is a proper subdomain of* $\mathbb{R}^n$, *then*

$$\mu_G(x,y) \le h_2\big(3k_G(x,y)\big) \tag{15}$$

for all $x, y \in G$. *Moreover, there are positive numbers* b_1, b_2 *depending only on* n *such that*

$$\mu_G(x,y) \le b_1 k_G(x,y) + b_2 \tag{16}$$

for all $x, y \in G$.

From (15) we have $\zeta_k^\mu(t) = h_2(3t)$.

Lemma 4. [**18**, Lemma 2.44] *If* $E, F \subseteq \mathbb{R}^n$ *are disjoint continua, then*

$$M\big(\Delta(E,F)\big) \ge \bar{c}_n \min\{j_{\mathbb{R}^n \setminus E}(F), j_{\mathbb{R}^n \setminus F}(E)\}$$

where $\bar{c}_n$ *is a positive number depending only on* n.

Corollary 1. [**18**, Corollary 2.46] *If* E *and* F *are disjoint continua in* $\overline{\mathbb{R}^n}$ *and* $\infty \in F$, *and* $c_n > 0$ *is as in* (11), *then*

$$M(\Delta(E,F)) \ge c_n j_{\mathbb{R}^n \setminus F}(E).$$

Corollary 2. [**21**, Lemma 6.23] *Let* $G \subseteq \mathbb{R}^n$ *be a domain* $G \neq \mathbb{R}^n$ *and with a connected boundary* ∂G. *Then*

$$\mu_G(a,b) \ge c_n j_G(a,b) \tag{17}$$

holds for $a, b \in G$. If, in addition, G is uniform, then

$$\mu_G(a,b) \ge B\, k_G(a,b) \tag{18}$$

for all $a, b \in G$.

The first part of Corollary 2 gives $\zeta_\mu^j(t) = t/c_n$ if ∂G is connected. Inequality (18) gives $\zeta_\mu^k(t) = c\,t$ if ∂G is connected and G is uniform.

Lemma 5. [**2**, Corollary 15.13] *Let G be a proper subdomain of $\mathbb{R}^n$, x and y distinct points in G and $m(x,y) = \min\{d(x), d(y)\}$. Then*

$$\lambda_G(x,y) \le \sqrt{2}\tau_n\left(\frac{|x-y|}{m(x,y)}\right). \tag{19}$$

From (19) using again $r = e^{j_G(x,y)} - 1$, $r = |x-y|/m(x,y)$, we have

$$\sqrt{2}\tau_n(e^{j_G(x,y)} - 1) \ge \lambda_G(x,y),$$

and then, since τ_n is decreasing, $e^{j_G(x,y)} \le \tau_n^{-1}(\lambda_G(x,y)/\sqrt{2}) + 1$ and from here

$$j_G(x,y) \le \log\left(1 + \tau_n^{-1}\left(\frac{1}{\sqrt{2}\,\lambda_G^{-1}(x,y)}\right)\right).$$

Finally we obtain $\zeta_{\lambda^{-1}}^j(t) = \log\big(1 + \tau_n^{-1}(1/(\sqrt{2}t))\big)$.

Definition 3. *A closed set E in $\mathbb{R}^n$ is called a c-quasiextremal distance set or c-QED exceptional or c-QED set, $c \in (0,1]$, if for each pair of disjoint continua $F_1, F_2 \subseteq \overline{\mathbb{R}^n} \setminus E$*

$$M\big(\Delta(F_1, F_2; \overline{\mathbb{R}^n} \setminus E)\big) \ge cM\big(\Delta(F_1, F_2)\big). \tag{20}$$

If G is a domain in $\overline{\mathbb{R}^n}$ such that $\overline{\mathbb{R}^n} \setminus G$ is a c-QED set, then we call G a c-QED domain.

Theorem 4. [**20**, Theorem 6.21] *Let G be a c-QED domain in $\mathbb{R}^n$. Then*

$$\lambda_G(x,y) \ge c\tau_n(s^2 + 2s) \ge 2^{1-n}c\tau_n(s) \tag{21}$$

where $s = |x-y|/\min\big(d(x), d(y)\big)$.

From the first inequality in (21), taking into account that $s = e^{j_G(x,y)} - 1$, we obtain

$$\lambda^{-1} = \frac{1}{\lambda} \le \frac{1}{c}\,\frac{1}{\tau_n((s+1)^2 - 1)} = \frac{1}{c}\,\frac{1}{\tau_n(e^{2j_G(x,y)} - 1)}.$$

This gives $\zeta_j^{\lambda^{-1}}(t) = \dfrac{1}{c}\,\dfrac{1}{\tau_n(e^{2t} - 1)}$ for a c-QED domain G.

Combining ζ_k^j and $\zeta_j^{\lambda^{-1}}$ we estimate $\lambda_G^{-1}(x,y)$ in terms of $k_G(x,y)$, so $\zeta_k^{\lambda^{-1}} = \zeta_j^{\lambda^{-1}} \circ \zeta_k^j = \zeta_j^{\lambda^{-1}}$. In fact, we have

$$\lambda_G^{-1}(x,y) \le \frac{1}{c}\,\frac{1}{\tau_n(e^{2j_G(x,y)}-1)} \le \frac{1}{c\tau_n(e^{2k_G(x,y)}-1)}\,.$$

The functions $\zeta_\mu^{\lambda^{-1}}$, $\zeta_{\lambda^{-1}}^k$, $\zeta_{\lambda^{-1}}^\mu$ are obtained in the same fashion as $\zeta_k^{\lambda^{-1}}$, namely as compositions of appropriate functions ζ_ρ^d. We use the following inequalities.

For $\zeta_\mu^{\lambda^{-1}}$ we have

$$\lambda_G^{-1}(x,y) \le \frac{1}{c\tau_n(e^{2j_G(x,y)}-1)} \le \frac{1}{c\tau_n(e^{2\mu/c_n}-1)} = \frac{1}{c\tau_n(e^{b\mu}-1)}\,,$$

where the second inequality follows from (17) and where $b = 2/c_n$ and where c_n is the constant from Corollary 1.

For $\zeta_{\lambda^{-1}}^k$ we have

$$k_G \le c\, j_G(x,y) \le c\log(1+u)\,, \quad u = \tau_n^{-1}\left(\frac{1}{\sqrt{2}\lambda_G^{-1}(x,y)}\right)$$

and for $\zeta_{\lambda^{-1}}^\mu$ we have

$$\mu_G \le \gamma_n\left(\frac{1}{e^{j_G(x,y)}-1}\right) \le \gamma_n\left(\frac{1}{e^{\log(1+u)}-1}\right) = \gamma_n\left(\frac{1}{u}\right).$$

Theorem 5. [**14**, Theorem 3.4] *The inequalities* $j_G(x,y) \le \delta_G(x,y) \le 2j_G(x,y)$ *hold for every open set* $G \subset \mathbb{R}^n$.

So, we deduce that $\zeta_\delta^j(t) = t$ and $\zeta_j^\delta(t) = 2t$.

Theorem 6. [**14**, Theorem 4.2] *Let* $G \subset \mathbb{R}^n$ *be a convex domain, then* $j_G(x,y) \le \alpha_G(x,y)$.

This means that $\zeta_\alpha^j(t) = t$ for convex domains.

Theorem 7. [**14**, Theorem 6.2] *Let* G *be a domain in* $\mathbb{R}^n$, *for which* $\operatorname{card}\partial G \ge 2$ *and* ∂G *is connected. Then, for distinct points* $x, y \in G$,

$$\mu_G(x,y) \ge \tau_n\left(\frac{1}{e^{\delta_G(x,y)}-1}\right).$$

Because τ_n is a decreasing homeomorphism we get

$$\tau_n^{-1}\big(\mu_G(x,y)\big) \le \frac{1}{e^{\delta_G(x,y)-1}}$$

and from here

$$\delta_G(x,y) \le \log\left(1 + \frac{1}{\tau_n^{-1}\big(\mu_G(x,y)\big)}\right).$$

Hence, $\zeta_\mu^\delta(t) = \log\big(1 + 1/\tau_n^{-1}(t)\big)$ if ∂G is connected and has at least two points.

Theorem 8. [**14**, Theorem 6.5] *Let $G \subset \overline{\mathbb{R}^n}$ be a domain with card $\partial G \geq 2$. If m_G is as in the definition of δ_G (8)*, then

$$\lambda_G(x,y) \leq \tau_n\left(\frac{m_G(x,y)}{2}\right).$$

Expressing $m_G(x,y)$ in terms of $\delta_G(x,y)$ we get:

$$\lambda_G(x,y) \leq \tau_n\left(\frac{e^{\delta_G(x,y)}-1}{2}\right)$$

and from here we obtain

$$\delta_G(x,y) \leq \log\left(1 + 2\tau_n^{-1}\left(\frac{1}{\lambda_G^{-1}(x,y)}\right)\right).$$

This means that $\zeta_{\lambda^{-1}}^\delta(t) = \log\big(1 + 2\tau_n^{-1}(1/t)\big)$ for domains with card$\,(\partial G) \geq 2$.

At first, we give Chart 1. Here the functions h_j are defined as follows:

$$h_3(t) = \gamma_n\left(\frac{1}{e^t-1}\right), \quad h_4(t) = \frac{1}{c\tau_n(e^{2t}-1)}, \quad h_{13}(t) = \log\left(1 + \tau_n^{-1}\left(\frac{1}{\sqrt{2}\,t}\right)\right)$$

and function $h_2(t)$ appears in Lemma 3.

1	2 $\zeta_j^k(t) = ct$ G uniform $\zeta_j^k(t) = \varphi(t)$ G φ domain	3 $\zeta_j^\mu(t) = h_3(t)$ locally	4 $\zeta_j^{\lambda^{-1}}(t) = h_4(t)$ G c-QED domain
5 $\zeta_k^j(t) = t$	6	7 $\zeta_k^\mu(t) = h_2(3t)$	8 $\zeta_k^{\lambda^{-1}} = \zeta_j^{\lambda^{-1}}$
9 $\zeta_\mu^j(t) = t/c_n$ ∂G connected	10 $\zeta_\mu^k(t) = ct$ G uniform ∂G connected	11	12 $\zeta_\mu^{\lambda^{-1}} = \zeta_\mu^j \circ \zeta_j^{\lambda^{-1}}$ G c-QED domain ∂G connected
13 $\zeta_{\lambda^{-1}}^j(t) = h_{13}(t)$	14 $\zeta_{\lambda^{-1}}^k = \zeta_{\lambda^{-1}}^j \circ \zeta_j^k$ G uniform	15 $\zeta_{\lambda^{-1}}^\mu = \zeta_{\lambda^{-1}}^j \circ \zeta_j^\mu$ locally	16

Chart 1

Function ζ_j^μ can be written in a different form using the estimate of γ_n function. We define functions Φ and Ψ as in [**19**, 7.19] by

$$\gamma_n(s) = \omega_{n-1}\big(\log(\Phi(s))\big)^{n-1}, \quad s > 1, \tag{22}$$

(23) $$\tau_n(t) = \omega_{n-1}\big(\log(\Psi(t))\big)^{n-1}, \quad t > 0.$$

Lemma 6. [**19**, Lemma 7.22] *For each* $n \geq 2$ *there exists a number* $\lambda_n \in [4, 2\,e^{n-1})$, $\lambda_2 = 4$, *such that*

(24) $$t \leq \Phi(t) \leq \lambda_n t, \quad t > 1,$$

(25) $$t + 1 \leq \Psi(t) \leq \lambda_n^{\,2}(t+1), \quad t > 0.$$

From (23) we have that $\omega_{n-1}\big(\log(\lambda_n^2(t+1))\big)^{1-n} \leq \tau_n(t) \leq \omega_{n-1}\big(\log(t+1)\big)^{1-n}$.

From (22) we have

(26) $$\omega_{n-1}\left(\log \lambda_n t\right)^{1-n} \leq \gamma_n(t) \leq \omega_{n-1}\left(\log t\right)^{1-n}, \quad t > 1.$$

Using the right side of inequality (26) we have

$$\gamma_n\left(\frac{1}{e^t - 1}\right) \leq \omega_{n-1}\left(\log\left(\frac{1}{e^t-1}\right)\right)^{1-n} \leq \omega_{n-1}\left(\log\left(\frac{1}{t}\right)\right)^{1-n}.$$

This gives $\zeta_j^{\mu}(t) \leq \omega_{n-1}\big(\log(1/t)\big)^{1-n}$ locally.

4. INCLUSION RELATIONS FOR BALLS

Each statement on modulus of continuity has its counterpart stated in terms of inclusions of balls. Namely, if for some metrics d_1 and d_2 there holds

$$d_1(x,y) < t \Rightarrow d_2(x,y) < \zeta(t),$$

then

$$D_{d_1}(x,t) \subset D_{d_2}(x, \zeta(x,t)),$$

where

(27) $$D_m(x,t) = \{z \in G \,|\, m(x,z) < t\},$$

when $x \in G$ and $t > 0$.

A related question is to find, for a given $x \in G$ and $t > 0$, minimal $\zeta(x,t)$ such that

$$D_{d_1}(x,t) \subset D_{d_2}\big(x, \zeta(x,t)\big),$$

This is circumscribed ball problem for a fixed $x \in G$.

By [**19**, (3.9)], we have the inclusions

(28) $$B^n\big(x, r\,d(x)\big) \subset D_k(x, M) \subset B^n\big(x, R\,d(x)\big),$$

where $r = 1 - e^{-M}$ and $R = e^M - 1$. It is easy to see that both numbers are best possible for the case of the half space H^n.

It was proved in [**2**, 15.13] that if G is a proper subdomain of $\mathbb{R}^n$ and if $x, y \in G$ with $x \neq y$, then

$$\lambda_G(x,y) \leq \inf_{z \in \partial G} \big(\lambda_{\mathbb{R}^n \setminus \{z\}}(x,y)\big) \leq \sqrt{2}\tau_n \left(\frac{|x-y|}{\min\{d(x), d(y)\}} \right). \tag{29}$$

Theorem 9. [**9**, Theorem 6.11] *Let G be a proper subdomain of $\mathbb{R}^n$ and let $t > 0$. If we denote $c_1 = 1/\big(1 + \tau_n^{-1}(t/\sqrt{2})\big)$, $c_2 = \sqrt{\tau_n^{-1}(2t)/\big(1 + \tau_n^{-1}(2t)\big)}$ and $c_3 = \tau_n^{-1}(t/\sqrt{2})$, then the inclusions*

$$D_{\lambda^{-1}}(a,t) \subset \{z \in G \,|\, d(z) > c_1 d(a)\}, \tag{30}$$

$$D_{\lambda^{-1}}(a,t) \supset B^n\big(a, c_2 d(a)\big) \supset D_k\big(a, \log(c_2 + 1)\big) \tag{31}$$

and

$$D_{\lambda^{-1}}(a,t) \subset B^n\big(a, c_3 d(a)\big) \cap G \tag{32}$$

are valid for all $a \in G$. If, in addition, $t > \sqrt{2}\tau_n(1)$, we have that

$$B^n(a, c_3 d(a)) \subset D_k\big(a, \log(1/(1 - c_3))\big). \tag{33}$$

To prove the inclusion (32), we apply (29) to obtain

$$\lambda_G(a,z) \leq \sqrt{2}\tau_n \left(\frac{|z-a|}{d(a)} \right).$$

From here with the assumption $t \leq \lambda_G(a,z)$ we have $|z - a| < \tau_n^{-1}(t/\sqrt{2}) d(a)$.

Since $D_{\lambda^{-1}} \subset G$, the inclusion (32) holds.

Inclusion (33) follows directly from (28) after we notice that the condition $t > \sqrt{2}\tau_n(1)$ implies that $c_3 < 1$ and hence that the ball $B^n\big(a, c_3 d(a)\big)$ is included in G.

Theorem 10. [**9**, Theorem 6.18] *Let G be a proper subdomain of $\mathbb{R}^n$ and assume that G has a connected, nondegenerate boundary. Let $t > 0$ and denote $d_1 = \tau_n^{-1}(t)/\big(1 + \tau_n^{-1}(t)\big)$, $d_2 = 1/\gamma_n^{-1}(t)$ and $d_3 = 1/\tau_n^{-1}(t)$. Then, for all $a \in G$, the following inclusions hold*

$$D_\mu(a,t) \subset \{z \in G \,|\, d(z) > d_1 d(a)\}, \tag{34}$$

$$D_\mu(a,t) \supset B^n\big(a, d_2 d(a)\big) \supset D_k\big(a, \log(d_2 + 1)\big), \tag{35}$$

$$D_\mu(a,t) \subset B^n\big(a, d_3 d(a)\big) \cap G. \tag{36}$$

If in addition $t < \tau_n(1)$, *then*

$$B^n\big(a, d_3 d(a)\big) \subset D_k\big(a, \log(1/(1-d_3))\big). \tag{37}$$

The numbers d_1, d_2 *and* d_3 *are best possible for these inclusions.*

We prove (35) only, because that part is used later on.

We assume that $a, z \in G$ and that $|z-a| \le d_2\, d(a)$. Then, since $\gamma_n^{-1}(t) > 1$, we have $d(z,a) < d(a)$. We consider the following curve families

$$\Gamma_J = \Delta(J_{az}, \partial G; G), \quad \Gamma = \Delta\Big(J_{az}, S^{n-1}\big(a, d(a)\big); \overline{B^n\big(a, d(a)\big)}\Big),$$

and

$$\widetilde{\Gamma} = \Delta([z', +\infty), S^{n-1}; \mathbb{R}^n \setminus B^n), \tag{38}$$

where $z' = d(a)/|z-a|\, e_1$. Since J_{az} is a continuum which joins a and z, we have

$$\mu_G(a,z) \le M(\Gamma_J) \tag{39}$$

and since $\Gamma < \Gamma_J$, we have that $M(\Gamma_J) < M(\Gamma)$.

Using Möbius transformations, we get

$$M(\Gamma) = M(\widetilde{\Gamma}) = \gamma_n\left(\frac{d(a)}{|z-a|}\right), \tag{40}$$

and since $|z-a| < d_2\, d(a)$ and γ_n is a strictly decreasing homeomorphism, it follows that

$$\gamma_n\left(\frac{d(a)}{|z-a|}\right) < \gamma_n\left(\frac{1}{d_2}\right) = t. \tag{41}$$

Combining all these inequalities, we get $\mu_G(a,z) < t$, which proves the left side of (35). The right side inclusion follows from (28).

Theorem 9 ((32) and (33)) gives

Theorem 11. *Let* G *be a proper subdomain of* $\mathbb{R}^n$ *and* $a, b \in G$ *distinct*

$$\lambda^{-1}(a,b) < \frac{1}{t} \Rightarrow k(a,b) < \log\frac{1}{1-\tau_n^{-1}\Big(\frac{t}{\sqrt{2}}\Big)}, \quad \textit{for } t > \sqrt{2}\tau_n(1)$$

$$\lambda^{-1}(a,b) < s \Rightarrow k(a,b) < \log\frac{1}{1-\tau_n^{-1}\Big(\frac{1}{\sqrt{2}s}\Big)},$$

$$\zeta^k_{\lambda^{-1}}(s) = \log\frac{1}{1-\tau_n^{-1}\Big(\frac{1}{\sqrt{2}s}\Big)}, \quad s < \frac{1}{\sqrt{2}\,\tau_n(1)}.$$

Furthermore, if G *is bounded, then we obtain* $\lambda^{-1}(x,a) < 1/t \Rightarrow |x-a| < c_3 d(a) <$ $\operatorname{diam}(G)\, c_3(1/t)$ *and from here* $\zeta^{|\cdot|}_{\lambda^{-1}}(t) = \tau_n^{-1}\big(1/(\sqrt{2}t)\big)\operatorname{diam}(G)$.

From Theorem 10 we deduce

Theorem 12. *In a domain* G *with connected nondegenerate boundary*:

$$D_\mu(a,t) \supset D_k\big(a, \log(d_2+1)\big), \quad d_2 = \frac{1}{\gamma_n^{-1}(t)}, \tag{42}$$

and $\mu(a,b) < t$ *if* $k(a,b) < \log(d_2+1)$.

Also, $\zeta^\mu_k(s) = \gamma_n(1/(e^s-1))$. *If we put*

$$s = \log\left(\frac{1}{\gamma_n^{-1}(t)} + 1\right), \qquad \text{we have } e^s - 1 = \frac{1}{\gamma_n^{-1}(t)}, \quad t = \gamma_n\left(\frac{1}{e^s-1}\right).$$

Theorem 13. [**14**, Theorem 3.8] *If* $G \subset \mathbb{R}^n$ *is open,* $x \in G$ *and* $t > 0$ *then*

$$D_j(x,t) \subset B^n(x,R)$$

where $R = (e^t-1)\, d(x)$. *This formula for* R *is the best possible expressed in terms of* t *and* $d(x)$ *only.*

Therefore, for a bounded domain G using $d(x) \le \operatorname{diam}(G)$, we get $\zeta^{|\cdot|}_j(t) = (e^t-1)\operatorname{diam}(G)$.

Theorem 14. [**14**, Theorem 3.10] *If* $G \subset \mathbb{R}^n$ *is an open set,* $x \in G$ *and* $t > 0$ *then* $D_\delta(x,t) \subset B^n(x,R)$ *where* $R = (e^t-1)\, d(x)$.

As above, we get, for a bounded domain G, $\zeta^{|\cdot|}_\delta(t) = (e^t-1)\operatorname{diam}(G)$.

In Chart 2 functions in cells 3,4,7 and 14 have been modified.

1	2 $\zeta^k_j(t) = ct$ G uniform $\zeta^k_j(t) = \varphi(t)$ G φ domain	3 $\zeta^\mu_j(t) = g_3(t)$ locally	4 $\zeta^{\lambda^{-1}}_j(t) = g_4(t)$ G c-QED domain
5 $\zeta^j_k(t) = t$	6	7 $\zeta^\mu_k(t) = g_7(t)$ ∂G connected, non-degenerate	8 $\zeta^{\lambda^{-1}}_k = \zeta^{\lambda^{-1}}_j$
9 $\zeta^j_\mu(t) = t/c_n$ ∂G connected	10 $\zeta^k_\mu(t) = ct$ G uniform ∂G connected	11	12 $\zeta^{\lambda^{-1}}_\mu = \zeta^j_\mu \circ \zeta^{\lambda^{-1}}_j$ G c-QED domain ∂G connected
13 $\zeta^j_{\lambda^{-1}}(t) = g_{13}(t)$	14 $\zeta^k_{\lambda^{-1}}(t) = g_{14}(t)$ $t < \frac{1}{\sqrt{2}\tau_2(1)}$	15 $\zeta^\mu_{\lambda^{-1}} = \zeta^j_{\lambda^{-1}} \circ \zeta^\mu_j$ locally	16

Chart 2

In Chart 2 the functions g_j are defined as follows:

$$g_3(t) = \omega_{n-1}\left(\log\left(\frac{1}{t}\right)\right)^{1-n}, \qquad g_4(t) = \frac{1}{c\tau_n(e^{2t}-1)},$$
$$g_7(t) = \gamma_n\left(\frac{1}{e^t-1}\right), \qquad g_{13}(t) = \log\left(1+\tau_n^{-1}\left(\frac{1}{\sqrt{2}\,t}\right)\right),$$
$$g_{14}(t) = \log\frac{1}{1-\tau_2^{-1}(1/(\sqrt{2}t))}.$$

EXAMPLE 1. For $G \subset \mathbb{R}^n$ we choose $z_0 \in \partial G$, a sequence $x_k \in G$ such that $x_k \to z_0$ and a sequence $y_k \in G$ such that

$$|y_k - z_0| < \frac{|x_k - z_0|}{k}. \tag{43}$$

Clearly $|x_k - y_k| \to 0$ and

$$|x_k - y_k| > |x_k - z_0| - |y_k - z_0| > |x_k - z_0|\left(1 - \frac{1}{k}\right). \tag{44}$$

But

$$j_G(x_k, y_k) \geq \log\left(1 + \frac{|x_k - y_k|}{|y_k - z_0|}\right) \geq \log\left(1 + \frac{1-\frac{1}{k}}{\frac{1}{k}}\right) = \log(k) \to +\infty.$$

Hence $id : (G, |\cdot|) \longrightarrow (G, j_G)$ is not uniformly continuous. By this reason, corresponding functions in the charts do not exist (see Chart 3c and Chart 3d).

Also, for a fixed small $d > 0$ we can find $x, y \in G$ such that $|x - y| = d$ and $d(x, \partial G)$ is as small as we like.

So we get $k_G(x, y)$ as large as we like and there is no upper bound of $k_G(x, y)$ in terms of $|x - y|$.

For a bounded domain G we see by Lemma 1 (1) that the modulus of continuity $id : (G, k_G) \to (G, |\cdot|)$ is $\zeta_k^{|\cdot|}(t) = t\,\mathrm{diam}(G)$.

All the remaining functions are obtained by composition of the above moduli of continuity.

And finally we have Chart 3, which is split in four subcharts: 3a, 3b, 3c and 3d, which are given in the Section 6.

Sharper results can be obtained for special domains, for example $G = \mathbb{R}^n \setminus \{0\}$ was studied by R. KLÉN **[11]** in relation to j_G metrics.

5. REMOVING A POINT

Let $\mathcal{M}$ be a collection of metrics on a domain $G \subset \mathbb{R}^n$ and $D_m(x, T)$ is as in (27) and $m \in \mathcal{M}$. Let

$$r_T = \sup\{r > 0 \,:\, B^n(x, r) \subset D_m(x, T)\},$$
$$R_T = \inf\{r > 0 \,:\, B^n(x, r) \cap D_m(x, T) = \emptyset\}.$$

A natural question is whether we can find a lower bound for r_T and an upper bound for R_T.

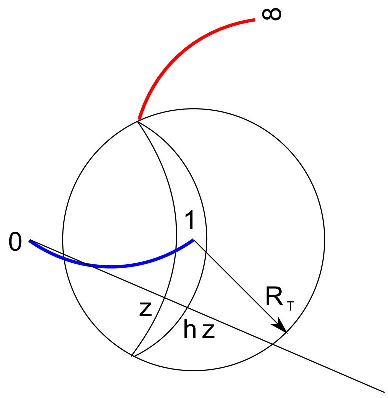

Figure 1. Radius of circumscribed ball

Problem 1. Find the radius of circumscribed ball R_T in the case $G = \mathbb{C} \setminus \{0\}$ and $m = \lambda_G^{-1}$ (see Figure 1).

It is evident from the definition of λ_G that adding new points, even isolated ones, to the boundary of G will affect the value of $\lambda_G(x, y)$ for fixed points $x, y \in G$. We study this phenomenon in the case when $G = \mathbb{C} \setminus \{0\}$.

Let $h(z) = z/|z|^2$ be an inversion. Since $h : G \to G$ (h is an isometry for λ metric) we have

$$\lambda_G(1, z) = \lambda_G(1, h(z)).$$

From [15, (3.3), (3.22)] we have

$$p(z) = \frac{2\pi}{\log M(2z-1)}, \quad z \in \mathbb{C} \setminus \{0, 1\} \quad \text{and} \tag{45}$$

$$\log M(2e^{i\theta} - 1) = \frac{2\pi\, \mathcal{K}\big(\sin(\theta/4)\big)\, \mathcal{K}\big(\cos(\theta/4)\big)}{\mathcal{K}^2\big(\sin(\theta/4)\big) + \mathcal{K}^2\big(\cos(\theta/4)\big)}, \tag{46}$$

where

$$\mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \quad \text{for } 0 < r < 1.$$

If we put $z = e^{i\theta}$ we have

$$p(e^{i\theta}) = \frac{\mathcal{K}^2\big(\sin(\theta/4)\big) + \mathcal{K}^2\big(\cos(\theta/4)\big)}{\mathcal{K}\big(\sin(\theta/4)\big)\, \mathcal{K}\big(\cos(\theta/4)\big)}.$$

For $|z| = 1$ we obtain $\lambda_G(1, z) = p(z)$.

Choose θ such that $\sin(\theta/2) = R_T/2$. From here $\theta = 2\arcsin(R_T/2)$. Now if we put

$$y = \frac{\mathcal{K}\big(\sin(\theta/4)\big)}{\mathcal{K}\big(\cos(\theta/4)\big)} = \frac{2}{\pi}\mu\big(\cos(\theta/4)\big) \tag{47}$$

we have

$$p(e^{i\theta}) = y + \frac{1}{y} = \frac{1}{T}.$$

We are interested for solutions $y < 1$ because we want $\theta < \pi$. From here $y = 2T/(1+\sqrt{1-4T^2})$. Since from (47)

$$\theta = 4\arccos\left(\mu^{-1}\left(\frac{\pi y}{2}\right)\right)$$

now we have

(48)
$$\theta = 4\arccos\left(\mu^{-1}\left(\frac{\pi}{2}\frac{2T}{1+\sqrt{1-4T^2}}\right)\right) = 4\arccos\left(\mu^{-1}\left(\frac{\pi T}{1+\sqrt{1-4T^2}}\right)\right).$$

Hence, the radius of the circumscribed sphere is

$$R_T = 2\sin(\theta/2), \quad T \in (0,1/2), \quad \theta \text{ from (48)}.$$

Open question.

(1) Can we find r_T in the case above?

(2) Can we estimate R_T, where G is now bounded subset of $\mathbb{C}$ (instead of $\mathbb{R}^2 \setminus \{0\}$)?

(3) Consider μ_G-balls where ∂G is connected, say $\partial G = [0, e_1]$. Can we find a lower bound for r_T (upper bound for R_T) in this case?

Problem 2. (Estimate for $\lambda_{B^2\setminus\{0\}}(x,y)$) Next we investigate the following situation: $G \subseteq \mathbb{R}^n$ is domain, $a \in G$, $G' = G \setminus \{a\}$. Is $\lambda_G(x,y) = \lambda_{G'}(x,y)$ true under some additional assumptions, like x, y close to ∂G?

We consider a special case where $G = B^2$ and $a = 0$.

In [**13**, Lemma 2.8] it is proven that if $\Gamma_0 = \Delta([0,x],[\tilde{y}, x/|x|]; B^2)$, where $\tilde{y} = (|y|/|x|)\,x$ and if we put $|x| = r$, $|\tilde{y}| = s$ (see Figure 2), then we have

(49)
$$M(\Gamma_0) = \tau_n\left(\frac{(s-r)(1-rs)}{r(1-s)^2}\right).$$

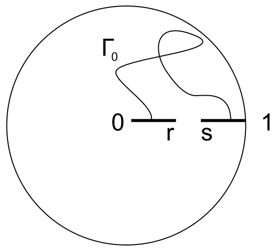

Figure 2. Family of curves Γ_0.

Further, from [**18, (2.6)**]] we have that if $\Delta_0 = \Delta([-x/|x|, -x], [x, x/|x|]; B^2)$ and if $|x| = r$ as before (see Figure 3), then

$$M(\Delta_0) = \frac{1}{2}\,\tau_n\left(\frac{4r^2}{(1-r^2)^2}\right).$$

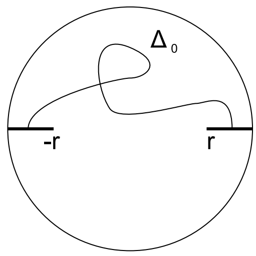

Figure 3. Family of curves Δ_0.

Also, using Möbius transformation $T_r : B^2 \to B^2$, $T(r) = 0$ we can map family of curves Δ_1 to family of curves Δ_1', where

$$\Delta_1 = \Delta([-x/|x|, -\tilde{y}], [0, x]; B^2) \text{ and } \Delta_1' = \Delta([-x/|x|, -\tilde{y}'], [-x, 0]; B^2)$$

.

We know that

$$\rho(-s,0)=\rho(-r,-t),$$

where r and s are as before and $-t=T_r(-s)$. Further, this is equivalent to

$$\log\frac{1+s}{1-s}=\log\frac{1+t}{1-t}\frac{1-r}{1+r}. \tag{50}$$

Solving (50) in t we obtain $t=(s+r)/(1+sr)$.

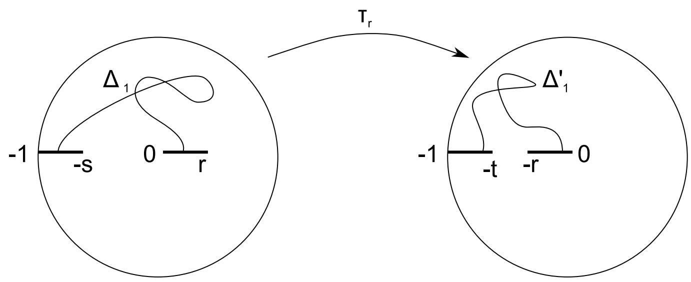

Figure 4. Möbius transformation T_r.

Now we have (see Figure 4)

$$M(\Delta_1)=M(\Delta_1')=\tau_n\left(\frac{(t-r)(1-tr)}{r(1-t)^2}\right)=\tau_n\left(\frac{s(1+r)^2}{r(1-s)^2}\right).$$

The first equality holds because T_r is a conformal map, the second one follows from (49) and the third one from the expression for t.

Now, if we put in last term that $r=s$, we obtain

$$M(\Delta_1)=\tau_n\Big(\Big(\frac{1+r}{1-r}\Big)^2\Big).$$

The question is when do we have $M(\Delta_1)\geq M(\Delta_0)$. In other words, when does the inequality

$$\tau_n\Big(\Big(\frac{1+r}{1-r}\Big)^2\Big)\geq\frac{1}{2}\tau_n\left(\frac{4r^2}{(1-r^2)^2}\right) \tag{51}$$

hold?

Applying formula [**2, 5.19 (5)**]:

$$\frac{1}{2}\tau_n(t)\geq\tau_n\big((\sqrt{t}+\sqrt{t+1}\,)^4-1\big)$$

for $t=4r^2/(1-r^2)^2$ we have

$$\frac{1}{2}\,\tau_n\left(\frac{4r^2}{(1-r^2)^2}\right)=\tau_n\left(\frac{8r(r^2+1)}{(1-r)^4}\right).$$

Then (51) is equivalent to

$$\left(\frac{1+r}{1-r}\right)^2 \le \frac{8r(r^2+1)}{(1-r)^4},$$

since τ_n is decreasing. The last inequality is equivalent to

$$r^4 - 8r^3 - 2r^2 - 8r + 1 \le 0.$$

This inequality holds for $r \in [2+\sqrt{5}-2\sqrt{2+\sqrt{5}}, 1)$ and

$$2+\sqrt{5}-2\sqrt{2+\sqrt{5}} < 0.12.$$

This gives the answer to the question: For which values of $|x|$ we have

$$\lambda_A(x,-x) = M\big(\Delta(E,-E;B^2)\big),$$

where $A = B^2 \setminus \{0\}$, $E = [x, x/|x|]$?

A related result can be found in HEIKKALA's dissertation, [**9**, Theorem 7.3]. In fact, this theorem deals with the more general situation: If x and y are close to the boundary and far apart then $\lambda_{B^n\setminus\{0\}}(x,y) = \lambda_{B^n}(x,y)$. His theorem is:

Theorem 15. *Let $G = B^n \setminus \{0\}$ and let $x, y \in G$ with $|x-y| \ge \delta > 0$. Then, if $\min\{|x|,|y|\} \in (r_1, 1)$ with $r_1 = (\sqrt{\delta^4+64}-\delta^2)/8$, we have that*

$$\lambda_G(x,y) = \lambda_{B^n}(x,y).$$

However, we have in the special case $x=-y$, better constant (letting $\delta = 2|x|$ and $r_1 = |x|$ in Theorem 7.3 gives equation $r_1^3 + r_1^2 - 1 = 0$, and its real root is larger than 0.75, and consequently larger than 0.12).

6. CHARTS FOR MODULI OF CONTINUITY

Here, as noted at the end of Section 4, we present Charts 3a, 3b, 3c and 3d.

1	2 $\zeta_\alpha^\delta(t) = 2t$ G convex	3 $\zeta_\alpha^j(t) = t$ G convex	4 $\zeta_\alpha^k(t) = ct$ G convex, uniform
5 $\zeta_\delta^\alpha(t) = t$	6	7 $\zeta_\delta^j(t) = t$	8 $\zeta_\delta^k(t) = ct$ G uniform
9 $\zeta_j^\alpha(t) = 2t$	10 $\zeta_j^\delta(t) = 2t$	11	12 $\zeta_j^k(t) = ct$ G uniform
13 $\zeta_k^\alpha(t) = 2t$	14 $\zeta_k^\delta(t) = 2t$	15 $\zeta_k^j(t) = t$	16

Chart 3a

1 $\zeta_\alpha^q(t) = u_1(t)$ G convex	2 $\zeta_\alpha^{\lvert\cdot\rvert}(t) = u_1(t)$ G convex	3 $\zeta_\alpha^\mu(t) = u_2(t)$ G convex, locally	4 $\zeta_\alpha^{\lambda^{-1}}(t) = u_3(t)$ G c-QED, convex
5 $\zeta_\delta^q(t) = u_1(t)$	6 $\zeta_\delta^{\lvert\cdot\rvert}(t) = u_1(t)$	7 $\zeta_\delta^\mu(t) = u_2(t)$ locally	8 $\zeta_\delta^{\lambda^{-1}}(t) = u_3(t)$ G c-QED
9 $\zeta_j^q(t) = u_1(t)$	10 $\zeta_j^{\lvert\cdot\rvert}(t) = u_1(t)$	11 $\zeta_j^\mu(t) = u_2(t)$ locally	12 $\zeta_j^{\lambda^{-1}}(t) = u_3(t)$ G c-QED
13 $\zeta_k^q(t) = t\,\mathrm{d}(G)$	14 $\zeta_k^{\lvert\cdot\rvert}(t) = t\,d(G)$	15 $\zeta_k^\mu(t) = u_2(t)$ ∂G connected nondegenerate	16 $\zeta_k^{\lambda^{-1}}(t) = u_3(t)$ G c-QED

Chart 3b

In Chart 3b for 1,2,5,6,9,10,13,14 the domain G is assumed to be bounded and the functions u_j are defined as follows:

$$u_1(t) = (e^t - 1)\,d(G),\ u_2(t) = \gamma\left(\frac{1}{e^t - 1}\right),\ u_3(t) = \frac{1}{c\tau(e^{2t} - 1)}.$$

Also, abbreviation $d(G) = \mathrm{diam}(G)$ is used.

1 Function ζ_q^α does not exist	2 Function ζ_q^δ does not exist	3 Function ζ_q^j does not exist	4 Function ζ_q^k does not exist
5 Function $\zeta_{\lvert\cdot\rvert}^\alpha$ does not exist	6 Function $\zeta_{\lvert\cdot\rvert}^\delta$ does not exist	7 Function $\zeta_{\lvert\cdot\rvert}^j$ does not exist	8 Function $\zeta_{\lvert\cdot\rvert}^k$ does not exist
9 $\zeta_\mu^\alpha(t) = v_1(t)$ ∂G connected	10 $\zeta_\mu^\delta(t) = v_1(t)$ ∂G connected $\mathrm{card}(\partial G) \geq 2$	11 $\zeta_\mu^j(t) = \dfrac{t}{c_n}$ ∂G connected	12 $\zeta_\mu^k(t) = ct$ G uniform ∂G connected
13 $\zeta_{\lambda^{-1}}^\alpha(t) = v_2(t)$	14 $\zeta_{\lambda^{-1}}^\delta(t) = v_2(t)$ $\mathrm{card}(\partial G) \geq 2$	15 $\zeta_{\lambda^{-1}}^j(t) = v_3(t)$	16 $\zeta_{\lambda^{-1}}^k(t) = c\,v_3(t)$ G uniform

Chart 3c

In Chart 3c the functions v_j are defined as follows:
$v_1(t) = \log\left(1 + \dfrac{1}{\tau^{-1}(t)}\right)$, $v_2(t) = \log\left(1 + 2\tau^{-1}\left(\dfrac{1}{t}\right)\right)$ and
$v_3(t) = \log\left(1 + \tau^{-1}\left(\dfrac{1}{\sqrt{2t}}\right)\right)$.

1	2 $\zeta_q^{\vert\cdot\vert}(t) = ct$ G bounded	3 Function ζ_q^{μ} does not exist	4 Function $\zeta_q^{\lambda^{-1}}$ does not exist
5 $\zeta_{\vert\cdot\vert}^{q}(t) = t$	6	7 Function $\zeta_{\vert\cdot\vert}^{\mu}$ does not exist	8 Function $\zeta_{\vert\cdot\vert}^{\lambda^{-1}}$ does not exist
9 $\zeta_{\mu}^{q}(t) = \dfrac{d(G)}{\tau^{-1}(t)}$ ∂G connected	10 $\zeta_{\mu}^{\vert\cdot\vert}(t) = \dfrac{d(G)}{\tau^{-1}(t)}$ ∂G connected	11	12 $\zeta_{\mu}^{\lambda^{-1}} = w_1(t)$ G c-QED domain ∂G connected
13 $\zeta_{\lambda^{-1}}^{q} = w_2(t)$	14 $\zeta_{\lambda^{-1}}^{\vert\cdot\vert} = w_2(t)$	15 $\zeta_{\lambda^{-1}}^{\mu} = w_3(t)$ locally	16

Chart 3d

In Chart 3d for 2,9,10,13,14 the domain G is assumed to be bounded and the functions w_j are defined as follows:

$$w_1(t) = \frac{1}{c\tau(e^{bt}-1)}, w_2(t) = \tau^{-1}(1/(\sqrt{2}t))\, d(G) \text{ and } w_3(t) = \gamma\left(\frac{1}{\tau^{-1}\left(1/(\sqrt{2}\,t)\right)}\right).$$

We use abbreviation $d(G) = \mathrm{diam}(G)$ as in Chart 3b.

Acknowledgments. I wish to thank Prof. Matti Vuorinen for his continuous guidance and support during my research on this topic, which is a part of my PhD thesis. My visits to Finland were supported by his research project “Quasiconformal Maps” funded by the Academy of Finland as well as by a grant of the Väisälä Foundation of the Finnish Academy of Sciences.

REFERENCES

1. L. Ahlfors, A. Beurling: *Conformal invariants and function-theoretic null-sets.* Acta Math., **83** (1950), 101–129.
2. G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen: *Conformal invariants, inequalities and quasiconformal mappings.* J. Wiley, 1997, 505 pp.
3. A. F. Beardon: *The geometry of discrete groups.* Graduate Texts in Math., Vol. **91**, Springer Verlag, Berlin – Heidelberg – New York, 1982.
4. A. F. Beardon: *The Apollonian metric of a domain in* $\mathbb{R}^n$*.* Quasiconformal mappings and analysis – a collection of papers honoring F. W. Gehring, ed. by P. L. Duren, J. M. Heinonen, B. G. Osgood and B. P. Palka, Springer Verlag 1998, 91-108.
5. F. W. Gehring: *Symmetrization of rings in space.* Trans. Amer. Math. Soc., **101** (1961), 499–519.

6. F. W. Gehring: *Rings and quasiconformal mappings in space.* Trans. Amer. Math. Soc., **103** (1962), 353–393.
7. F. W. Gehring: *Quasiconformal mappings in Euclidean spaces.* Handbook of complex analysis: geometric function theory, Vol. 2, ed. by R. Kühnau, 1-29, Elsevier, Amsterdam, 2005.
8. F. W. Gehring, B. G. Osgood: *Uniform domains and the quasi-hyperbolic metric.* J. Anal. Math., **36** (1979), 50–74.
9. V. Heikkala: *Inequalities for conformal capacity, modulus, and conformal invariants.* Ann. Acad. Sci. Fenn. Math. Diss., No. 132 (2002), 62 pp.
10. V. Heikkala, M. Vuorinen: *Teichmüller's extremal ring problem.* Math. Z., **254**, No. 3 (2006), 509–529.
11. R. Klén: *Local convexity properties of j-metric balls.* Ann. Acad. Sci. Fenn. Math., **33**, No. 1 (2008), 281–293.
12. R. Kühnau, ed.: *Handbook of complex analysis: geometric function theory.* Vol. 1 and Vol. 2, Elsevier Science B.V., Amsterdam, 2002 and 2005.
13. M. Lehtinen, M. Vuorinen: *On Teichmüller's modulus problem in the plane.* Rev. Roumaine Math. Pures Appl., **33** (1988), 97–106.
14. P. Seittenranta: *Möbius-invariant metrics.* Math. Proc. Cambridge Philos. Soc., **125**, No. 3 (1999), 511–533.
15. A. Yu. Solynin, M. Vuorinen: *Extremal problems and symmetrization for plane ring domains.* Trans. Amer. Math. Soc., **348** (1996), 4095–4112.
16. J. Väisälä: *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Math. 229, Springer-Verlag, Berlin, 1971.
17. J. Väisälä: *Quasisymmetric embeddings in Euclidean spaces.* Trans. Amer. Math. Soc., **264**, No. 1 (1981), 191–204.
18. M. Vuorinen: *Conformal invariants and quasiregular mappings.* J. Anal. Math., **45** (1985), 69–115.
19. M. Vuorinen: *Conformal geometry and quasiregular mappings.* Lecture Notes in Math., 1319, Springer, Berlin, 1988.
20. M. Vuorinen: *Conformally invariant extremal problems and quasiconformal maps.* Quart. J. Math. Oxford Ser. (2), **43** (1992), 501–514.
21. M. Vuorinen: *Metrics and quasiregular mappings.* Proc. Int. Workshop on Quasiconformal Mappings and their Applications, IIT Madras, Dec 27, 2005 - Jan 1, 2006, ed. by S. Ponnusamy, T. Sugawa, M. Vuorinen: *Quasiconformal Mappings and their Applications.* Narosa Publishing House, 291–325, New Delhi, India, 2007.
22. V. Manojlovic, M. Vuorinen: *On quasiconformal maps with identity boundary values.* Transactions of the AMS (in press)

University of Belgrade,
Faculty of Organizational Sciences,
Jove Ilića 154, Belgrade,
Serbia
E–mail: vesnak@fon.bg.ac.yu

(Received September 11, 2008)
(Revised January 29, 2009)