Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 177–197. doi:10.2298/AADM0902177S

AN EXTENSION OF A NORM INEQUALITY FOR A SEMI-DISCRETE $\mathbf{g}^*_\lambda$ FUNCTION

Caroline Sweezy

A norm inequality for a semi-discrete $g^*_\lambda(f)$ function is obtained for functions, f, that can be written as a sum whose terms consist of a numerical coefficient multiplying a member of a family of functions that have properties of geometric decay, minimal smoothness and almost orthogonality condition. The theorem is applied to the rate of change of u, a solution to $Lu = \operatorname{div} \overrightarrow{f}$ in a bounded, nonsmooth domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$, $u = 0$ on $\partial\Omega$.

1. INTRODUCTION

In classical harmonic analysis the various different "square" functions, the Lusin area integral, the g^*_λ-function and the g-function, were useful in obtaining norm estimates for functions that were not themselves easily estimated. In probability square functions are even more widely employed. These auxiliary functions continue to be useful in obtaining information about solutions to more general second order partial differential equations. However, one needs to find the "right" definition for a square function that can be useful in this setting. The lack of smoothness for solutions is an obstacle, as it means one cannot prove pointwise estimates for derivatives of weak solutions. In this paper a weighted norm inequality is proved for a "Littlewood-Paley" type function that is defined for functions that can be represented as sums of coefficients multiplying members of a family of functions that have minimal smoothness, some cancellation properties and geometric decay. This theorem can be applied to solutions of second order elliptic equations to obtain conditions on two measures sufficient for obtaining weighted norm inequalities for the gradients (or, more usefully, local Hölder norms) of solutions to homogeneous and non-homogeneous elliptic equations on bounded domains. The

2000 Mathematics Subject Classification. 42B25,35J25.
Keywords and Phrases. Minimal smoothness, semi-discrete Littlewood-Paley type inequalities, good-λ inequalities, strictly elliptic operators, non-smooth bounded domains.

Appendix contains the proof that a family of functions $\varphi_J(x)$ satisfying such conditions can be derived using averages of the Green function generated by a strictly elliptic operator on a rough boundary domain. Theorem A, the square function result, was announced in [**7**] and [**8**]; these papers contained the weighted norm inequality for pde solutions, Theorem B, and a sketch of the proof of Theorem B. The proof of Theorem A appears here for the first time.

Before Theorem A is stated, first recall what it means for a measure to be A^∞ with respect to another measure:

A measure σ defined on a domain D, is said to be A^∞ with respect to Lebesgue measure if for any cube $Q \subset D$ and any measurable subset E of Q, there are fixed constants C_0 and $\kappa > 0$ so that $\left(\frac{\sigma(E)}{\sigma(Q)}\right)^\kappa \leq C_0 \frac{|E|}{|Q|}$, [**1**].

Theorem A will be proved for functions of the form $f(x) = \sum_{J \in \mathcal{F}} \lambda_J \varphi_{(J)}(x)$, with $\mathcal{F}$ being a finite set of dyadic cubes that are also Whitney-type cubes with respect to the domain Ω, the $\varphi_{(J)}$ are functions as described above. $\mathcal{W}$ will denote the collection of certain Whitney-type dyadic cubes (these are dyadic cubes whose dimension compares with the cube's distance from the boundary of Ω) that lie in Ω. These cubes have the property that their interiors are pairwise disjoint; a fixed dilate of any cube will also be Whitney-type with respect to Ω, and $\Omega = \bigcup_{Q_j \in \mathcal{W}} \overline{Q_j}$. For technical reasons we need to take Q_0 as a large dyadic cube that contains Ω. The cubes in $\mathcal{W}$ are also dyadic subcubes of Q_0. We will refer to the collection of all dyadic subcubes of Q_0 as $\mathcal{D}$, so $\mathcal{F} \subsetneqq \mathcal{W} \subsetneqq \mathcal{D}$.

Next we need to define the function $g^*(f)(x) = g^*(x)$ and to give the conditions **a**), **a**$'$), **b**), and **c**). When $f(x) = \sum_{J \in \mathcal{F}} \lambda_J \varphi_{(J)}(x)$,

$$g^*(x) = \left(\sum_{J \in \mathcal{F}} \frac{\lambda_J^2}{|J|} \left(1 + \frac{|x - x_J|}{\ell(J)}\right)^{-d+\varepsilon} \right)^{1/2}.$$

$g^*(x)$ is a discrete version of the g_λ^* function of classical Littlewood-Paley theory.

$\mathcal{S}(Q) = \{J \in \mathcal{F} : J \nsubseteq Q\}$. $\delta(x) =$ distance $(x, \partial\Omega)$. $x_J =$ the center point of J.

The four conditions that will be assumed to hold for the family $\left\{\varphi_{(J)}(x)\right\}$ are:

a) $\left|\varphi_{(J)}(x)\right| \leq C\ell(J)^{2-d/2} \left(1 + \frac{|x - x_J|}{\ell(J)}\right)^{2-d}$ *for all* $x \in \Omega$.

a$'$) $\left|\varphi_{(J)}(x)\right| \leq C\delta(x)^\alpha \ell(J)^{2-d/2-\alpha} \left(1 + \frac{|x - x_J|}{\ell(J)}\right)^{2-d-\alpha}$ *for all* $x \in \Omega$.

b) *There is an absolute constant* η, $0 < \eta < 1$, *so that*

$$\left|\varphi_{(J)}(x) - \varphi_J(y)\right| \leq C\,|x - y|^\alpha\, \ell(J)^{2-d/2-\alpha} \left(1 + \frac{|x - x_J|}{\ell(J)} + \frac{|y - x_J|}{\ell(J)}\right)^{2-d-\alpha}$$

for all x, y in ηQ_j and $J \in \mathcal{S}(Q_j)$.

c) $\displaystyle\int \Big|\sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)\Big|^2 \mathrm{d}x \le C \sum_{J\in\mathcal{F}} \lambda_J^2.$

Theorem A. *Suppose that $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$ is a function defined on Ω, where $\mathcal{F}$ is a finite set of dyadic cubes from $\mathcal{W}$, and the $\{\varphi_{(J)}\}_{J\in\mathcal{F}}$ are a family of functions that satisfy conditions* **a**), **a′**), **b**), *and* **c**), *and such that $\varphi_{(J)}(x) = 0$ if $x \in Q_0 \setminus \Omega$. Then, if $d\sigma \in A^\infty(Q_0, dx)$, there is a constant $C = C(d, \alpha, p, \Omega, \varepsilon, \kappa, C_0)$ such that, for any $0 < p < \infty$,*

$$\|f\|_{L^p(Q_0,d\sigma)} \le C \|g^*\|_{L^p(Q_0,d\sigma)} .$$

Theorem A is proved in the next two sections. Section 2 contains the local estimates needed to prove a good-λ inequality, which is contained in the third section. The fourth section describes briefly an application of Theorem A. The application, Theorem B, consists of determining sufficient conditions for two measures, μ and η, defined on a bounded rough domain Ω in $\mathbb{R}^n$ so that, for solutions $u(x)$ to the inhomogeneous equation $Lu = \operatorname{div} \overrightarrow{f}$ in Ω, $u|_{\partial\Omega} = 0$ with $L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{i,j}(x)\frac{\partial}{\partial x_j}\Big)$, symmetric and strictly elliptic,

$$\left(\int_\Omega (\|u\|_{H^\alpha}(x))^q \, \mathrm{d}\mu(x)\right)^{1/q} \le C \left(\int_\Omega \left(\left|\overrightarrow{f}(x)\right|^p + \left|\operatorname{div} \overrightarrow{f}(x)\right|^p\right) \mathrm{d}\nu(x)\right)^{1/p}.$$

C is independent of u and f. The local Hölder norm of u is defined by $\|u\|_{H^\alpha}(x)$

$:= \sup_{0<|x-y|<\delta(x)/50} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$, $\delta(x) =$ distance of x from $\partial\Omega$.

2. LOCAL ESTIMATES

To prove Theorem A we follow the method of Wilson [**11**] (see also [**5**], [**6**], [**9**], and [**10**]) in using the following "cube-skipping" functions $F(I, x) = \sum_{J\in\mathcal{S}(I)} \lambda_J \varphi_{(J)}(x)$, $F(I) = F(I, x_I)$, $F^*(x) = \sup_{I\ni x} F(I)$ and

$$G(I, x) = \left(\sum_{J\in\mathcal{S}(I)} \frac{\lambda_J^2}{|J|}\left(1 + \frac{|x - x_J|}{\ell(J)}\right)^{-(d-\varepsilon)}\right)^{1/2},$$

$G(I) = G(I, x_I)$, $G^*(x) = \sup_{I\ni x} G(I)$, for $I \in \mathcal{D}$. Notice that I can be any dyadic cube from $\mathcal{D}$, although the cubes J in $\mathcal{S}(I)$ are taken from $\mathcal{W}$. The functions $F(I, x)$, $G(I, x)$, etc. are always generated by a given function $f(x) =$

$\sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$, where $\mathcal{F}$ is a finite family of dyadic cubes from $\mathcal{W}$. $\mathcal{S}(I) = \{J \in \mathcal{F} : J \supsetneqq I\}$, and $\ell(I)$ is the side length of the dyadic cube I. $F(I,x)$ and $G(I,x)$ are only defined for $x \in I$. We note some special properties of the particular functions $\varphi_{(J)}(x)$ that are defined below. These properties will be crucial in proving the estimates in Lemmas 1–7 and the Central Lemma, Lemma 8. We have $\varphi_{(J)}(x) = 0$ whenever x lies outside Ω. Also each $\varphi_{(J)}$ is chosen so that $J \in \mathcal{W}$. As in [**11**] we obtain local estimates relating the functions $F(I,x)$, $G(I,x)$, etc. in order to use these functions to prove the crucial good-λ inequality of the Corollary to Lemma 8. The good-λ inequality then yields the result of Theorem A by standard methods. The local estimates are established in Lemmas 1–7 below.

For the remaining part of the paper we take $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$, where $\mathcal{F}$ is a finite family of dyadic cubes; the $\varphi_{(J)}$ satisfy properties **a**), **a′**), **b**), and **c**), and they have all the properties mentioned in the previous paragraph. We note that many of the constants obtained in Lemmas 1–7 depend on $\operatorname{diam}(\Omega)$. For the functions $\varphi_{(J)}$ that appear in the proof of Theorem B, i.e. for

$$\varphi_{(J)}(y) = \frac{1}{\sqrt{|J|}} \left(\int_{(3/2)J} \left(G(x,y) - \widetilde{G}(x,y) \right) \mathrm{d}x \right),$$

the constants in **a′**) and **b**) for these functions also depend on $\operatorname{diam}(\Omega)$ and Ω (see [**3**]), so this is no new restriction. We also note that having the $\varphi_{(J)}(x) = 0$ whenever x lies outside Ω means that $F(I,x) = 0$ when $x \in \Omega^C$. However, $F^*(x)$, $G(I,x)$, $G^*(x)$ are not necessarily zero for x outside Ω. Following Wilson [**11**] we start with

Lemma 1. *$|f(x)| \le F^*(x)$ for a.e. $x \in Q_0$.*

Proof. This follows from the definition of $F^*(x)$, the fact that $\mathcal{F}$ is a finite family, and that the boundaries of all the dyadic cubes in $\mathcal{D}$ form a set of measure zero.

Lemma 2. *There is a constant C so that $G^*(x) \le Cg^*(x)$ for a.e. $x \in Q_0$.*

Proof. $G^*(x) = \sup_{Q\ni x} G(Q)$. If $x \in Q$ and $I \in \mathcal{S}(Q)$, then either $I \supsetneqq Q$ or I lies outside Q. In both cases, $|x_I - x_Q| \ge c\ell(Q)$ and $|x - x_Q| \le c'\ell(Q)$. Therefore $|x - x_I| \le |x - x_Q| + |x_Q - x_I| \le C\,|x_Q - x_I|$. So $(1 + |x - x_I|/\ell(I)) \le C'(1 + |x_Q - x_I|/\ell(I))$ or

$$(1 + |x_Q - x_I|/\ell(I))^{-d+\varepsilon} \le C''\,(1 + |x - x_I|/\ell(I))^{-d+\varepsilon}.$$

For $I \in \mathcal{F}$ whenever the term on the left is in $G(Q)$, the term on the right appears in $g^*(x)$, $x \in Q$, multiplied by $1/C''$. This is true for all dyadic cubes Q with the same constant $C = \max(1, C'')$, so $G(Q) \le Cg^*(x)$

Lemma 3. *For any η, $0 < \eta < 1$, if $x \in \eta Q$, then there is a constant $C_1 = C(d,\eta)$ so that $C_1^{-1}G(Q) \le G(Q,x) \le C_1 G(Q)$.*

Proof. For any cube $I \in \mathcal{S}(Q)$, $|x - x_I|/|x_Q - x_I|$ is bounded above and below by constants that depend on η and d.

Lemma 4. *For η as in* **b**), $0 < \eta < 1$, *if* $x, y \in \eta Q$, *then there is a constant* $C_2 = C(d, \lambda, \eta, \operatorname{diam}(\Omega), \Omega, \alpha)$ *so that* $|F(Q,x) - F(Q,y)| \leq C_2 G(Q)$.

Proof. As in [**11**] we write

$$\begin{aligned}|F(Q,x) - F(Q,y)| &= \left|\sum_{J\in\mathcal{S}(Q)} \lambda_J(\varphi_{(J)}(x) - \varphi_{(J)}(y))\right| \\ &\leq \sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)} |\lambda_J| \left|\varphi_{(J)}(x) - \varphi_{(J)}(y)\right| \\ &\quad + \sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)} |\lambda_J| \left|\varphi_{(J)}(x) - \varphi_{(J)}(y)\right| = I. + II.\end{aligned}$$

When x and y both lie inside Ω, $I.$ will be shown to be bounded by $CG(Q)$ using the Hölder continuity of the $\varphi_{(J)}$'s, property **b**). $II.$ must also be bounded using Hölder continuity. When both x and y lie outside Ω, $F(Q,x)$ and $F(Q,y)$ are both 0, so the estimate of Lemma 4 is trivially valid. However, the situation when $x \in \Omega$ but $y \in \Omega^C$ needs to be considered separately. We are not guaranteed that **b**) is valid when one point, x or y, lies outside the domain Ω. $I.$ and $II.$ must be estimated using **a′**) in this case.

We start with the proof for x and $y \in \eta Q \subset \Omega$. Then by **b**) and Cauchy-Schwarz, (remember that $d \geq 3$),

$$\begin{aligned}I. &= \sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)} |\lambda_J| \left|\varphi_{(J)}(x) - \varphi_{(J)}(y)\right| \\ &\lesssim \sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)} |\lambda_J|\,|x-y|^\alpha\,(\ell(J))^{2-(d/2)-\alpha}\left(1+\frac{|x-x_J|}{\ell(J)}+\frac{|y-x_J|}{\ell(J)}\right)^{2-d-\alpha} \\ &= \sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)} |\lambda_J|\left(\frac{|x-y|}{\ell(J)}\right)^\alpha(\ell(J))^{2-(d/2)}\left(1+\frac{|x-x_J|}{\ell(J)}+\frac{|y-x_J|}{\ell(J)}\right)^{2-d-\alpha} \\ &\lesssim \left(\sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)} \frac{\lambda_J^2}{|J|}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{-d+\varepsilon}\right)^{1/2} \times \\ &\qquad\times\left(\sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)}\left(\frac{|x-y|}{\ell(J)}\right)^{2\alpha}(\ell(J))^4\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{4-d-\varepsilon-2\alpha}\right)^{1/2} \\ &\lesssim G(Q,x)C(d(\Omega))^2\left(\sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)}\left(\frac{|x-y|}{\ell(J)}\right)^{2\alpha}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{-d-\varepsilon-2\alpha}\right)^{1/2}.\end{aligned}$$

The last inequality follows from the fact that $\ell(J)\left(1+\dfrac{|x-x_J|}{\ell(J)}\right) \leq C(\operatorname{diam}(\Omega)) = C(d(\Omega))$. From Lemma 3, $G(Q,x) \leq C(d,\eta)G(Q)$ because $x \in \eta Q$. So we need only show that

$$\left(\sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)} \left(\frac{|x-y|}{\ell(J)}\right)^{2\alpha} \left(1+\frac{|x-x_J|}{\ell(J)}\right)^{-d-\varepsilon-2\alpha} \right) \leq C.$$

The sum on the left can be divided into two parts:

$$\sum_{\substack{J\supsetneq Q\\ J\in\mathcal{S}(Q),\ell(J)\gneqq\ell(Q)}} + \sum_{\substack{J\cap Q=\varnothing\\ J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)}}.$$

It is easy to see that the first sum reduces to a geometric series. For the sum over cubes that lie outside Q, we know that Q lies inside some dyadic cube, J_Q, of side length $2^k\ell(Q)$ for each $k=0,1,2,\ldots$. The other dyadic cubes, call them J, of side length $2^k\ell(Q)$, will lie in cubic annular regions $A_Q^{j,k}$, whose distance from J_Q ranges from 0 to $C(d,\operatorname{diam}(\Omega))\ell(J_Q)$ or $2^{j-1}\ell(J_Q)\leq \operatorname{dist}(J_Q,J) < C2^j\ell(J_Q)$ for $j\geq 1$. This means that $2^{(j-1)}\ell(J)\leq \ell(J)+\operatorname{dist}(J_Q,J)\leq C_2 2^{(j+1)}\ell(J_Q)$ for $j\geq 1$. Now $x\in\eta Q$ implies $\operatorname{dist}(J_Q,J)\lesssim |x-x_J|\sim|x_Q-x_J|$, with constants depending on the dimension d. We can say that if $J\subset A_Q^{j,k}$, then $2^{(j-1)}\ell(J)\lesssim \ell(J)+|x-x_J|\leq C2^{(j+1)}\ell(J)$ where $C=C(d,\operatorname{diam}(\Omega))>1$.

Since x and y lie in ηQ, the second sum can be written as

$$\begin{aligned}
&\sum_{k=0}^{\infty} \sum_{\substack{\ell(J)=2^k\ell(Q)\\ J\in\mathcal{S}(Q)}} \left(\frac{\ell(Q)}{\ell(J)}\right)^{2\alpha}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{-d-\varepsilon-2\alpha}\\
&\quad=\sum_{k=0}^{\infty}2^{-2\alpha k}\left\{C+\sum_{j=1}\sum_{\substack{J\subset A_Q^{j,k}\\ \ell(J)=2^k\ell(Q),J\in\mathcal{S}(Q)}}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{-d-\varepsilon-2\alpha}\right\}\\
&\quad\lesssim C+\sum_{k=0}^{\infty}2^{-2\alpha k}\sum_{j=0}\left(\sum_{\substack{J\subset A_Q^{j,k}\\ \ell(J)=2^k\ell(Q),J\in\mathcal{S}(Q)}}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{-d-\varepsilon-2\alpha}\right)\\
&\quad\lesssim C+C(d)\sum_{k=0}^{\infty}2^{-2\alpha k}\sum_{j=k}^{\infty}2^{d(j-k)}\left(1+\frac{|x_Q-x_{J_0}|}{\ell(J_0)}\right)^{-d-\varepsilon-2\alpha}.
\end{aligned}$$

J_0 is a cube of size $2^k\ell(Q)$. The last estimate follows from counting the number of cubes J of side length $2^k\ell(Q)$ that can exist in the annular region $A_Q^{j,k}$ if $j\geq k$. It is easy to see that the last sum is bounded by

$$\begin{aligned}
C(d)\sum_{k=0}^{\infty}2^{-2\alpha k}\sum_{j=k}^{\infty}2^{d(j-k)}\left(2^{j-k-1}\right)^{-(d+\varepsilon+2\alpha)} &\leq C(d,\alpha,\varepsilon)\sum_{k=0}^{\infty}2^{\varepsilon k}\sum_{j=k}^{\infty}2^{-(\varepsilon+2\alpha)j}\\
&\leq C(d,\alpha,\varepsilon)\sum_{k=0}^{\infty}2^{-2\alpha k}\sum_{j=0}^{\infty}2^{-(2\alpha+\varepsilon)j}\leq C(d,\alpha,\varepsilon).
\end{aligned}$$

Now to bound II, still keeping x, $y \in \eta Q$ inside Ω, we have

$$II. \le \sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)} |\lambda_J| \left(\left|\varphi_{(J)}(x) - \varphi_{(J)}(y)\right|\right),$$

so by Lemma 3 it is enough to show this sum is $\le CG(Q,x)$. Using **b**) gives $\sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)} |\lambda_J| \left|\varphi_{(J)}(x) - \varphi_{(J)}(y)\right| \lesssim$

$$(\mathbf{A}) \quad \sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)} |\lambda_J|\,|x-y|^{\alpha}\,(\ell(J))^{2-d/2-\alpha}\left(1+\frac{|x-x_J|}{(\ell(J))}+\frac{|y-x_J|}{(\ell(J))}\right)^{2-d-\alpha}$$

$$\lesssim \left(\sum_{J\in\mathcal{S}(Q)} \frac{\lambda_J^2}{|J|}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{-d+\varepsilon}\right)^{1/2}\times$$

$$\times\left\{\sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)} (\ell(Q))^{2\alpha}\,(\ell(J))^{4-2\alpha}\left(1+\frac{|x-x_J|}{(\ell(J))}\right)^{4-d-\varepsilon-2\alpha}\right\}^{1/2}$$

$$= CG(Q,x)\left\{\sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)} \left(\frac{\ell(Q)}{\ell(J)}\right)^{2\alpha}(\ell(J))^{4}\left(1+\frac{|x-x_J|}{(\ell(J))}\right)^{4-d-\varepsilon-2\alpha}\right\}^{1/2}$$

$$\le CG(Q)\cdot H_{(Q)}(x).$$

Now, (remember that $\ell(J)\left(1+\frac{|x-x_J|}{\ell(J)}\right) \le C\,\mathrm{diam}\,(\Omega)$),

$$H_{(Q)}(x) \le C(\,\mathrm{diam}\,\Omega)^2\left(\sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)} \left(\frac{\ell(Q)}{\ell(J)}\right)^{2\alpha}\left(1+\frac{|x-x_J|}{(\ell(J))}\right)^{-d-\varepsilon-2\alpha}\right)^{1/2}$$

$$\le C\left(\sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)} \left(\frac{\ell(Q)}{\ell(J)}\right)^{-d-\varepsilon-2\alpha+2\alpha}\left(1+\frac{|x_Q-x_J|}{(\ell(Q))}\right)^{-d-\varepsilon-2\alpha}\right)^{1/2}$$

The last inequality follows from the fact that for

$$J\cap Q=\varnothing,\ J\in\mathcal{S}(Q),\ x\in\eta Q,\ \frac{|x-x_J|}{|x_Q-x_J|}\sim C$$

and, since $|x_Q-x_J| \gtrsim \ell(Q)$, we have

$$1+\frac{|x-x_J|}{\ell(J)} \gtrsim \frac{\ell(Q)}{\ell(J)}\frac{|x_Q-x_J|}{\ell(Q)} \gtrsim \frac{\ell(Q)}{\ell(J)}\left(1+\frac{|x_Q-x_J|}{\ell(Q)}\right).$$

This means that

$$\left(1+\frac{|x-x_J|}{(\ell(J))}\right)^{-d-\varepsilon-2\alpha} \lesssim \left(\frac{\ell(Q)}{\ell(J)}\left(1+\frac{|x_Q-x_J|}{\ell(Q)}\right)\right)^{-d-\varepsilon-2\alpha}.$$

Finally, to estimate

$$\left(\sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)}\left(\frac{\ell(Q)}{\ell(J)}\right)^{-d-\varepsilon}\left(1+\frac{|x_Q-x_J|}{(\ell(Q))}\right)^{-d-\varepsilon-2\alpha}\right)^{1/2}$$

we can proceed as in [**11**] to divide $Q_0\backslash Q$ into dyadic cubes Q' whose size is the same as that of Q. We write the sum as

$$\left(\sum_{Q'\subset Q_0\backslash Q}\sum_{J\in\mathcal{S}(Q),J\subset Q'}\left(\frac{\ell(J)}{\ell(Q)}\right)^{d+\varepsilon}\left(1+\frac{|x_Q-x_J|}{(\ell(Q))}\right)^{-d-\varepsilon-2\alpha}\right)^{1/2}$$
$$\lesssim\left(\sum_{Q'\subset Q_0\backslash Q}\sum_{J\in\mathcal{S}(Q),J\subset Q'}\left(\frac{\ell(J)}{\ell(Q)}\right)^{d+\varepsilon}\left(1+\frac{|x_Q-x_{Q'}|}{(\ell(Q'))}\right)^{-d-\varepsilon-2\alpha}\right)^{1/2},$$

which is valid since $|x_Q-x_J|\gtrsim|x_Q-x_{Q'}|$. Now the J are Whitney cubes from $\mathcal{F}$, so they are disjoint. Consequently, for each Q',

$$\sum_{J\subset Q'}\left(\frac{\ell(J)}{\ell(Q)}\right)^{d+\varepsilon}\le\sum_{J\subset Q'}\frac{|J|}{|Q'|}\le 1.$$

Therefore we can write

$$H_Q^2(x)\lesssim\sum_{k=0}^{\infty}\sum_{\substack{Q'\subset Q_0\backslash Q\\ 2^{k-1}\ell(Q)\lesssim|x_Q-x_{Q'}|<C(d)2^k\ell(Q)}}\left(1+\frac{|x_Q-x_{Q'}|}{(\ell(Q'))}\right)^{-d-\varepsilon-2\alpha}$$
$$\lesssim\sum_{k=0}^{\infty}2^{kd}2^{-k(d+\varepsilon+2\alpha)}\le C(\alpha,d,\varepsilon),$$

by counting the maximum number of cubes Q' that can lie inside the annular region $2^{k-1}\ell(Q)\lesssim|x_Q-x_{Q'}|\lesssim 2^k\ell(Q)$.

We have shown that $|F(Q,x)-F(Q,y)|\le C_2G(Q)$, when both x and y lie inside Ω, or when both points lie in $Q_0\backslash\Omega$. The remaining case is for one point lying inside Ω and the other point lying outside Ω. This implies of course that the dyadic cube Q is such that $\eta Q\cap\Omega\neq\varnothing$ and $\eta Q\cap\Omega^C\neq\varnothing$. Without loss of generality $x\in\Omega$, and $y\notin\Omega$. So $F(Q,y)=0$. Here we cannot use **b**), since the decay in **b**) is not necessarily valid for points outside Ω. However, we note that **a′**) is useful. Since Q overlaps the boundary of Ω, and $x\in Q\cap\Omega$, we have that $\delta(x)=$ distance$(x,\partial\Omega)\lesssim\ell(Q)$. So $|F(Q,x)-F(Q,y)|=|F(Q,x)|\le$

$$\sum_{J\in\mathcal{S}(Q),\ell(J)\ge\ell(Q)}|\lambda_J|\,|\varphi_J(x)|+\sum_{J\in\mathcal{S}(Q),\ell(J)<\ell(Q)}|\lambda_J|\,|\varphi_J(x)|=I'+II'.$$

Now,

$$(I')\lesssim\sum_{J\in\mathcal{S}(Q),\ell(J)\ge\ell(Q)}|\lambda_J|\,\delta(x)^{\alpha}\ell(J)^{2-d/2-\alpha}\left(1+\frac{|x-x_J|}{(\ell(J))}\right)^{2-d-\alpha}$$

from using $\mathbf{a}'$) on the functions $|\varphi_J(x)|$ in I'. The last sum is bounded by

$$C(\operatorname{diam}(\Omega)^2) \sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)} |\lambda_J| \left(\frac{\ell(Q)}{\ell(J)}\right)^{\alpha} \ell(J)^{-d/2} \left(1+\frac{|x-x_J|}{(\ell(J))}\right)^{-d-\alpha}$$

$$\leq CG(Q)\left(\sum_{J\in\mathcal{S}(Q),\ell(J)\geq\ell(Q)} \left(\frac{\ell(Q)}{\ell(J)}\right)^{2\alpha} \left(1+\frac{|x-x_J|}{(\ell(J))}\right)^{-d-\varepsilon-2\alpha}\right)^{1/2}.$$

Dominating the last sum by a constant follows as before. Estimating II' follows from almost the same proof that gave the bound for II in the first case, in which x and y were both located inside Ω. Here the fact that $\delta(x) \lesssim \ell(Q)$ replaces the similar estimate for $|x-y|$ in ($\mathbf{A}$). After that the calculations are identical.

For the next four Lemmas we define $N(I) = \{I^* \in \mathfrak{D} : I^* \subset I \text{ and } \ell(I^*) = 0.5\,\ell(I)\}$ for any dyadic cube $I \in \mathfrak{D}$. We have

Lemma 5. $G(I) \leq CG(I^*)$.

Proof. $x_{I^*} \in \eta I$ if $0 < \eta < 1$ is sufficiently large, depending on d. By Lemma 3, $G(I) \leq CG(I, x_{I^*})$, and by definition $G(I, x_{I^*}) \leq G(I^*, x_{I^*}) = G(I^*)$.

Lemma 6. *For $I^* \in N(I)$, $G(I^*) \leq CG^*(x)$ whenever $x \in I$.*

Proof. By definition $G^*(x) = \sup_{J\ni x} G(J)$, so if $x \in I^*$, $G(I^*) \leq G^*(x)$. Suppose that x lies in $I \setminus I^*$. For any $J \subset I \setminus I^*$ such that $x \in J$, then $G(I^*)^2 \leq CG(J)^2 + B$, where

$$B = \sum_{K\subset J, K\in\mathcal{S}(I^*)} \frac{\lambda_K^2}{|K|} \left(1+\frac{|x_{I^*}-x_K|}{\ell(K)}\right)^{-d+\varepsilon}.$$

All the terms in B occur in $G(I^*)^2$. If $L \in \mathcal{S}(I^*)\backslash\{K \subset J, K \in \mathcal{S}(I^*)\}$, then $L \in \mathcal{S}(J)$. We also have $|x_L - x_J| \leq |x_L - x_{I^*}| + |x_J - x_{I^*}| \leq |x_L - x_{I^*}| + c\ell(I) \leq c'\,|x_L - x_{I^*}|$ since $|x_L - x_{I^*}| \gtrsim \ell(I)$. We may assume $c' \geq 1$; this implies that

$$\left(1+\frac{|x_L-x_J|}{\ell(L)}\right)^{-d+\varepsilon} \geq C'\left(1+\frac{|x_L-x_{I^*}|}{\ell(L)}\right)^{-d+\varepsilon}.$$

So each term in $G(I^*)$ that does not occur in B is less than or equal to a constant times a term that occurs in $G(J)$. Now $\mathcal{F}$ is a finite family, so for $|J|$ sufficiently small, the sum in B will be empty, and $G(J) \leq G^*(x)$.

Lemma 7. $|F(I^*) - F(I)| \leq CG(I^*)$.

Proof. Lemmas 4 and 5 imply that $|F(I, x_{I^*}) - F(I)| \leq CG(I) \leq C'G(I^*)$; consequently it is enough to show that $|F(I, x_{I^*}) - F(I^*)| \leq CG(I^*)$. If $x_{I^*} \in \Omega^C$, then both functions on the left are zero, so we can assume that $x_{I^*} \in \Omega$. We have

$$|F(I, x_{I^*}) - F(I^*)| = \left|\sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} \lambda_J\varphi_J(x_{I^*})\right|$$

$$\leq \sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} |\lambda_J|\,\ell(J)^{2-d/2}\left(1+\frac{|x_{I^*}-x_J|}{\ell(J)}\right)^{2-d}$$

from **a**). Cauchy-Schwarz gives

$$|F(I,x_{I^*})-F(I^*)| \le \left(\sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} \frac{|\lambda_J|^2}{|J|}\left(1+\frac{|x_{I^*}-x_J|}{\ell(J)}\right)^{-d+\varepsilon}\right)^{1/2} \times$$

$$\times\left(\sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} \ell(J)^4\left(1+\frac{|x_{I^*}-x_J|}{\ell(J)}\right)^{4-d-\varepsilon}\right)^{1/2}$$

$$\le CG(I^*)\cdot C(d(\Omega))^2\left(\sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} \left(1+\frac{|x_{I^*}-x_J|}{\ell(J)}\right)^{-d-\varepsilon}\right)^{1/2}.$$

If we can show that $\left(\sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} \left(1+\frac{|x_{I^*}-x_J|}{\ell(J)}\right)^{-d-\varepsilon}\right)^{1/2}$ is bounded by a constant, we will be done. Notice that

$$1+\frac{|x_{I^*}-x_J|}{\ell(J)} \ge \frac{|x_{I^*}-x_J|}{\ell(J)} = \frac{\ell(I)}{\ell(J)}\cdot\frac{|x_{I^*}-x_J|}{\ell(I)},$$

and $|x_{I^*}-x_J| \sim \ell(I)$ because $J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)$. So

$$\frac{\ell(I)}{\ell(J)}\cdot\frac{|x_{I^*}-x_J|}{\ell(I)} \ge C\,\frac{\ell(I)}{\ell(J)}\left(1+\frac{|x_{I^*}-x_J|}{\ell(I)}\right).$$

We have

$$\left(1+\frac{|x_{I^*}-x_J|}{\ell(J)}\right)^{-d-\varepsilon} \le C\left(\frac{\ell(J)}{\ell(I)}\right)^{d+\varepsilon}\left(1+\frac{|x_{I^*}-x_J|}{\ell(I)}\right)^{-d-\varepsilon}.$$

This gives

$$\sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} \left(1+\frac{|x_{I^*}-x_J|}{\ell(J)}\right)^{-d-\varepsilon}$$

$$\le C\sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} \left(\frac{\ell(J)}{\ell(I)}\right)^{d+\varepsilon}\left(1+\frac{|x_{I^*}-x_J|}{\ell(I)}\right)^{-d-\varepsilon}.$$

Now remember that the cubes J originally came from $\mathcal{F}$ so they are disjoint. Also $J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)$ means that either $J=I$ or $J\subset I\setminus I^*$. As a result

$$\sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} \left(\frac{\ell(J)}{\ell(I)}\right)^{d+\varepsilon}\left(1+\frac{|x_{I^*}-x_J|}{\ell(I)}\right)^{-d-\varepsilon} \le \sum_{J\in\mathcal{S}(I^*)\backslash\mathcal{S}(I)} \left(\frac{\ell(J)}{\ell(I)}\right)^{d} \le 2.$$

The purpose of establishing Lemmas 1–7 is to prove Lemma 8, the Central Lemma.

3. THE GOOD-λ INEQUALITIES

Lemma 8 (Central Lemma). *Let* $f(x) = \sum_{I\in\mathcal{F}} \lambda_I \varphi_{(I)}(x)$, *where* $\mathcal{F}$ *is a finite family of cubes from* $\mathcal{W}$, *the* $\varphi_{(I)}$ *satisfy* **a**), **a′**), **b**) *and* **c**), *and* $\lambda_I = 0$ *for any* $I \subsetneqq I_0$. I_0 *is a fixed cube in* $\mathfrak{D}$. *For any* β, $0 < \beta < 1$, *there is a* $\gamma = \gamma(\beta, d, \lambda, \alpha, \Omega, \eta)$ *such that*

$$|\{x \in I_0 : F^*(x) > 1 \text{ and } G^*(x) \le \gamma\}| \le \beta\, |I_0|\,.$$

Proof. Let I_j be the dyadic cubes for which one of the subcubes, $I_j^* \in N(I_j)$, is a maximal dyadic cube in I_0 so that $G(I_j^*) > A\gamma$, for A large enough so $AC^{-1} > 1$, C being the constant in Lemma 6. Notice that $G(I_0) = 0$ (and so is $F(I_0, x)$ for any $x \in I_0$), and $I_j \subseteq I_0$. We have that $G(I_j) \le A\gamma$, $x \in I_j$ implies that $G^*(x) > AC^{-1}\gamma > \gamma$ from Lemma 6, and $G^*(x) \le A\gamma$ for almost all $x \in I_0 \setminus \cup I_j$.

Let $E = \{x \in I_0 : F^*(x) > 1 \text{ and } G^*(x) \le \gamma\}$. For any $x \in E$ there is a maximal dyadic cube Q_i such that $F(Q_i) > 1$. $Q_i \subset I_0$ and $Q_i \subsetneqq I_j$ for any of the maximal cubes defined in the previous paragraph, because $G^*(x) \le \gamma$ means that x can't lie in I_j. Following the argument in the proof of the Main Lemma in [**11**], we create the family of dyadic cubes $\mathcal{G}=\{P_k\}$ which consists of the maximal disjoint cubes that result from combining the I_j and the Q_i. So $E \subset \cup_k P_k$. In fact $x \in E$ implies that $x \in P_{k'}$ for some maximal cube in $\mathcal{G}$ for which $F(P_{k'}) > 1$. It is also true that $G(P_{k'}) \le \gamma$, since $G^*(x) \le \gamma$. We proceed to divide the cubes in $\mathcal{F}$ into two sets, $\mathcal{F}_1 = \{J : J \subsetneqq P_k \text{ for any } P_k \in \mathcal{G})\}$ and $\mathcal{F}_2 = \{J : J \subseteqq P_k \text{ for some } P_k \in \mathcal{G})\}$. Writing $f(x) = \sum_{J\in\mathcal{F}_1} \lambda_I \varphi_{(I)}(x) + \sum_{J\in\mathcal{F}_2} \lambda_I \varphi_{(I)}(x) = f_1(x) + f_2(x)$, we can define $F_i(Q,x)$, $F_i(Q), F_i^*(x), G_i(Q,x), G_i(Q)$,and $G_i^*(x)$ for $i = 1,2$ just as we did for $f(x)$. $F(Q,x) = F_1(Q,x) + F_2(Q,x)$, while $G_i(Q,x) \le G(Q,x) \le G_1(Q,x) + G_2(Q,x) \le CG(Q,x)$.

The facts that $E \subset \cup P_{k'}$ and that Lebesgue measure is a doubling measure mean $|E| \le C(d) \sum_{k'} |c(P_{k'})|$, where $c(P_k) = \left\{x \in P_k : x \in \frac{1}{10} P_k\right\}$. For $x \in P_{k'}$, we must have either $F_1(P_{k'}) > 0.5$ or $F_2(P_{k'}) > 0.5$. For $x \in c(P_{k'})$, Lemma 4 says that either $F_1(P_{k'}, x) > 0.25$ or $F_2(P_{k'}, x) > 0.25$ whenever γ is small enough. Also

$$\begin{aligned}\sum_{k'} |c(P_{k'})| &\le \sum_{F_1(P_{k'})>0.5} |c(P_{k'})| + \sum_{F_2(P_{k'})>0.5} |c(P_{k'})| \\ &\le \sum_{k'} |\{x \in c(P_{k'}) : F_1(P_{k'}, x) > 0.25\}| + \sum_{k'} |\{x \in c(P_{k'}) : F_2(P_{k'}, x) > 0.25\}|\,.\end{aligned}$$

Using Chebyshev's inequality we can see we only need to estimate

$$\sum_{k'} 16 \int_{c(P_{k'})} |F_1(P_{k'}, x)|^2 \,\mathrm{d}x \text{ and } \sum_{k'} 16 \int_{c(P_{k'})} |F_2(P_{k'}, x)|^2 \,\mathrm{d}x.$$

In fact, for the second sum we will estimate each integral taken over a smaller set than $c(P_{k'})$. This will be explained after we obtain a bound for the first sum.

Notice that the definition of $\mathcal{F}_1$ gives that $F_1(P_{k'},x) = f_1(x)$ for any $x \in P_{k'}$. Then $\sum_{k'} 16 \int_{c(P_{k'})} |F_1(P_{k'},x)|^2 \,\mathrm{d}x = \sum_{k'} 16 \int_{c(P_{k'})} |f_1(x)|^2 \,\mathrm{d}x \le C \int_{I_0} |f_1(x)|^2 \,\mathrm{d}x$. By the almost orthogonality property **c**) for the $\varphi_{(I)}$'s,

$$\int_{I_0} |f_1(x)|^2 \,\mathrm{d}x \le \sum_{J\in\mathcal{F}_1} \lambda_J^2 = \int_{I_0} \sum_{\substack{J\in\mathcal{F}_1\\ J\ni x}} \frac{\lambda_J^2}{|J|} \mathrm{d}x \le C(A\gamma)^2 |I_0| \le \frac{\beta}{3} |I_0|$$

for γ sufficiently small. The second to the last estimate follows from the fact that for $x \in I_0 \setminus \cup I_j$, ($I_j$ are the maximal cubes defined in the beginning of the proof), $\sum_{J\ni x} \lambda_J^2 / |J| \le CG^*(x)^2 \le (A\gamma)^2$, and for $x \in I_j$, $\sum_{J\ni x, J\in\mathcal{F}_1}$ is empty.

Next we bound $\sum_{k'} |\{x \in c(P_{k'}) : F_2(P_{k'},x) > 0.25\}|$. As in [**11**] we cut out a thin annular region around each of the $P_{k'}$'s to handle edge effects. Choosing $\tau > 1$, so that $|\tau P_{k'} \setminus P_{k'}| \le \frac{\beta}{3} |P_{k'}|$, and letting $D = \cup\{\tau P_{k'} \setminus P_{k'}\}$, then $|D| \le \frac{\beta}{3} |I_0|$ (remember the $P_{k'}$ are disjoint). Also

$$\sum_{k'} |\{x \in c(P_{k'}) : F_2(P_{k'},x) > 0.25\}| \le |D| + \sum_{k'} 16 \int_{c(P_{k'})\setminus D} |F_2(P_{k'},x)|^2 \,\mathrm{d}x.$$

We need only prove that $\sum_{k'} 16 \int_{c(P_{k'})\setminus D} |F_2(P_{k'},x)|^2 \,\mathrm{d}x \le C'(A\gamma)^2 |I_0|$, and take γ small enough so that $C'(A\gamma)^2 \le \frac{\beta}{3}$.

If k' is temporarily fixed and $x \in c(P_{k'}) \setminus D$, then

$$F_2(P_{k'},x) = \sum_{J\in\mathcal{F}_2, J\nsubseteq P_{k'}} \lambda_J \varphi_{(J)}(x),$$

so

$$\begin{aligned} |F_2(P_{k'},x)|^2 &\le \Big| \sum_{J\in\mathcal{F}_2, J\nsubseteq P_{k'}} \lambda_J \varphi_{(J)}(x) \Big|^2 \\ &\le \left(\sum_{J\in\mathcal{F}_2, J\nsubseteq P_{k'}} |\lambda_J|\, \ell(J)^{2-d/2} \left(1 + \frac{|x-x_J|}{\ell(J)}\right)^{2-d} \right)^2 \end{aligned}$$

by **a**). Again, Cauchy-Schwarz gives

$$\begin{aligned} |F_2(P_{k'},x)|^2 &\le \left(\sum_{J\subseteq P_j, j\ne k'} \frac{\lambda_J^2}{|J|} \left(1 + \frac{|x-x_J|}{\ell(J)}\right)^{-d+\varepsilon} \right) \times \\ &\qquad \left(\sum_{J\subseteq P_j, j\ne k'} \ell(J)^4 \left(1 + \frac{|x-x_J|}{\ell(J)}\right)^{4-d-\varepsilon} \right) \\ &\le CG(P_{k'})^2\, C(\operatorname{diam}(\Omega)^4) \left(\sum_{J\subseteq P_j, j\ne k'} \left(1 + \frac{|x-x_J|}{\ell(J)}\right)^{-d-\varepsilon} \right). \end{aligned}$$

To bound the last sum by a constant we note that $|x - x_J| \geq C \left|x - x_{P_j}\right|$ whenever $x \in c(P_{k'}) \setminus D$ and $J \subseteq P_j$, $j \neq k'$. So as above

$$1 + \frac{|x - x_J|}{\ell(J)} \geq \frac{|x - x_J|}{\ell(J)} \geq C\,\frac{\left|x - x_{P_j}\right|}{\ell(P_j)} \cdot \frac{\ell(P_j)}{\ell(J)} \geq C'\,\frac{\ell(P_j)}{\ell(J)} \left(1 + \frac{\left|x - x_{P_j}\right|}{\ell(P_j)}\right),$$

since also $\left|x - x_{P_j}\right| \geq C''\ell(P_j)$. We have

$$\left(\sum_{J\subset P_j, j\neq k'} \left(1 + \frac{|x - x_J|}{\ell(J)}\right)^{-d-\varepsilon}\right) \leq C \sum_{J\subset P_j, j\neq k'} \left(\frac{\ell(P_j)}{\ell(J)}\right)^{-d-\varepsilon} \left(1+\frac{\left|x - x_{P_j}\right|}{\ell(P_j)}\right)^{-d-\varepsilon}.$$

This means that

$$\begin{aligned}
&\sum_{k'} \int_{c(P_{k'})\setminus D} |F_2(P_{k'}, x)|^2 \,\mathrm{d}x \\
&\quad\leq C \sum_{k'} G(P_{k'})^2 \int_{c(P_{k'})\setminus D} \sum_{j\neq k'} \sum_{J\subset P_j} \frac{|J|}{|P_j|} \left(1 + \frac{\left|x - x_{P_j}\right|}{\ell(P_j)}\right)^{-d-\varepsilon} \mathrm{d}x \\
&\quad\leq C(A\gamma)^2 \int_{I_0} \sum_j \left(1 + \frac{\left|x - x_{P_j}\right|}{\ell(P_j)}\right)^{-d-\varepsilon} \mathrm{d}x \\
&\quad\leq C(A\gamma)^2 \sum_j \int_{I_0} \left(1 + \frac{\left|x - x_{P_j}\right|}{\ell(P_j)}\right)^{-d-\varepsilon} \ell(P_j)^d \mathrm{d}\left(\frac{(x - x_{P_j})}{\ell(P_j)}\right) \\
&\quad\leq C(A\gamma)^2 \sum_j |P_j| \int \frac{r^{d-1}}{(1+r)^{d+\varepsilon}}\,\mathrm{d}r\,d\omega_{d-1} \leq C(A\gamma)^2\,|I_0|
\end{aligned}$$

using polar coordinates and the fact that the P_j's are disjoint in I_0. $C = C(\tau, \varepsilon, \alpha, d, \lambda, \eta, \Omega)$. The Central Lemma is proved.

Corollary. *Suppose $\sigma \in A^\infty(Q_0, dx)$ and $f(x) = \sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)$ with $\mathcal{F}$ a finite family of cubes from $\mathcal{W}$ and the $\varphi_{(J)}$ satisfying* **a**), **a′**), **b**) *and* **c**). *Then for any $\beta > 0$ there exists a $\gamma = \gamma(d, \lambda, \varepsilon, \Omega, \alpha, \beta)$ so that, for every $\xi > 0$,*

$$\sigma\left(\{x \in Q_0 : F^*(x) > 2\xi, G^*(x) \leq \gamma\xi\}\right) \leq \beta\sigma(\{x \in Q_0 : F^*(x) > \xi\}).$$

Proof. Let $\{I_j\}$ be the maximal dyadic cubes in Q_0 such that $F(I_j) > \xi$. We need only show that

$$|\{x \in I_j : F^*(x) > 2\xi, G^*(x) \leq \gamma\xi\}| \leq \widehat{\beta}\,|\{x \in I_j : F^*(x) > \xi\}|$$

for some $\widehat{\beta}$, determined by β and the constants in the condition that $\sigma \in A^\infty(Q_0, dx)$. Notice that $\{x \in Q_0 : F^*(x) > \xi\} = \cup I_j$. Once again we cut out a small annular region for each cube I_j, but here the region lies inside I_j. We take $\varepsilon > 0$ so small that $\left|\left\{x \in I_j : \text{distance}(x, I_j^C) \leq \varepsilon\right\}\right| \leq (\widehat{\beta}/3)\,|I_j|$. For $x \in (1 - \varepsilon)I_j$ we have $|F(I_j) - F(I_j,x)| \le CG(I_j)$, by Lemma 4. It is also true that for $\widehat{I_j} \supset I_j$, $\ell(I_j) = 0.5\,\ell(\widehat{I_j})$, Lemma 7 implies that $\left|F(\widehat{I_j}) - F(I_j)\right| \le C'G(I_j)$. By maximality $F(\widehat{I_j}) \le \xi$. We also have

$$\begin{aligned} E_j &= \{x \in I_j : F^*(x) > 2\xi, G^*(x) \le \gamma\xi\} \\ &\subseteq \{x \in (1-\varepsilon)I_j : F^*(x) > 2\xi, G^*(x) \le \gamma\xi\} \cup \left\{x \in I_j : \text{distance}(x, I_j^C) \le \varepsilon\right\}. \end{aligned}$$

For any I_j such that $E_j \neq \varnothing$, then $G(I_j) \le \gamma\xi$. From the previous calculations we have, for any $x \in \eta I_j$, that $|F(I_j,x)| \le F(\widehat{I_j}) + cG(I_j)$. So if γ is small enough $|F(I_j,x)| \le 1.1\xi$. Writing

$$f(x) = \sum_{J \not\subseteq I_j, J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x) + \sum_{J\subset I_j, J\in\mathcal{F}} \lambda_J\varphi_{(J)}(x) = F(I_j,x) + h(x),$$

and taking $H(I,x) = \sum_{J\in\mathcal{S}(I)\cap\mathcal{F}, J\subset I_j} \lambda_J\varphi_{(J)}(x)$, $H^*(x) = \sup_{I\ni x} H(I,x_I)$, then $F^*(x) - 1.2\xi \le H^*(x)$, This happens since I_j is maximal so that $F(I_j) > \xi$; consequently any dyadic cube $Q \ni x$ such that $F(Q) > 2\xi$ must be contained in the I_j that contains x. Setting $F_j(x) = F(I_j,x)$, we have $\sup_{\substack{J\ni x\\ J\subset I_j}} F_j(J,x_J) = \sup_{\substack{J\ni x\\ J\subset I_j}} F_j(x_J)$. Also, $x \in (1-\varepsilon)I_j$ means that for any dyadic $J \subset I_j$ such that $x\in J$, $\text{distance}(x_J, I_j^C) \ge \frac{\varepsilon}{2}\,\ell(I_j)$. Taking $\eta = (1-\varepsilon/2)$, we have $F_j(x_J) \le 1.2\xi$, for any such J, so

$$\begin{aligned} &\{x \in (1-\varepsilon)I_j : F^*(x) > 2\xi, G^*(x) \le \gamma\xi\} \\ &\qquad \subseteq \{x\in(1-\varepsilon)I_j : H^*(x) > 0.8\xi, G^*(x) \le \gamma\xi\}. \end{aligned}$$

After rescaling, the Central Lemma can be applied to the function $h(x)$.

The full result of Theorem A follows from the Corollary by a standard argument because Ω is bounded and, $f(x)$ being a finite sum, means that $F^* \in L^p(\Omega, d\sigma)$. To prove Theorem B for infinite sums we can use Fatou's Lemma on $|f_n(x)|^p$, for $f_n(x) = \sum_{\substack{J\in\mathcal{F}\\ \ell(J)\ge(1/n)}} \lambda_J\varphi_{(J)}(x)$, taking $\mathcal{F}$ to be an infinite family of dyadic cubes from $\mathcal{W}$.

4. AN APPLICATION

In this section questions concerning the rate of change of solutions to the inhomogeneous equation $Lu = \operatorname{div}\overrightarrow{f}$ in Ω, $u\,|_{\partial\Omega} = 0$ will be stated for operators of the form $L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{i,j}(x)\frac{\partial}{\partial x_j}\Big)$, with L being symmetric and strictly elliptic, i.e. there exists a positive constant λ such that $\frac{1}{\lambda}|\xi|^2 \le \sum_{i,j=1}^{d} \xi_i a_{i,j}(x)\xi_j \le \lambda|\xi|^2$, and $a_{i,j}(x) = a_{j,i}(x)$ for all $x \in \Omega$ and for $i = 1,\dots,d, j = 1,\dots,d$. Ω is a bounded domain in $\mathbb{R}^d$ whose boundary satisfies an exterior cone condition. In [**6**] the question of finding conditions on two measures μ and ν, defined on Ω, to give the inequality

$$\left(\int_\Omega (|\nabla u(x)|)^q \,\mathrm{d}\mu(x)\right)^{1/q} \le C\left(\int_\Omega \left(\left|\overrightarrow{f}(x)\right|^p + \left|\operatorname{div}\overrightarrow{f}(x)\right|^p\right)\mathrm{d}\nu(x)\right)^{1/p},$$

was considered. Also it was shown that a condition involving a singular potential of the measure μ gives the same kind of norm inequality for a local Hölder norm of the solution $u(x)$ instead of $|\nabla u(x)|$. Here, we can use Theorem A to prove that a condition on the cubes in $\mathcal{W}$ gives a better result for the local Hölder norm, $\|u\|_{H_\alpha}$, of the solution $u(x)$ instead of $|\nabla u(x)|$. Recall that $\|u\|_{H_\alpha}(x) = \sup\limits_{0<|x-y|<\delta(x)/50} \dfrac{|u(x)-u(y)|}{|x-y|^\alpha}$.

In Theorem B $\mathcal{W}$ and Q_0 are as defined above. The measures μ and ν will be taken to be Borel measures; μ is defined on Ω, with ν defined on Q_0, absolutely continuous with Lebesgue measure.

Next we define $M(Q_j)$ for any dyadic cube Q_j, so that $4Q_j$ lies inside Ω, and for $\mathrm{d}\sigma(y) = \left(\dfrac{\mathrm{d}\nu}{\mathrm{d}y}(y)\right)^{1-p'} \mathrm{d}y$,

$$M(Q_j) = \max\left\{\left(\frac{1}{|Q_j|}\int_{4Q_j}\left(\frac{\mathrm{d}\nu}{\mathrm{d}x}(x)\right)^{s/(s-p)}\mathrm{d}x\right)^{1/s-1/p}\ell(Q_j)^{d/p'+1};\right.$$
$$\left.\left(\int_{Q_0}\left(1+\frac{|y-x_{Q_j}|}{\ell(Q_j)}\right)^{-(d-\varepsilon)p'/2}\mathrm{d}\sigma(y)\right)^{1/p'}\right\}.$$

Theorem B. *For* $3 \le d < s < p \le q < \infty$, *if for any* $Q_j \in \mathcal{W}$,

$$\mu(Q_j)^{1/q} M(Q_j)\ell(Q_j)^{-d-\alpha} \le C_0 \tag{1}$$

then for any u, *a solution to* $Lu = \operatorname{div}\overrightarrow{f}$ *in* Ω, $u\,|_{\partial\Omega} = 0$, *there is a constant* C *independent of* u, $\overrightarrow{f}$, μ *and* ν *so that*

$$\left(\int_\Omega (\|u\|_{H^\alpha}(x))^q \,\mathrm{d}\mu(x)\right)^{1/q} \le C\left(\int_\Omega \left(\left|\overrightarrow{f}(x)\right|^p + \left|\operatorname{div}\overrightarrow{f}(x)\right|^p\right)\mathrm{d}\nu(x)\right)^{1/p}. \tag{2}$$

Remark. By allowing C to depend on $\mu(\Omega)$, on $\nu(\Omega)$ and on q_0 and p_0, the range of p and of q can be extended to $0 < q \le q_0$ and $d < s < p_0 \le p < \infty$ for any fixed pair of indices p_0 and q_0 having $s < p_0 \le q_0$. This follows from using Holder's inequality on both integrals in (2).

The proof of Theorem B follows the same general outline initiated in [**12**], (see also [**5**], [**6**], [**9**], and [**10**]), namely one employs a dual operator argument which depends on a Littlewood-Paley type inequality, Theorem A in this situation.

Theorem A is applied to the family of functions defined in the appendix, which also contains the proof that these functions satisfy the necessary decay, smoothness and cancellation conditions.

5. APPENDIX

In Lemma 9 we establish the fact that the family of functions $\varphi_{(J)}(x)$,

$$\varphi_{(J)}(x) = \frac{C}{\sqrt{|J|}} \int_{3J/2} \Big(G(x,y) - \widetilde{G}(x,y) \Big) \, \mathrm{d}y$$

satisfy the conditions **a**), **a′**), **b**) and **c**). Here $G(x,y)$, respectively $\widetilde{G}(x,y)$, is the Green function for L on Ω, respectively $4J$, and is zero outside the domain of definition. Among other results, Lemma 9 justifies the validity of applying Theorem A to obtain the application given in Theorem B of the last section.

Lemma 9. *Given* $\varphi_{(J)}(x)$ *as above,* $J \in \mathcal{F}$, *then*

a) $$\Big|\varphi_{(J)}(x)\Big| \le C\ell(J)^{2-d/2} \left(1 + \frac{|x - x_J|}{\ell(J)}\right)^{2-d} \quad \text{for all } x \in \Omega.$$

a′) $$\Big|\varphi_{(J)}(x)\Big| \le C\delta(x)^{\alpha}\ell(J)^{2-d/2-\alpha} \left(1 + \frac{|x - x_J|}{\ell(J)}\right)^{2-d-\alpha} \quad \text{for all } x \in \Omega.$$

b) *There is an absolute constant* η, $0 < \eta < 1$, *so that*

$$\Big|\varphi_{(J)}(x) - \varphi_J(y)\Big| \le C\,|x-y|^{\alpha}\,\ell(J)^{2-d/2-\alpha} \left(1 + \frac{|x - x_J|}{\ell(J)} + \frac{|y - x_J|}{\ell(J)}\right)^{2-d-\alpha}$$

for all x, y *in* ηQ_j *and* $J \in \mathcal{S}(Q_j)$.

c) $$\int \Big|\sum_{J\in\mathcal{F}} \lambda_J \varphi_{(J)}(x)\Big|^2 \mathrm{d}x \le C \sum_{J\in\mathcal{F}} \lambda_J^2.$$

Proof. To establish **a**), **a′**), **b**) for these functions, we use the fact that $H(x,y) = \Big(G(x,y) - \widetilde{G}(x,y)\Big)$ is a non-negative solution to $Lv = 0$ on $4J$, ([**2**], [**4**])) so Harnack's inequality and the maximum principle can be applied to this function. Since the coefficients of L are symmetric, this is valid for both the forward and the adjoint variables. Three estimates of Grüter and Widman [**3**] are also used.

First we prove **a**):

If $x \in \Omega \backslash 4J$, then for $y \in \frac{3}{2}J$, distance$(x,y) \simeq$ distance$(x, x_J) \gtrsim l(J)$, where x_J is the center point in the dyadic cube J. By Theorem 1.1, (1.8), of Grüter and Widman, we have

$$G(x,y) \le C\,|x-y|^{2-d}$$

so by the distance estimates,

$$|x-y|^{2-d} \le C'\left(\ell(J)+|x-x_J|\right)^{2-d} \le C'\ell(J)^{2-d}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{2-d},$$

and

$$G(x,y) \le C''\ell(J)^{2-d}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{2-d}.$$

This gives **a**) for $\varphi_{(J)}(x)$ when $x \in \Omega\backslash 4J$.

If $x \in 4J$, then the maximum principle implies that $H(x,y) = G(x,y) - \widetilde{G}(x,y) \le \max_{x\in\partial 4J} H(x,y) \le \max_{x\in\partial 4J} G(x,y) = G(x^*,y)$ for some point x^* in $\partial 4J$. Harnack's inequality gives that $G(x^*,y) \le CG(x^*,x_J)$ since $y \in \frac{3}{2}J$. Now distance $(x^*,x_J) \gtrsim \ell(J)$, so the first case applies to show that

$$G(x^*,y) \le C''\ell(J)^{2-d}\left(1+\frac{|x^*-x_J|}{\ell(J)}\right)^{2-d}.$$

Since $|x^*-x_J| \gtrsim |x-x_J|$, $\left(1+\frac{|x^*-x_J|}{\ell(J)}\right)^{2-d} \le C\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{2-d}$ for any $x \in 4J$. This means that

$$\varphi_{(J)}(x) \le C\ell(J)^{2-(d/2)}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{2-d}.$$

To prove **a′**) we use Theorem 1.8 of Grüter and Widman: This says that for all x, y in Ω,

$$G(x,y) \le C\delta(x)^\alpha\, |x-y|^{2-d-\alpha}.$$

(Notice that $G(x,y) = G(y,x)$ since we assume that the coefficients of L are symmetric.)

Again assuming first that $x \in \Omega\backslash 4J$, we have the same distance estimates as above; Theorem 1.8 gives

$$\begin{aligned}
\varphi_{(J)}(x) &= \frac{C}{\sqrt{|J|}}\int_{3J/2}\left(G(x,y)-\widetilde{G}(x,y)\right)\mathrm{d}y \\
&\le \frac{1}{\sqrt{|J|}}\int_{3J/2} C\delta(x)^\alpha\,|x-y|^{2-d-\alpha}\,\mathrm{d}y \\
&\le C\delta(x)^\alpha\ell(J)^{2-d-\alpha}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{2-d-\alpha}\cdot\frac{|J|}{\sqrt{|J|}} \\
&= C\delta(x)^\alpha\ell(J)^{2-d/2-\alpha}\left(1+\frac{|x-x_J|}{\ell(J)}\right)^{2-d-\alpha}.
\end{aligned}$$

If $x \in 4J$, as above the maximum principle and Harnack's inequality give

$$\begin{aligned}\varphi_{(J)}(x) &= \frac{C}{\sqrt{|J|}} \int_{3J/2} \Big(G(x,y) - \widetilde{G}(x,y)\Big)\, \mathrm{d}y \\ &\le \frac{C}{\sqrt{|J|}} \int_{3J/2} (H(x^*,y))\, \mathrm{d}y \le C\ell(J)^{d/2} G(x^*, x_J) \\ &\le C\ell(J)^{d/2}\delta(x)^\alpha \ell(J)^{2-d-\alpha}\left(1 + \frac{|x - x_J|}{\ell(J)}\right)^{2-d-\alpha}.\end{aligned}$$

Finally to prove the smoothness estimate, **b**), Theorem 1.9 of Grüter and Widman is useful; it says that

$$|G(x,y) - G(z,y)| \le C\, |x - z|^\alpha \left(|x - y|^{2-d-\alpha} + |z - y|^{2-d-\alpha}\right)$$

for all x, z, and $y \ne x$ or z in Ω.

b) will be proved for x and z in ηQ, with Q and J being cubes from $\mathcal{D}$, J is a cube in $\mathcal{W}$, and $J \in \mathcal{S}(Q)$. Recall that $0 < \eta < 1$. The definition of $\mathcal{S}(Q)$ means that either J straddles Q, or, J and Q are disjoint. There is a technical problem that arises from the fact that $4J$ may not be a cube in $\mathcal{D}$, so Q and $4J$ are not always nested or disjoint. If J straddles Q, then $4J$ will contain Q in its interior with distance$(Q, \partial(4J)) \simeq \ell(J)$; this is covered in case 2 below. For Q and J being disjoint, there is a $k \in \mathbb{Z}$, such that $\ell(J) = 2^k \ell(Q)$. Notice that $4J$ is in fact a union of dyadic cubes that are half J's size, so if $k \ge 1$, $4J$ and Q are nested or disjoint. In the following analysis we are thinking of $4J$ as being a union of several annular regions, centered around $3J/2$, of dimension $\sim \ell(J)/4$. The outer corridor of $4J$ is the outermost annular region in $4J$; part of the outer corridor's boundary coincides with the boundary of $4J$ and all points in the outer corridor lie a distance $\sim \ell(J)$ from $3J/2$. The interior of $4J$ is $4J$ minus its outer corridor. If $Q \subset 4J$, then ηQ will lie in the interior of $4J$ (case 2.) or in an outer corridor of $4J$ (case 3.). If $k \le -2$, even when J is adjacent to Q, $4J$ will be a distance $\simeq \ell(Q)$ from ηQ for η sufficiently small, and we can use the estimate in case 1. (Here, for convenience, η is assumed to be very much smaller than 1, say $\eta < 1/10$.) If $k = -1$ or 0, one may have ηQ in the interior of $4J$, in the outer corridor of $4J$, or straddling part of $\partial(4J)$. The last scenario can mean that $x \in 4J$, but $z \notin 4J$; this eventuality is dealt with in case 4.

For η closer to 1 (see the proof of Lemma 7), the constants will depend on η. It is possible that $x \in \mathrm{int}(4J)$, but z lies in or outside the outer corridor of $4J$. This can happen only for a fixed range of sizes for Q. One can take a strategically placed intermediate point, x^*, and, using the estimates from cases 2. and 3. or from 2. and 4., a short calculation shows that $|\varphi_J(x) - \varphi_J(x^*)| + |\varphi_J(z) - \varphi_J(x^*)| \le C\, |x - z|^\alpha\, \ell(J)^{2-d/2-\alpha}\left(1 + \frac{|x - x_J|}{\ell(J)} + \frac{|z - x_J|}{\ell(J)}\right)^{2-d-\alpha}$.

Again we first take case 1. where Q lies outside of $4J$, but x and z both lie inside ηQ. Then $|x - y| \simeq |x - x_J| \gtrsim \ell(J)$, $|z - y| \simeq |z - x_J| \gtrsim \ell(J)$, $|x - x_J| \simeq$

$|z - x_J|$, and $\widetilde{G}(x, y) = 0 = \widetilde{G}(z, y)$. This means that

$$\begin{aligned}\left|\varphi_{(J)}(x) - \varphi_{(J)}(z)\right| &\leq \frac{C}{\sqrt{|J|}} \int_{3J/2} |G(x, y) - G(z, y)| \,\mathrm{d}y \\ &\leq \frac{C}{\sqrt{|J|}} \int_{3J/2} \left(|x - z|^\alpha \left(|x - y|^{2-d-\alpha} + |z - y|^{2-d-\alpha}\right)\right) \mathrm{d}y \\ &\leq C\ell(J)^{d/2} \left(|x - z|^\alpha \left(|x - x_J|^{2-d-\alpha} + |z - x_J|^{2-d-\alpha}\right)\right) \\ &\leq C\ell(J)^{2-d/2-\alpha} |x - z|^\alpha \left(1 + \frac{|x - x_J|}{\ell(J)} + \frac{|z - x_J|}{\ell(J)}\right)^{2-d-\alpha}.\end{aligned}$$

If x and z both lie inside $4J$, then $H(x, y)$ is a non-negative solution to $Lu = 0$ in $4J$. In fact we will only need to consider the case where x, $z \in \eta Q$ and $J \in \mathcal{S}(Q)$. First we consider case 2. when both x and z lie away from the outer corridor of $4J$, so that $\operatorname{dist}(x, (4J)^C) \simeq \ell(J)$ and $\operatorname{dist}(z, (4J)^C) \simeq \ell(J)$. The usual Holder continuity result for a non-negative solution on a region which lies securely inside the domain and the maximum principle give

$$|H(x, y) - H(z, y)| \leq C \max_{z^* \in \partial(4J)} H(z^*, y) \cdot \left(\frac{|x - z|^\alpha}{\ell(J)^\alpha}\right).$$

Then Harnack's inequality and Theorem 1.8 imply that (remember that $4J$ is still a Whitney-type cube),

$$\max_{y \in 3J/2} H(z^*, y) \leq CG(z^*, x_J) \leq C\ell(J)^\alpha |z^* - x_J|^{2-d-\alpha},$$

and, since $|z^* - x_J| \simeq \ell(J) \simeq \operatorname{dist}(J, \partial\Omega)$, we can say that $|z^* - x_J|^{2-d-\alpha} \leq C(\ell(J) + |x - x_J| + |z - x_J|)^{2-d-\alpha} \leq C\ell(J)^{2-d-\alpha}(1 + \frac{|x - x_J|}{\ell(J)} + \frac{|z - x_J|}{\ell(J)})^{2-d-\alpha}$. This implies the estimate of **b**).

Next we consider case 3. in which both x and z lie in the outer corridor of $4J$ which has width $\sim \frac{1}{2}\,\ell(J)$. Since $y \in \frac{3}{2}\,J$ we have $|x - y| \sim \ell(J) \sim |z - y| \sim |x - x_J| \sim |z - x_J|$. We have $H(x, y) - H(z, y) = G(x, y) - G(z, y) + \widetilde{G}(z, y) - \widetilde{G}(x, y)$. Since the pole of both Green functions, y, is a safe distance from x and z, we can use the estimate in Theorem 1.9 of Grüter and Widman for both Green functions taken separately. So $|H(x, y) - H(z, y)| \leq C |x - z|^\alpha (|x - y|^{2-d-\alpha} + |z - y|^{2-d-\alpha})$. Now the same estimates as above give

$$|H(x, y) - H(z, y)| \leq C |x - z|^\alpha \ell(J)^{2-d-\alpha} \left(1 + \frac{|x - x_J|}{\ell(J)} + \frac{|z - x_J|}{\ell(J)}\right)^{2-d-\alpha},$$

and this estimate yields **b**) for $\left|\varphi_{(J)}(x) - \varphi_{(J)}(z)\right|$.

Finally we consider case 4. in which, say, x lies in the outer corridor of $4J$ and z lies just outside $4J$. This case will only occur when $\ell(J) \sim \ell(Q)$, so we can assume

$\mathrm{dist}(x,z) \lesssim \eta\,\ell(J)$. We also have that $|x-y| \simeq |x-x_J| \simeq |z-x_J| \simeq |z-y| \simeq \ell(J)$ for $y \in \frac{3}{2}Q$. We have $H(x,y) - H(z,y) = G(x,y) - G(z,y) - \widetilde{G}(x,y)$. We estimate $G(x,y)-G(z,y)$ as in the last paragraph. For $\widetilde{G}(x,y)$ we can use Theorem 1.8 of Grüter and Widman applied to the domain $4J$.

$$\widetilde{G}(x,y) \le C\delta_J(x)^\alpha\, |x-y|^{2-d-\alpha}.$$

Here $\delta_J(x) = \mathrm{dist}(x, \partial(4J) \le \mathrm{dist}(x,z)$ and $|x-y| \simeq |x-x_J| \simeq |z-x_J|$, so we have $\widetilde{G}(x,y) \le C\,|x-z|^\alpha\,\ell(J)^{2-d-\alpha}\left(1 + \frac{|x-x_J|}{\ell(J)} + \frac{|z-x_J|}{\ell(J)}\right)^{2-d-\alpha}$ as needed.

That the family of functions φ_J satisfy condition **c**) is easy to establish: we may assume that $\lambda_j \ge 0$ and that $\varphi_{(Q_j)}(y) \ge 0$ on Ω. We can write

$$\begin{aligned}
\int_\Omega |h(y)|^2\,\mathrm{d}y &= \int_\Omega \Big|\sum_{Q_j\in\mathcal{F}} \lambda_j \varphi_{(Q_j)}(y)\Big|^2 \mathrm{d}y = \sum_{Q_j\in\mathcal{F}} \lambda_j \int_\Omega h(y)\varphi_{(Q_j)}(y)\mathrm{d}y \\
&= \sum_{Q_j\in\mathcal{F}} \lambda_j \int_\Omega h(y)\bigg(\frac{1}{\sqrt{|Q_j|}}\int_{(3/2)Q_j}\Big(G(x,y) - \widetilde{G}(x,y)\Big)\,\mathrm{d}x\bigg)\mathrm{d}y \\
&\le \sum_{Q_j\in\mathcal{F}} \lambda_j \frac{1}{\sqrt{|Q_j|}}\int_{(3/2)Q_j} v(x)\,\mathrm{d}x \\
&\le \bigg(\sum_{Q_j\in\mathcal{F}} \lambda_j^2\bigg)^{1/2}\bigg(\sum_{Q_j\in\mathcal{F}}\bigg(\frac{1}{\sqrt{|Q_j|}}\int_{(3/2)Q_j} v(x)\,\mathrm{d}x\bigg)^2\bigg)^{1/2} \\
&\le \bigg(\sum_{Q_j\in\mathcal{F}} \lambda_j^2\bigg)^{1/2}\bigg(\sum_{Q_j\in\mathcal{F}}\int_{(3/2)Q_j} v(x)^2\mathrm{d}x\bigg)^{1/2} \\
&\le C\bigg(\sum_{Q_j\in\mathcal{F}} \lambda_j^2\bigg)^{1/2}\bigg(\int_\Omega v(x)^2\mathrm{d}x\bigg)^{1/2} \\
&\le C'\bigg(\sum_{Q_j\in\mathcal{F}} \lambda_j^2\bigg)^{1/2}\bigg(\int_\Omega h(x)^2\mathrm{d}x\bigg)^{1/2}.
\end{aligned}$$

We have taken $v(x) = \int_\Omega G(x,y)h(y)\,\mathrm{d}y$ to be the solution to $Lv = h$ in Ω. Dividing by $\left(\int_\Omega h(x)^2 dx\right)^{1/2}$ gives the property of almost orthogonality.

REFERENCES

1. R. Coifman, C. Fefferman: *Weighted norm inequalities for maximal functions and singular integrals.* Studia Math., **51** (1974), 241–250.
2. R. Fefferman, C. Kenig, J. Pipher: *The theory of weights and the Dirichlet problem for elliptic equations.* Annals of Math., **134** (1991), 65–124.

3. M. Grüter, K-O. Widman: *The Green function for uniformly elliptic equations.* Manuscripta Mathematica, **37** (1982), 303–342.
4. C. Kenig: *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems.* American Mathematical Society (1994), Providence, RI.
5. C. Sweezy: *Weights and Holder norms for solutions to a second order elliptic Dirichlet problem on nonsmooth domains.* American Mathematical Society Contemporary Mathematics, **428**, *Harmonic Analysis, Partial Differential Equations, and Related Topics* 2007, 127–138.
6. C. Sweezy: *A Littlewood Paley type inequality with applications to the elliptic Dirichlet problem.* Annales Polonici Mathematici, **90**, (2) (2007), 105–130.
7. C. Sweezy: *Harmonic gradients, Hölder norms for elliptic functions, and solutions to Poisson's equation on non-smooth domains.* IJPAM, **51**, No. 2 (2009), 225–232.
8. C. Sweezy: *Weighted inequalities with Hölder norms of solutions to Lu=divf in nonsmooth domains.* WSEAS Recent Advances in Applied Mathematics and Computational and Information Sciences, Proceedings of the 15th American Conference on Applied Mathematics, 2009, Houston, TX.
9. C. Sweezy, J. M. Wilson: *Weighted inequalities for gradients on nonsmooth domains*, preprint.
10. C. Sweezy, J. M. Wilson: *Weighted norm inequalities for parabolic gradients on nonsmooth domains.* International Journal of Pure and Applied Mathematics, **24**, No. 1 (2005), 61–109.
11. J. M. Wilson: *Global orthogonality implies local almost orthogonality.* Revista Mathematica Iberoamericana, **16** (2000), 29–48.
12. R. L. Wheeden, J. M. Wilson: *Weighted norm estimates for gradients of half-space extensions.* Indiana Univ. Math. Journal, **44** (1995), 917–969.

Department of Mathematical Sciences,
New Mexico State University,
Box 30001, 3MB,
Las Cruces, New Mexico,
U.S.A. 88003-8001
E–mail: csweezy@math.nmsu.edu

(Received June 2, 2009)
(Revised August 7, 2009)