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OSCILLATION CRITERIA FOR FIRST ORDER FORCED DYNAMIC EQUATIONS

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We obtain some new oscillation criteria for solutions to certain first order forced dynamic equations on a time scale $\mathbb{T}$ of the form

$$x^{\Delta}(t) + r(t)\Phi_{\gamma}\left(x^{\sigma}(t)\right) + p(t)\Phi_{\alpha}\left(x^{\sigma}(t)\right) + q(t)\Phi_{\beta}\left(x^{\sigma}(t)\right) = f(t),$$

with $\Phi_{\eta}(u) := |u|^{\eta-1}u,\ \eta > 0$. Here $r(t),\ p(t),\ q(t)$ and $f(t)$ are rd-continuous functions on $\mathbb{T}$ and the forcing term $f(t)$ is not required to be the derivative of an oscillatory function. Our results in the special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ involve and improve some previous oscillation results for first-order differential and difference equations. An example illustrating the importance of our results is also included.

1. INTRODUCTION

Following Hilger's landmark paper [**26**], a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus, where a time scale $\mathbb{T}$ is any nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [**7**]). Not only does the new theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations, but it also extends these classical cases to cases "in between", e.g., to the so-called $q-$difference equations when $\mathbb{T} = q^{\mathbb{N}_0}$ (which has important applications in quantum theory [**27**]) and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{N}_0^2$ and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers. Throughout this work a knowledge and understanding of time scales and time scale

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notation is assumed; for an excellent introduction to the calculus on time scales, see [**7**, **8**, **26**]. In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales, we refer the reader to [**1**, **4**, **9**–**19**, **21**–**25**], and the references cited therein. To the best of our knowledge, there is very little known about the oscillatory behavior of first order dynamic equations. Agarwal and Bohner [**2**] (see also [**3**]) considered the first order dynamic equations

$$x^{\Delta}(t) - q(t)x^{\sigma}(t) + q_1(t)\left(x^{\sigma}(t)\right)^{\alpha_1} = f(t),$$

$$x^{\Delta}(t) = q_2(t)\left(x^{\sigma}(t)\right)^{\beta_1} + f(t)$$

and

$$x^{\Delta}(t) + q_1(t)\left(x^{\sigma}(t)\right)^{\alpha_1} = q_2(t)\left(x^{\sigma}(t)\right)^{\beta_1} + f(t)$$

where α_1 and β_1 are ratios of odd positive integers with $0 < \alpha_1 < 1$ and $\beta_1 > 1$ and f, q, q_1 and q_2 are rd-continuous functions such that q_1 and q_2 are positive on $\mathbb{T}$. Recently, Bohner and Hassan [**6**] improved the above results for the first order dynamic equations

$$x^{\Delta}(t) + p(t)x^{\gamma}(h(t)) + q_3(t)x^{\alpha}(h(t)) = f(t)$$

and

$$x^{\Delta}(t) + p(t)x^{\gamma}(h(t)) + q_3(t)x^{\alpha}(h(t)) + q_4(t)x^{\beta}(h(t)) = f(t),$$

where γ, α and β are ratios of odd positive integers with $\alpha > \gamma > 0$ and $\beta > \gamma > 0$, p is an rd-continuous function on $\mathbb{T}$, and f, q_3 and q_4 are positive rd-continuous functions on $\mathbb{T}$ and the function $h : \mathbb{T} \to \mathbb{T}$ satisfies $\lim_{t\to\infty} h(t) = \infty$. Agarwal and Zafer [**5**] and Erbe et al [**12**] consider oscillation criteria for the related second order forced dynamic equation

$$(r(t)\Phi_{\alpha}(x^{\Delta}))^{\Delta} + a(t)\Phi(x^{\sigma}(t)) + \sum_{i=1}^{n} q_i(t)\Phi_{\beta_i}(x^{\sigma}(t)) = f(t), \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

We are concerned with the oscillatory behavior of the following forced first order dynamic equation with mixed arguments

$$x^{\Delta}(t) + r(t)\Phi_{\gamma}\left(x^{\sigma}(t)\right) + p(t)\Phi_{\alpha}\left(x^{\sigma}(t)\right) + q(t)\Phi_{\beta}\left(x^{\sigma}(t)\right) = f(t), \tag{1.1}$$

with $\Phi_{\eta}(u) := |u|^{\eta-1} u$, $\eta > 0$, on an arbitrary time scale $\mathbb{T}$, where $r(t)$, $p(t)$, $q(t)$ and $f(t)$ are rd-continuous functions on $\mathbb{T}$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$ (unbounded above), and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1) we mean a nontrivial real–valued function $x \in C^1_{rd}[T_x, \infty)$, $T_x \geq t_0$ which satisfies equation (1.1) on $[T_x, \infty)$, where C_{rd} is the space of $rd-$continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

2. MAIN RESULTS

Before stating our main results, we begin with the following lemma which will play an important role in the proof of our main results.

Lema 2.1 *Let $a, b \in \mathbb{R}$ with $a > 0$ and $b \geq 0$. Then*

(i) $$a^{\alpha-\gamma} - ba^{\beta-\gamma} \geq -\delta_1 b^{(\alpha-\gamma)/(\alpha-\beta)} \quad \textit{for all } \alpha > \beta > \gamma > 0;$$

(ii) $$a^{\alpha-\gamma} + ba^{\beta-\gamma} \geq \delta_2 b^{(\alpha-\gamma)/(\alpha-\beta)} \quad \textit{for all } \alpha > \gamma > \beta > 0,$$

where

$$\delta_1 := (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\beta-\gamma)^{(\beta-\gamma)/(\alpha-\beta)}$$

and

$$\delta_2 := (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\gamma-\beta)^{(\beta-\gamma)/(\alpha-\beta)}.$$

Proof. We let

$$f(u) := u^{\alpha-\gamma} - bu^{\beta-\gamma} \quad \text{for} \quad u > 0,$$

where $\alpha > \beta > \gamma > 0$. It is easy to see that f obtains its minimum at $u = (\alpha-\gamma)^{1/(\beta-\alpha)}(\beta-\gamma)^{1/(\alpha-\beta)}b^{1/(\alpha-\beta)}$ and

$$f_{\min} = (\beta-\alpha)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\beta-\gamma)^{(\beta-\gamma)/(\alpha-\beta)}b^{(\alpha-\gamma)/(\alpha-\beta)}.$$

Also, we let

$$g(u) := u^{\alpha-\gamma} + bu^{\beta-\gamma} \quad \text{for} \quad b,\ u > 0,$$

where $\alpha > \gamma > \beta > 0$ and we find that g obtains its minimum at $u = (\alpha-\gamma)^{1/(\beta-\alpha)}$ $(\gamma-\beta)^{1/(\alpha-\beta)}b^{1/(\alpha-\beta)}$ and

$$g_{\min} = (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\gamma-\beta)^{(\beta-\gamma)/(\alpha-\beta)}b^{(\alpha-\gamma)/(\alpha-\beta)}. \qquad \square$$

In the following, let $\mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\}$ and consider the functions $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$ such that

$$H(t,t) = 0, \quad t \geq t_0, \quad H(t,s) > 0, \quad t > s \geq t_0, \tag{2.1}$$

and where H has a nonpositive continuous Δ- partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable and satisfies

$$H^{\Delta_s}(t,s) = -h(t,s)\, H^{1/\gamma}(t,s)\,. \tag{2.2}$$

Throughout this paper, we let

$$d_+ := \max\{0, d\}, \quad d_- := \max\{0, -d\},$$

and define

$$g_k(t,s) := (\gamma-1)\gamma^{\gamma/(1-\gamma)} \left|l_k(s)\right|^{1/(1-\gamma)} h^{\gamma/(\gamma-1)}(t,s)\,, \quad k = 1,2,3,4, \tag{2.3}$$

where

$$\begin{aligned} l_1(t) &:= r_-(t) - \delta_1 (p(t))^{(\gamma-\beta)/(\alpha-\beta)} (q_-(t))^{(\alpha-\gamma)/(\alpha-\beta)}, \\ l_2(t) &:= r_-(t) - \delta_2 (p(t))^{(\gamma-\beta)/(\alpha-\beta)} (q(t))^{(\alpha-\gamma)/(\alpha-\beta)}, \\ l_3(t) &:= r(t) + \delta_1 (-p(t))^{(\gamma-\beta)/(\alpha-\beta)} (q_+(t))^{(\alpha-\gamma)/(\alpha-\beta)}, \\ l_4(t) &:= r(t) - \delta_2 (-p(t))^{(\gamma-\beta)/(\alpha-\beta)} (-q(t))^{(\alpha-\gamma)/(\alpha-\beta)}, \end{aligned}$$

and

$$\begin{aligned} \delta_1 &:= (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\beta-\gamma)^{(\beta-\gamma)/(\alpha-\beta)}, \\ \delta_2 &:= (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\gamma-\beta)^{(\beta-\gamma)/(\alpha-\beta)}. \end{aligned}$$

Theorem 2.1. *Assume that* $0 < \gamma < 1$, $\alpha > \beta > \gamma$ *and* $p(t) > 0$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *If* $g_1(t,s)$ *is given by* (2.3),

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\, f(s) + g_1(t,s)\right) \Delta s = \infty \tag{2.4}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\, f(s) - g_1(t,s)\right) \Delta s = -\infty, \tag{2.5}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. Multiplying both sides of (1.1), with t replaced by s, by $H(t,s)$, for $t \geq s \geq T$ and integrating with respect to s from T to t, we have

$$\begin{aligned} \int_T^t H(t,s) f(s)\Delta s &= \int_T^t H(t,s)x^{\Delta}(s)\Delta s + \int_T^t H(t,s)r(s)x^{\sigma\gamma}(s)\Delta s \\ &\quad + \int_T^t H(t,s)\left(p(s)x^{\sigma\alpha}(s) + q(s)x^{\sigma\beta}(s)\right)\Delta s. \end{aligned}$$

Integrating by parts and using (2.1) and (2.2), we obtain

$$\begin{aligned} \int_T^t H(t,s) f(s)\Delta s = &-H(t,T)x(T) + \int_T^t h(t,s)\, H^{1/\gamma}(t,s)\, x^{\sigma}(s)\Delta s \\ &+ \int_T^t H(t,s)r(s)x^{\sigma\gamma}(s)\Delta s \\ &+ \int_T^t H(t,s)\left(p(s)x^{\sigma\alpha}(s) + q(s)x^{\sigma\beta}(s)\right)\Delta s \end{aligned}$$

$$
\begin{aligned}
\geq & - H(t,T)x(T) + \int_T^t h(t,s)\, H^{1/\gamma}(t,s)\, x^\sigma(s)\Delta s \\
& - \int_T^t H(t,s) r_-(s) x^{\sigma\gamma}(s)\Delta s \\
& + \int_T^t H(t,s)\left(p(s)x^{\sigma\alpha}(s) - q_-(s)x^{\sigma\beta}(s)\right)\Delta s.
\end{aligned}
$$

Define

$$a := p^{1/(\alpha-\gamma)}x^\sigma \quad \text{and} \quad b := p^{(\gamma-\beta)/(\alpha-\gamma)}q_-$$

and using Lemma 2.1 (i), we have

$$p\,(x^\sigma)^{\alpha-\gamma} - q_-\,(x^\sigma)^{\beta-\gamma} \geq -\delta_1 p^{(\gamma-\beta)/(\alpha-\beta)} q_-^{(\alpha-\gamma)/(\alpha-\beta)},$$

where $\delta_1 = (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\beta-\gamma)^{(\beta-\gamma)/(\alpha-\beta)}$. It follows that

$$
\begin{aligned}
\int_T^t H(t,s)f(s)\Delta s \geq & - H(t,T)x(T) \\
& + \int_T^t \left[h(t,s)\, H^{1/\gamma}(t,s)\, x^\sigma(s) - H(t,s)\,|l_1(s)|\, x^{\sigma\gamma}(s)\right]\Delta s.
\end{aligned}
$$

Define $X \geq 0$ and $Y \geq 0$ by

$$X^\gamma := H\,|l_1|\, x^{\sigma\gamma}, \quad Y^{\gamma-1} := h\gamma^{-1}\,|l_1|^{-1/\gamma}.$$

Then, using the inequality (see [**20**])

$$\gamma XY^{\gamma-1} - X^\gamma \geq (\gamma-1)Y^\gamma, \qquad 0<\gamma<1,$$

we get that

$$h(t,s)\, H^{1/\gamma}(t,s)\, x^\sigma(s) - H(t,s)\,|l_1(s)|\, x^{\sigma\gamma}(s) \geq g_1(t,s).$$

Then

$$\frac{1}{H(t,T)}\int_T^t \left(H(t,s)f(s) - g_1(t,s)\right)\Delta s \geq -x(T),$$

which leads to a contradiction to (2.5). Next, we assume that x satisfies (1.1) and is eventually negative. If we put $y=-x$, then y is an eventually positive solution of (1.1), where f is replaced by $-f$. This case therefore leads to a contradiction as in the first case, provided

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \left(-H(t,s)f(s) - g_1(t,s)\right)\Delta s = -\infty,$$

which indeed holds due to the condition (2.4). $\square$

Theorem 2.2. *Assume that* $0 < \gamma < 1$, $\alpha > \gamma > \beta > 0$, $p(t) > 0$ *and* $q(t)$, $l_2(t) \geq 0$, *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *If* $g_2(t,s)$ *is given by* (2.3),

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s) f(s) + g_2(t,s)\right) \Delta s = \infty \tag{2.6}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s) f(s) - g_2(t,s)\right) \Delta s = -\infty, \tag{2.7}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a solution x of (1.0) and a $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. As in the proof of Theorem 2.1, we get

$$\begin{aligned}
&\int_T^t H(t,s) f(s) \Delta s = -H(t,T)x(T) + \int_T^t h(t,s) H^{1/\gamma}(t,s) x^\sigma(s) \Delta s \\
&\quad + \int_T^t H(t,s) r(s) x^{\sigma\gamma}(s) \Delta s + \int_T^t H(t,s) \left(p(s) x^{\sigma\alpha}(s) + q(s) x^{\sigma\beta}(s)\right) \Delta s \\
\geq\ & -H(t,T)x(T) + \int_T^t h(t,s) H^{1/\gamma}(t,s) x^\sigma(s) \Delta s \\
&\quad - \int_T^t H(t,s) r_-(s) x^{\sigma\gamma}(s) \Delta s + \int_T^t H(t,s) \left(p(s) x^{\sigma\alpha}(s) + q(s) x^{\sigma\beta}(s)\right) \Delta s.
\end{aligned}$$

By using Lemma 2.1 (ii), we have

$$p\,(x^\sigma)^{\alpha-\gamma} + q\,(x^\sigma)^{\beta-\gamma} \geq \delta_2 p^{(\gamma-\beta)/(\alpha-\beta)} q^{(\alpha-\gamma)/(\alpha-\beta)},$$

where $\delta_2 = (\alpha-\beta)(\alpha-\gamma)^{(\gamma-\alpha)/(\alpha-\beta)}(\gamma-\beta)^{(\beta-\gamma)/(\alpha-\beta)}$. The rest of the proof is the same as the proof of Theorem 2.1 with l_1 replaced by l_2. $\square$

Theorem 2.3. *Assume that* $\alpha > \beta > \gamma > 1$ *and* $p(t)$, $l_3(t) < 0$, *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *If* $g_3(t,s)$ *is given by* (2.3),

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s) f(s) - g_3(t,s)\right) \Delta s = \infty \tag{2.8}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s) f(s) + g_3(t,s)\right) \Delta s = -\infty, \tag{2.9}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. As in the proof of Theorem 2.1, we get

$$(2.10)\qquad \int_T^t H(t,s)f(s)\Delta s = -H(t,T)x(T) + \int_T^t h(t,s)\,H^{1/\gamma}(t,s)\,x^\sigma(s)\Delta s$$
$$+\int_T^t H(t,s)r(s)x^{\sigma\gamma}(s)\Delta s + \int_T^t H(t,s)\left(p(s)x^{\sigma\alpha}(s)+q(s)x^{\sigma\beta}(s)\right)\Delta s$$
$$\le\ -H(t,T)x(T) + \int_T^t h(t,s)\,H^{1/\gamma}(t,s)\,x^\sigma(s)\Delta s + \int_T^t H(t,s)r(s)x^{\sigma\gamma}(s)\Delta s$$
$$+\int_T^t H(t,s)\left(p(s)x^{\sigma\alpha}(s)+q_+(s)x^{\sigma\beta}(s)\right)\Delta s.$$

Again, by using Lemma 2.1 (i), we have

$$(2.11)\qquad -p\,(x^\sigma)^{\alpha-\gamma} - q_+\,(x^\sigma)^{\beta-\gamma} \ge -\delta_1(-p)^{(\gamma-\beta)/(\alpha-\beta)}\,(q_+)^{(\alpha-\gamma)/(\alpha-\beta)}.$$

Using (2.11) in (2.10), we get

$$(2.12)\qquad \int_T^t H(t,s)f(s)\Delta s \le -H(t,T)x(T) + \int_T^t h(t,s)\,H^{1/\gamma}(t,s)\,x^\sigma(s)\Delta s$$
$$+\int_T^t H(t,s)\left[r(s)+\delta_1(-p\,(s))^{(\gamma-\beta)/(\alpha-\beta)}\,(q_+(s))^{(\alpha-\gamma)/(\alpha-\beta)}\right]x^{\sigma\gamma}(s)\Delta s$$
$$= -H(t,T)x(T) + \int_T^t \left[h(t,s)\,H^{1/\gamma}(t,s)\,x^\sigma(s) - H(t,s)\,|l_3(s)|\,x^{\sigma\gamma}(s)\right]\Delta s.$$

Define $X\ge 0$ and $Y\ge 0$ by

$$X^\gamma := H\,|l_3|\,x^{\sigma\gamma},\quad Y^{\gamma-1} := h\gamma^{-1}\,|l_3|^{-1/\gamma}.$$

Then, using the inequality (see [**20**])

$$\gamma XY^{\gamma-1} - X^\gamma \le (\gamma-1)Y^\gamma,\qquad \gamma>1,$$

we get that

$$\begin{aligned}h(t,s)H^{1/\gamma}(t,s)x^\sigma(s) &- H(t,s)a_1(s)\,x^{\sigma\gamma}(s)\\ &\le (\gamma-1)\gamma^{\gamma/(1-\gamma)}\,|l_3(s)|^{1/(1-\gamma)}\,h^{\gamma/(\gamma-1)}(t,s)\\ &= g_3(t,s).\end{aligned}$$

From this last inequality and (2.12) we get

$$\frac{1}{H(t,T)}\int_T^t \left[H(t,s)f(s)-g_3(t,s)\right]\Delta s \le -x(T),$$

which leads to a contradiction to (2.8). □

Theorem 2.4. *Assume that* $\gamma>1,\ \alpha>\gamma>\beta>0,\ p(t),\ l_4(t)<0$ *and* $q(t)\le 0$, *for* $t\in[t_0,\infty)_{\mathbb{T}}$. *If* $g_4(t,s)$ *is given by* (2.3),

$$(2.13)\qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \left(H(t,s)f(s)-g_4(t,s)\right)\Delta s = \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)f(s) + g_4(t,s)\right)\Delta s = -\infty, \tag{2.14}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. As in the proof of Theorem 2.1, we get

$$\begin{aligned}\int_T^t H(t,s)f(s)\Delta s = & -H(t,T)x(T) + \int_T^t h(t,s)\, H^{1/\gamma}(t,s)\, x^{\sigma}(s)\Delta s \\ & + \int_T^t H(t,s)r(s)x^{\sigma\gamma}(s)\Delta s \\ & + \int_T^t H(t,s)\left(p(s)x^{\sigma\alpha}(s) + q(s)x^{\sigma\beta}(s)\right)\Delta s.\end{aligned}$$

Also, by Lemma 2.1 (ii), we have

$$(-p)\,(x^{\sigma})^{\alpha-\gamma} + (-q)\,(x^{\sigma})^{\beta-\gamma} \geq \delta_2\,(-p)^{(\gamma-\beta)/(\alpha-\beta)}\,(-q)^{(\alpha-\gamma)/(\alpha-\beta)},$$

and the rest of the proof is line by line the same as the proof of Theorem 2.3 with l_3 replaced by l_4. □

Theorem. *Assume $\gamma = 1$ holds and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)f(s)ds = \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)f(s)ds = -\infty.$$

Furthermore, assume one of the following conditions is satisfied, for $t \geq s \geq t_0$

(i) $\alpha > \beta > 1 > 0$, $p(s) > 0$, $h(t,s)\, H^{1/\gamma}(t,s) - H(t,s)l_1(s) \geq 0$;

(ii) $\alpha > 1 > \beta > 0$, $p(s) > 0$, $q(s) \geq 0$, $h(t,s)\, H^{1/\gamma}(t,s) - H(t,s)l_2(s) \geq 0$;

(iii) $\alpha > \beta > 1 > 0$, $p(s) < 0$, $h(t,s)\, H^{1/\gamma}(t,s) + H(t,s)l_3(s) \leq 0$;

(iv) $\alpha > 1 > \beta > 0$, $p(s) < 0$, $q(s) \leq 0$, $h(t,s)\, H^{1/\gamma}(t,s) + H(t,s)l_4(s) \leq 0$.

Then every solution of equation (1.1) *is oscillatory.*

Theorem 2.6. *Assume that $r(t)$, $p(t)$ and $q(t)$ are nonnegative functions on $t \in [t_0,\infty)_{\mathbb{T}}$. If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)f(s)ds = \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)f(s)ds = -\infty,$$

then every solution of equation (1.1) *is oscillatory.*

Remark 2.1. The results are in a form with a high degree of generality. Thus with an appropriate choice of $H(t,s)$ in Theorems 2.3–2.6, we can get several sufficient conditions for oscillation of equation (1.1). For instance, suppose we choose $H(t,s) = (t-s)^n$, for $n \in \mathbb{N}$ and $t \geq s \geq t_0$. Then, by using [**7**, Theorem 1.24], we get

$$h(t,s) = \frac{(-1)^{n+1}}{(t-s)^{\alpha/\gamma}} \sum_{\upsilon=0}^{n-1} (\sigma(s)-t)^{\upsilon} (s-t)^{n-\upsilon-1}, \ t \geq s \geq t_0.$$

Remark 2.2. The idea behind the proofs of Theorems 2.1–2.6 in this paper is to use an appropriate inequality in Lemma 2.1 to combine the terms $p(s)x^{\sigma\alpha}(s)$ and $q(s)x^{\sigma\beta}(s)$ to get a term involving $x^{\sigma}(s)$ and then to combine the term involving $x^{\sigma\gamma}(s)$ with $h(t,s)H^{\frac{1}{\gamma}}(t,s)x^{\sigma}(s)$ by one of the inequalities

$$\gamma XY^{\gamma-1} - X^{\gamma} \geq (\gamma-1)Y^{\gamma}, \quad 1 < \gamma < 1,$$

and

$$\gamma XY^{\gamma-1} - X^{\gamma} \leq (\gamma-1)Y^{\gamma}, \quad \gamma > 1.$$

By repeated applications of this process one can get results for the more general equation

$$x^{\Delta}(t) + r(t)\Phi_{\gamma}(x^{\sigma}(t)) + \sum_{i=1}^{n} [p_i(t)\Phi_{\alpha_i}(x^{\sigma}(t)) + q_i(t)\Phi_{\beta_i}(x^{\sigma}(t))] = f(t).$$

We leave these details to the interested reader.

Example 2.1. Let $\mathbb{T}$ be a time scale satisfying $\mu(t) > 0$ for all $t \in [t_0,\infty)_{\mathbb{T}}$, $t_0 > 0$, and consider on $[t_0,\infty)_{\mathbb{T}}$ the equation

$$x^{\Delta}(t)+r(t)\Phi_{\gamma}(x^{\sigma}(t))+p(t)\Phi_{\alpha}(x^{\sigma}(t))+q(t)\Phi_{\beta}(x^{\sigma}(t)) = \frac{t+\sigma(t)}{\mu(t)} e_{-2/\mu}(\sigma(t),t_0), \tag{2.15}$$

for any $0 < \gamma < 1$, $\alpha > \beta > \gamma$ and for any positive function $p(t)$ on $[t_0,\infty)_{\mathbb{T}}$ and for functions $r(t)$ and $q(t)$ on $[t_0,\infty)_{\mathbb{T}}$. As in [**2**, **6**, Example 2.1], we get

$$F^{\Delta}(t) = \frac{t+\sigma(t)}{\mu(t)} e_{-2/\mu}(\sigma(t),t_0) =: f(t)$$

for $F(t) := te_{-2/\mu}(t,t_0)$ and $e_{-2/\mu}(\sigma(t),t_0) = -e_{-2/\mu}(t,t_0)$,let us take $H(t,s) = 1$, for $t > s \geq t_0$ and $H(t,t) = 0$, then

$$\limsup_{t\to\infty} \int_{t_0}^{t} f(\tau)\Delta\tau = \limsup_{t\to\infty} \left(te_{-2/\mu}(t,t_0) - t_0\right) = \infty$$

and

$$\liminf_{t\to\infty} \int_{t_0}^{t} f(\tau)\Delta\tau = \liminf_{t\to\infty} \left(te_{-2/\mu}(t,t_0) - t_0\right) = -\infty.$$

Then by Theorem 2.1, every solution of (2.15) is oscillatory.

The important point to note here is that the recent results due to Agarwal and Bohner [**2**] and Agarwal, Bohner and Grace [**3**] and Bohner and Hassan

[**6**] do not apply to equation (2.15), since no sign conditions are imposed on r and q.

Additional examples may be readily given. We leave this to the interested reader.

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