Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 310–329. doi:10.2298/AADM0902310W

δ-FIBONACCI NUMBERS

Roman Wituła, Damian Słota

The scope of the paper is the definition and discussion of the polynomial generalizations of the Fibonacci numbers called here δ-Fibonacci numbers. Many special identities and interesting relations for these new numbers are presented. Also, different connections between δ-Fibonacci numbers and Fibonacci and Lucas numbers are proven in this paper.

1. INTRODUCTION

Rabinowitz in [**9**] and Grzymkowski and Wituła in [**4**] independently discovered and studied the following two identities:

$$(1 + \xi + \xi^4)^n = F_{n+1} + F_n(\xi + \xi^4) \tag{1.1}$$

and

$$(1 + \xi^2 + \xi^3)^n = F_{n+1} + F_n(\xi^2 + \xi^3), \tag{1.2}$$

where F_n, $n \in \mathbb{N}$ denotes the Fibonacci numbers (see, for example [**7**]) and $\xi \in \mathbb{C}$, $\xi \neq 1$, $\xi^5 = 1$. Corresponding to these identities, many classical identities and relations for Fibonacci and Lucas numbers could be proved in a new elegant way (see for example [**12**]).

In this paper we generalize the identities (1.1) and (1.2) to the following forms:

$$\big(1 + \delta(\xi + \xi^4)\big)^n = a_n(\delta) + b_n(\delta)(\xi + \xi^4) \tag{1.3}$$

and

$$\big(1 + \delta(\xi^2 + \xi^3)\big)^n = a_n(\delta) + b_n(\delta)(\xi^2 + \xi^3) \tag{1.4}$$

2000 Mathematics Subject Classification. 11B39, 11B83, 11A07, 39A10.
Keywords and Phrases. Fibonacci numbers, Lucas numbers.

for $\delta \in \mathbb{C}$, $n \in \mathbb{N}$. Simultaneously with these new identities, we introduce the polynomials $a_n(\delta), b_n(\delta) \in \mathbb{Z}[\delta]$, which by (1.1) and (1.2), can be treated as the generalization of the FIBONACCI numbers and, therefore, which are called here the δ-FIBONACCI numbers. The scope of this paper is to investigate basic properties of these new kinds of numbers. For example, BINET's formulae for δ-FIBONACCI numbers and formulae connecting $a_n(\delta)$ and $b_n(\delta)$, $n \in \mathbb{N}$, with FIBONACCI and LUCAS numbers are presented.

It is important to emphasize that our δ-FIBONACCI numbers belong to the family of the so called quasi-FIBONACCI numbers of (k-th, δ-as) orders for $k \in 2\mathbb{N} - 1$, $\delta \in \mathbb{C}$. These numbers are, by definition, the elements of the following sequences of polynomials:

$$\{a_{n,i}(\delta)\}_{n\in\mathbb{N}} \subset \mathbb{Z}[\delta], \quad \delta \in \mathbb{C}, \quad i = 1, 2, \ldots, \frac{1}{2}\,\varphi(k),$$

where φ is the EULER's totient function. If k is an odd prime number, then the quasi-FIBONACCI numbers of (k-th, δ-as) order are determined by the following identities:

$$\big(1 + \delta\,(\xi^{\ell} + \xi^{k-\ell})\big)^n = a_{n,1}(\delta) + \sum_{i=1}^{(k-3)/2} a_{n,i+1}(\delta)\,\big(\xi^{i\,\ell} + \xi^{i(k-\ell)}\big), \tag{1.5}$$

for $\ell = 1, 2, \ldots, (k-1)/2$, $n \in \mathbb{N}$ and where $\xi := \exp(2\,\pi\, i/k)$. More information and general definitions of the quasi-FIBONACCI numbers are given in papers [**12**, **14**].

We note that δ-FIBONACCI numbers, are the simplest members of the family of the quasi-FIBONACCI numbers of (k-th, δ-as) order. Moreover, in paper [**12**] and [**4**] the basic properties of the quasi-FIBONACCI numbers of the (7-th, δ-as) and (11-th, δ-as) order, respectively, are presented. In paper [**14**], which is a continuation of paper [**12**], the applications of the quasi-FIBONACCI numbers of the (7-th, δ-as) order to the decomposition of some polynomials of the third degree are studied. They make it possible to generate new RAMANUJAN-type trigonometric identities!

REMARK 1.1. Most identities and relations discussed in the paper are proved by an immediate application of identities (1.3) and (1.4). The use of BINET's formulae for δ-FIBONACCI numbers (which were proved in Section 4) is restricted in this paper only to some cases. By doing so, we want to promote an alternative, more creative method of generating and verifying the identities for δ-FIBONACCI numbers.

Our paper is divided into seven sections. In the second section, the definitions and notations are given. In the third section, the basic formulae and properties of δ-FIBONACCI numbers are derived. In the fourth section of the paper, the BINET formulae for δ-FIBONACCI numbers are proven. The relationships between δ-FIBONACCI numbers for different values of δ's are studied in the fifth section of the paper. In the sixth section, the reduction formulae for indices are given. Some summation and convolution type formulae for δ-FIBONACCI numbers are derived in the last section of the paper.

2. DEFINITIONS AND NOTATIONS

Let $\xi = \exp(i2\pi/5)$. Let us start with the following basic result:

Lemma 2.1. a) *Any two among the following numbers*:

$$1, \quad -\beta := \xi + \xi^4 = 2\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{2}, \quad -\alpha := \xi^2 + \xi^3 = -2\cos\frac{\pi}{5} = -\frac{\sqrt{5}+1}{2}$$

are linearly independent over $\mathbb{Q}$. *Moreover, we have* [**6**, **7**]:

$$\alpha + \beta = \alpha\,\beta = -1, \qquad \alpha^2 = \alpha + 1 \qquad \textit{and} \qquad \beta^2 = \beta + 1.$$

b) *Let* $f_k \in \mathbb{Q}[\delta]$ *and* $g_k \in \mathbb{Q}[\delta]$, $k = 1, 2$. *Then, for any* $a, b \in \mathbb{R}$ *linearly independent over* $\mathbb{Q}$, *if*

$$f_1(\delta)a + g_1(\delta)b = f_2(\delta)a + g_2(\delta)b, \quad \textit{for} \quad \delta \in \mathbb{Q}$$

then

$$f_1(\delta) = f_2(\delta) \quad \textit{and} \quad g_1(\delta) = g_2(\delta), \quad \textit{for} \quad \delta \in \mathbb{C}.$$

Now, let us set

$$\big(1 + \delta(\xi + \xi^4)\big)^n = a_n(\delta) + b_n(\delta)(\xi + \xi^4) \tag{2.1}$$

for $\delta \in \mathbb{C}$, $n \in \mathbb{N}$. Then by Lemma 2.1 the following recurrence relations hold true:

$$a_0(\delta) = 1, \qquad b_0(\delta) = 0,$$

$$a_{n+1}(\delta) = a_n(\delta) + \delta b_n(\delta), \tag{2.2}$$

$$b_{n+1}(\delta) = \delta a_n(\delta) + (1-\delta) b_n(\delta), \tag{2.3}$$

or

$$\begin{bmatrix} a_{n+1}(\delta) \\ b_{n+1}(\delta) \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ \delta & 1-\delta \end{bmatrix} \begin{bmatrix} a_n(\delta) \\ b_n(\delta) \end{bmatrix}. \tag{2.4}$$

The elements of sequences $\{a_n(\delta)\}$ and $\{b_n(\delta)\}$ will be called the δ-*Fibonacci numbers.*

Table 1: The most important cases (for *Axxxxxx* see [**10**])

δ	-2	-1	$1/2$	$2/3$	1	2
$a_n(\delta)$	*A*015448	*A*001519	$2^{-n}F_{2n+1}$	$3^{-n}F_{3n+1}$	F_{n+1}	$5^{\lfloor n/2\rfloor}$
$b_n(\delta)$	$-F_{3n}$	$-F_{2n}$	$2^{-n}F_{2n}$	$3^{-n}F_{3n}$	F_n	$(1-(-1)^n)5^{\lfloor n/2\rfloor}$

Remark 2.2. We note that, also by Lemma 2.1, the following identity holds:

$$\big(1+\delta\,(\xi^2+\xi^3)\big)^n = a_n(\delta) + b_n(\delta)\,(\xi^2+\xi^3). \tag{2.5}$$

Hence, by (2.1) we get the identity:

$$\big(1+\delta\,(\xi+\xi^4)\big)^n + \big(1+\delta\,(\xi^2+\xi^3)\big)^n = 2\,a_n(\delta) - b_n(\delta) := \mathbb{A}_n(\delta). \tag{2.6}$$

The new auxiliary sequence $\{\mathbb{A}_n(\delta)\}$ shall be used in many applications of numbers $\{a_n(\delta)\}$ and $\{b_n(\delta)\}$, especially for the decomposition of certain polynomials.

Remark 2.3. Let us highlight that from definition (2.2) and (2.3) the previously mentioned fact can be inferred that $a_n(\delta), b_n(\delta) \in \mathbb{Z}[\delta]$, $n = 0, 1, 2, \ldots$

Thus, in view of Lemma (2.1), it may be concluded that, if $F_i, G_i \in \mathbb{Q}[x_0, x_1, \ldots, x_{2n+1}]$, $i = 1, 2$, and

$$\begin{aligned} &F_1\big[a_0(\delta), a_1(\delta), \ldots, a_n(\delta), b_0(\delta), b_1(\delta), \ldots, b_n(\delta)\big] \\ &\qquad + (\xi+\xi^4)\,F_2\big[a_0(\delta), a_1(\delta), \ldots, a_n(\delta), b_0(\delta), b_1(\delta), \ldots, b_n(\delta)\big] \\ &= G_1\big[a_0(\delta), a_1(\delta), \ldots, a_n(\delta), b_0(\delta), b_1(\delta), \ldots, b_n(\delta)\big] \\ &\qquad + (\xi+\xi^4)\,G_2\big[a_0(\delta), a_1(\delta), \ldots, a_n(\delta), b_0(\delta), b_1(\delta), \ldots, b_n(\delta)\big], \end{aligned}$$

for $\delta \in \mathbb{Q}$, then

$$F_i\big[a_0(\delta), \ldots, b_n(\delta)\big] = G_i\big[a_0(\delta), \ldots, b_n(\delta)\big],$$

for every $i = 1, 2$ and $\delta \in \mathbb{R}$. A similar fact holds when $\xi + \xi^4$ is replace by $\xi^2 + \xi^3$.

The two facts discussed above and referred to as *reduction rules*, constitute a principal technical trick used almost throughout our paper. Once again, it should be emphasized that this method of proving the identities for elements of recurrence sequences "practically" has not been used in the literature. It is our hope that this paper shall contribute to the popularization of this method, because of its clarity and ease of operation (in contrast to the generating functions based method and Binet's formulae based method).

3. BASIC FORMULAE AND PROPERTIES

Lemma 3.1. *The following relations hold*:

a) $b_n(0) = 0$,

$$a_n(2) = 5^{\lfloor n/2 \rfloor}, \quad b_{2n}(2) = 0, \quad b_{2n+1}(2) = 2\cdot 5^n \quad \textit{and} \quad b_2(\delta) | b_{2n}(\delta). \tag{3.1}$$

b) *We have* (*for the proof see Corollary* 6.3) :

$$F_{k+1}^n\; a_n\Big(\frac{F_k}{F_{k+1}}\Big) = F_{kn+1} \qquad \textit{and} \qquad F_{k+1}^n\; b_n\Big(\frac{F_k}{F_{k+1}}\Big) = F_{kn}. \tag{3.2}$$

c)

$$b_{n+1}(\delta) - b_n(\delta) = \delta\big(a_n(\delta) - b_n(\delta)\big); \tag{3.3}$$

$$a_{n+k}(\delta) = a_k(\delta)a_n(\delta) + b_k(\delta)b_n(\delta); \tag{3.4}$$

$$b_{n+k}(\delta) = b_k(\delta)a_n(\delta) + \big[a_k(\delta) - b_k(\delta)\big]b_n(\delta); \tag{3.5}$$

$$a_{n+1}(\delta) - b_{n+1}(\delta) = \big[a_2(\delta) - b_2(\delta)\big]\big(a_{n-1}(\delta) - b_{n-1}(\delta)\big) + b_2(\delta)b_{n-1}(\delta) \tag{3.6}$$

$$= \big[(1-\delta)^2 + \delta^2\big]\big(a_{n-1}(\delta) - b_{n-1}(\delta)\big) + \delta(2-\delta)b_{n-1}(\delta) \tag{3.7}$$

and

$$\delta b_{n+1}(\delta) = (1-\delta)a_{n+1}(\delta) + (\delta^2 + \delta - 1)a_n(\delta). \tag{3.8}$$

d) *The recurrence formulae for elements of* $\{a_n(\delta)\}$*-as and* $\{b_n(\delta)\}$*-as*:

$$a_{n+2}(\delta) + (\delta - 2)a_{n+1}(\delta) + (1 - \delta - \delta^2)a_n(\delta) = 0 \tag{3.9}$$

and

$$b_{n+2}(\delta) + (\delta - 2)b_{n+1}(\delta) + (1 - \delta - \delta^2)b_n(\delta) = 0. \tag{3.10}$$

e) *For* $a, b, c, d \in \mathbb{C}$, $c + d(\xi + \xi^4) \neq 0$ *and* $d^2 - c^2 + cd \neq 0$:

$$\frac{a + b(\xi + \xi^4)}{c + d(\xi + \xi^4)} = \frac{bd + ad - ac}{d^2 - c^2 + cd} + \frac{ad - bc}{d^2 - c^2 + cd}\,(\xi + \xi^4). \tag{3.11}$$

Proof. a) We have

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^2 = 5 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, by (2.4) we get:

$$\begin{bmatrix} a_{2n+1}(2) \\ b_{2n+1}(2) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^{2n} \begin{bmatrix} a_1(2) \\ b_1(2) \end{bmatrix} = \begin{bmatrix} 5^n \\ 2 \cdot 5^n \end{bmatrix},$$

and

$$\begin{bmatrix} a_{2n}(2) \\ b_{2n}(2) \end{bmatrix} = \begin{bmatrix} 5^{n-1}\, a_2(2) \\ 5^{n-1}\, b_2(2) \end{bmatrix} = \begin{bmatrix} 5^n \\ 0 \end{bmatrix}.$$

Since $b_2(\delta) = \delta\,(2 - \delta)$, we have

$$b_2(\delta) \,\Big|\, b_{2n}(\delta) \quad \Longleftrightarrow \quad b_{2n}(0) = b_{2n}(2) = 0.$$

c) Identity (3.3) follows from (2.3).

For the proof of (3.4) and (3.5) first, we note that:

$$\big(1 + \delta\,(\xi + \xi^4)\big)^{m+n} = a_{m+n}(\delta) + b_{m+n}(\delta)\,(\xi + \xi^4). \tag{3.12}$$

Next, the following decomposition could be generated:

$$\big(1 + \delta\,(\xi + \xi^4)\big)^{m+n} = \big(1 + \delta\,(\xi + \xi^4)\big)^m \big(1 + \delta\,(\xi + \xi^4)\big)^n \tag{3.13}$$

$$= \big(a_m(\delta) + b_m(\delta)\,(\xi + \xi^4)\big)\,\big(a_n(\delta) + b_n(\delta)\,(\xi + \xi^4)\big) = a_m(\delta)\,a_n(\delta)$$
$$+\, b_m(\delta)\,b_n(\delta) + \big(a_m(\delta)\,b_n(\delta) + a_n(\delta)\,b_m(\delta) - b_m(\delta)\,b_n(\delta)\big)\,(\xi + \xi^4).$$

Comparing (3.12) with (3.13) we see that by reduction rules, identities (3.4) and (3.5) hold.

Identity (3.7) is derived from (3.4) and (3.5).

Next, by (2.2) and (2.3) again, we get:

$$\delta b_{n+1}(\delta) = \delta^2 a_n(\delta) + (1-\delta)\delta b_n(\delta) = \delta^2 a_n(\delta) + (1-\delta)\big(a_{n+1}(\delta) - a_n(\delta)\big)$$

which implies (3.8).

d) By (3.8) and (2.2) we obtain:

$$a_{n+2}(\delta) - a_{n+1}(\delta) = (1-\delta)a_{n+1}(\delta) + (\delta^2 + \delta - 1)a_n(\delta),$$

which implies (3.9). On the other hand, by (3.3) we obtain:

$$\begin{aligned} &b_{n+2}(\delta) + (\delta - 2)b_{n+1}(\delta) + (1 - \delta - \delta^2)b_n(\delta) = \\ &\quad = b_{n+2}(\delta) - b_{n+1}(\delta) + (\delta - 1)\big(b_{n+1}(\delta) - b_n(\delta)\big) - \delta^2 b_n(\delta) \\ &\quad = \delta\big(a_{n+1}(\delta) - b_{n+1}(\delta)\big) + (\delta - 1)\delta\big(a_n(\delta) - b_n(\delta)\big) - \delta^2 b_n(\delta) \\ &\quad = \delta\big[\big(a_{n+1}(\delta) - a_n(\delta) - \delta b_n(\delta)\big) + \big(\delta a_n(\delta) + (1-\delta)b_n(\delta) - b_{n+1}(\delta)\big)\big], \end{aligned}$$

which is equal to zero, by (2.2) and (2.3) again. □

The δ-Fibonacci numbers are described in an explicit form in the following:

Theorem 3.2. *The following decompositions hold* ($F_{k+1} = F_k + F_{k-1}$, $k \in \mathbb{Z}$);

(3.14)
$$a_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} F_{k-1}\,(-\delta)^k$$

and

(3.15)
$$b_n(\delta) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1}\,F_k\,\delta^k.$$

(*see also* (5.6) *and* (5.7) *where alternative formulae are given*).

Proof. Both formulae follow from (2.2) and (2.3). □

Corollary 3.3. *We have*:

(3.16)
$$\mathbb{A}_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} (-\delta)^k\,L_k,$$

where L_k denotes the k-th Lucas number.

Remark 3.4. The following formulae could easily be derived:

$$a_n'(\delta) = nb_{n-1}(\delta), \ \ b_n'(\delta) = n\big(a_{n-1}(\delta) - b_{n-1}(\delta)\big), \ \ \left(\frac{b_n(\delta)}{a_n(\delta)}\right)' = \frac{n(1-\delta-\delta^2)^{n-1}}{\big(a_n(\delta)\big)^2}.$$

Hence, the following relations could be deduced:

$$a_{2n}(\delta) > 0; \tag{3.17}$$

$$b_2(\delta) b_{2n}(\delta) \geqslant 0, \tag{3.18}$$

but, $b_{2n}(\delta) = 0 \Leftrightarrow b_2(\delta) = 0 \Leftrightarrow \delta = 0 \vee \delta = 2$;

$$a_{2n}(\delta) > b_{2n}(\delta); \tag{3.19}$$

$$b_{2n+1}(\delta) > b_{2n}(\delta) \quad \text{for} \ \ \delta > 0 \tag{3.20}$$

and

$$b_{2n+1}(\delta) < b_{2n}(\delta) \quad \text{for } \delta < 0; \tag{3.21}$$

$$a_{n+1}(\delta) > a_n(\delta) > 0 \quad \text{for } \delta < 0; \tag{3.22}$$

$$\delta b_{2n+1}(\delta) \geqslant 0, \tag{3.23}$$

but $b_{2n+1}(\delta) = 0 \Leftrightarrow \delta = 0$;

$$a_{2n+1}(\delta) > 0 \quad \text{for } \delta \in \left[\frac{\sqrt{5}-1}{2}, 2\right]. \tag{3.24}$$

4. BINET FORMULAE FOR δ-FIBONACCI NUMBERS

The characteristic equation of recurrence formulae (3.9) and (3.10) has the following decomposition:

$$\begin{aligned} \mathbb{X}^2 + (\delta-2)\,\mathbb{X} + (1-\delta-\delta^2) &= \left(\mathbb{X} - 1 + \delta\,\frac{1-\sqrt{5}}{2}\right)\left(\mathbb{X} - 1 + \delta\,\frac{1+\sqrt{5}}{2}\right) \\ &= \left(\mathbb{X} - 1 - 2\,\delta\,\cos\left(\frac{2\,\pi}{5}\right)\right)\left(\mathbb{X} - 1 + 2\,\delta\,\cos\left(\frac{\pi}{5}\right)\right). \end{aligned} \tag{4.1}$$

Hence, the following two identities follow:

Theorem 4.1 (Binet formulae for $a_n(\delta)$-as and $b_n(\delta)$-as). *We have*:

$$a_n(\delta) = \frac{5+\sqrt{5}}{10}\left(\frac{2-\delta+\sqrt{5}\,\delta}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{2-\delta-\sqrt{5}\,\delta}{2}\right)^n \tag{4.2}$$

and

$$b_n(\delta) = \frac{\sqrt{5}}{5}\left(\frac{2-\delta+\sqrt{5}\,\delta}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{2-\delta-\sqrt{5}\,\delta}{2}\right)^n \tag{4.3}$$

for every $n = 1, 2, \ldots$.

Corollary 4.2. *Using formulae* (4.2) *and* (4.3) *the following identities can be generated*:

$$(\sqrt{5}+1)\,a_n(\delta)+2\,b_n(\delta)=(\sqrt{5}+1)\left(\frac{2-\delta+\sqrt{5}\,\delta}{2}\right)^n, \tag{4.4}$$

$$2\,b_n(\delta)-(\sqrt{5}-1)\,a_n(\delta)=(1-\sqrt{5})\left(\frac{2-\delta-\sqrt{5}\,\delta}{2}\right)^n. \tag{4.5}$$

The next identity is derived by multiplying (4.4) *and* (4.5) :

$$b_n^2(\delta)-a_n^2(\delta)+a_n(\delta)\,b_n(\delta)=-(1-\delta-\delta^2)^n. \tag{4.6}$$

Furthermore from (4.2) *and* (4.3) *we get*:

$$a_n^2(\delta)+b_n^2(\delta)=a_{2n}(\delta), \tag{4.7}$$

and

$$5\,a_n(\delta)\,b_n(\delta)=a_{2n}(\delta)+2\,b_{2n}(\delta)-(1-\delta-\delta^2)^n. \tag{4.8}$$

Corollary 4.3. *We have*:

$$\begin{aligned} a_{n-k}(\delta)\,a_{n+k}(\delta)-a_n^2(\delta) &= b_n^2(\delta)-b_{n-k}(\delta)\,b_{n+k}(\delta)\\ &= b_k^2(\delta)\,(1-\delta-\delta^2)^{n-k} \quad (\textit{the Catalan's-type formula}),\\ &= \begin{cases} \delta^2\,(1-\delta-\delta^2)^{n-1} & \textit{for } k=1 \ (\textit{the Simson's-type formula}),\\ \delta^2\,(\delta^2-5\,\delta+5)\,(1-\delta-\delta^2)^{n-2} & \textit{for } k=2. \end{cases} \end{aligned} \tag{4.9}$$

Furthermore, the Tagiuri's-type formula [**3**] *holds*:

$$\begin{aligned} a_{n+k}(\delta)\,a_{n+\ell}(\delta)-a_n(\delta)\,a_{n+k+\ell}(\delta) &= b_n(\delta)\,b_{n+k+\ell}(\delta)-b_{n+k}(\delta)\,b_{n+\ell}(\delta)\\ &= -(1-\delta-\delta^2)^n\,b_k(\delta)\,b_\ell(\delta). \end{aligned}$$

Moreover, we note that the following expression:

$$a_{n-2}(\delta)\,a_{n-1}(\delta)\,a_{n+1}(\delta)\,a_{n+2}(\delta)-a_n^4(\delta)$$

is independent from $a_n^2(\delta)$ *iff* $\delta=0$ *or* $\delta=1$. *For* $\delta=1$ *we get the Gelin-Cesáro identity* (see [**3**, **5**, **8**]):

$$F_{n-2}\,F_{n-1}\,F_{n+1}\,F_{n+2}-F_n^4=-1.$$

Proof. Let us set $x_n(\delta)=r\,u^n+s\,v^n$, $n\in\mathbb{N}$. Then, we obtain:

$$x_n^2(\delta)-x_{n-k}(\delta)\,x_{n+k}(\delta)=-r\,s\,(u^k-v^k)^2\,(u\,v)^{n-1}.$$

Hence, using (4.2) and (4.3) the formula (4.9) follows. □

Corollary 4.4. *We have*:

$$5\,a_n^2(\delta) = 3\,a_{2n}(\delta) + b_{2n}(\delta) + 2\,(1-\delta-\delta^2)^n, \tag{4.10}$$

$$5\,b_n^2(\delta) = 2\,a_{2n}(\delta) - b_{2n}(\delta) - 2\,(1-\delta-\delta^2)^n, \tag{4.11}$$

$$5\,a_n^3(\delta) = 2\,a_{3n}(\delta) + b_{3n}(\delta) + 3\,(1-\delta-\delta^2)^n\,a_n(\delta), \tag{4.12}$$

$$5\,b_n^3(\delta) = b_{3n}(\delta) - 3\,(1-\delta-\delta^2)^n\,b_n(\delta), \tag{4.13}$$

and

$$5^k\,b_n^{2k} = \sum_{\ell=0}^{k}\binom{2k}{\ell}(-1)^\ell\,(1-\delta-\delta^2)^{\ell n}\,(2a_{2(k-\ell)n} - b_{2(k-\ell)n}). \tag{4.14}$$

Remark 4.5. From (3.14) and Corollaries 3.3 and 4.3 the following identities (for every $0 \le m \le 2n$) can be derived:

$$5\sum_{k=0}^{m}\binom{n}{k}\binom{n}{m-k}F_{k-1}\,F_{m-k-1} = \binom{2\,n}{m}L_{m-2} + 2\,\frac{(-1)^m}{m!}\,\frac{d^m}{d\delta^m}(1-\delta-\delta^2)^n\Big|_{\delta=0},$$

$$5\sum_{k=0}^{m}\binom{n}{k}\binom{n}{m-k}F_k\,F_{m-k} = \binom{2\,n}{m}L_m - 2\,\frac{(-1)^m}{m!}\,\frac{d^m}{d\delta^m}(1-\delta-\delta^2)^n\Big|_{\delta=0},$$

$$\sum_{k=0}^{m}\binom{n}{k}\binom{n}{m-k}L_k\,L_{m-k} = \binom{2\,n}{m}L_m + 2\,\frac{(-1)^m}{m!}\,\frac{d^m}{d\delta^m}(1-\delta-\delta^2)^n\Big|_{\delta=0},$$

$$\sum_{k=0}^{m}\binom{n}{k}\binom{n}{m-k}F_{k-1}\,L_{m-k} = \binom{2\,n}{m}F_{m-1} + \frac{(-1)^m}{m!}\,\frac{d^m}{d\delta^m}(1-\delta-\delta^2)^n\Big|_{\delta=0},$$

$$\sum_{k=0}^{m}\binom{n}{k}\binom{n}{m-k}F_k\,L_{m-k} = \binom{2\,n}{m}F_m.$$

Also, see papers [**1**, **2**] for interesting recurrence relations for the coefficients of polynomials $(1+\delta+\delta^2)^n$, $n \in \mathbb{N}$.

5. THE RELATIONSHIP BETWEEN δ-FIBONACCI NUMBERS FOR DIFFERENT VALUES OF δ's

Let us start with the following fundamental formulae, which make it possible to define numbers $a_n(\delta)$ and $b_n(\delta)$ by a given $a_n(\Delta)$ and $b_n(\Delta)$ with $\Delta \neq \delta$.

Theorem 5.1. *The following identities hold*:

$$\zeta^n a_n\!\left(\frac{\delta}{\zeta}\right) = \sum_{k=0}^{n}\binom{n}{k}(\zeta-1)^{n-k}a_k(\delta) \tag{5.1}$$

and

$$\zeta^n b_n\!\left(\frac{\delta}{\zeta}\right) = \sum_{k=0}^{n}\binom{n}{k}(\zeta-1)^{n-k}b_k(\delta) \tag{5.2}$$

for $\zeta \neq 1 \ \wedge \ \zeta \neq 0 \ \wedge \ \delta \neq 0$.

Proof. By (2.1) we have (for $\zeta \neq 0$):

$$(5.3) \quad \big(\zeta + \delta(\xi + \xi^4)\big)^n = \zeta^n \bigg(1 + \frac{\delta}{\zeta}(\xi + \xi^4)\bigg)^n = \zeta^n a_n\bigg(\frac{\delta}{\zeta}\bigg) + \zeta^n b_n\bigg(\frac{\delta}{\zeta}\bigg)(\xi + \xi^4).$$

On the other hand, we obtain:

$$\begin{aligned}\big(\zeta + \delta(\xi + \xi^4)\big)^n &= \big((\zeta - 1) + 1 + \delta(\xi + \xi^4)\big)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (\zeta - 1)^{n-k} \big(1 + \delta(\xi + \xi^4)\big)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (\zeta - 1)^{n-k} a_k(\delta) + \bigg(\sum_{k=0}^{n} \binom{n}{k} (\zeta - 1)^{n-k} b_k(\delta)\bigg)(\xi + \xi^4). \qquad (5.4)\end{aligned}$$

Using (5.3), (5.4) and the reduction rules identities (5.1) and (5.2) follow. □

Corollary 5.2. *For* $\zeta = 2$, $\delta = -2$ *we obtain* (*a new identity – as it seems*) :

$$(5.5) \qquad 2^n F_{2n-r} = \sum_{k=0}^{n} \binom{n}{k} F_{3k-r}, \qquad \text{for every } r \in \mathbb{Z}.$$

Corollary 5.3. *For* $\delta = 1$ *and* $\zeta = 1/\eta$ *we get*:

$$(5.6) \qquad a_n(\eta) = \sum_{k=0}^{n} \binom{n}{k} (1 - \eta)^{n-k} \eta^k F_{k+1}$$

and

$$(5.7) \qquad b_n(\eta) = \sum_{k=0}^{n} \binom{n}{k} (1 - \eta)^{n-k} \eta^k F_k.$$

Hence, for $\eta = -1$, *the following two known formulae follow*:

$$F_{2n-1} = 2^n \sum_{k=0}^{n} \binom{n}{k} \Big(-\frac{1}{2}\Big)^k F_{k+1}, \qquad -F_{2n} = 2^n \sum_{k=0}^{n} \binom{n}{k} \Big(-\frac{1}{2}\Big)^k F_k.$$

Remark 5.4. Comparing formulae (5.6) and (5.7) with formulae (3.14) and (3.15) respectively, two new interesting identities can be observed:

$$(5.8) \qquad \sum_{k=0}^{n} \binom{n}{k} (1 - \eta)^{n-k} \eta^k F_{k+1} = \sum_{k=0}^{n} \binom{n}{k} (-\eta)^k F_{k-1},$$

and

$$(5.9) \qquad \sum_{k=0}^{n} \binom{n}{k} (1 - \eta)^{n-k} \eta^k F_k = -\sum_{k=0}^{n} \binom{n}{k} (-\eta)^k F_k,$$

i.e.,

$$0 = \sum_{k=0}^{n} \binom{n}{k} \big[(1-\eta)^{n-k} + (-1)^k\big] \eta^k F_k. \tag{5.10}$$

Theorem 5.5. *The following four identities hold*:

$$a_n(\delta)\, F_{n+1} + b_n(\delta)\, F_n = (1+\delta)^n\, a_n\Big(\frac{1}{1+\delta}\Big), \tag{5.11}$$

$$a_n(\delta)\, F_n + b_n(\delta)\, F_{n-1} = (1+\delta)^n\, b_n\Big(\frac{1}{1+\delta}\Big), \tag{5.12}$$

$$F_{n+1}\, a_n(\delta) - F_{n+2}\, b_n(\delta) = (1-2\,\delta)^n\, a_n\Big(\frac{1-\delta}{1-2\delta}\Big) \tag{5.13}$$

and

$$F_n\, a_n(\delta) - F_{n+1}\, b_n(\delta) = (1-2\,\delta)^n\, b_n\Big(\frac{1-\delta}{1-2\delta}\Big). \tag{5.14}$$

Proof. Immediately from the identity ($\delta \neq -1$):

$$(1+\xi+\xi^4)\big(1+\delta(\xi+\xi^4)\big) = (1+\delta)\Big(1+\frac{1}{1+\delta}(\xi+\xi^4)\Big) \tag{5.15}$$

we obtain:

$$(1+\xi+\xi^4)^n\big(\big(1+\delta(\xi+\xi^4)\big)^n = (1+\delta)^n\Big(1+\frac{1}{1+\delta}(\xi+\xi^4)\Big)^n,$$

i.e.,

$$\begin{aligned}
&\big(F_{n+1} + F_n\,(\xi+\xi^4)\big)\big(a_n(\delta) + b_n(\delta)\,(\xi+\xi^4)\big) \\
&\qquad\qquad = (1+\delta)^n\Big(a_n\Big(\frac{1}{1+\delta}\Big) + b_n\Big(\frac{1}{1+\delta}\Big)\,(\xi+\xi^4)\Big), \\
&a_n(\delta)\, F_{n+1} + b_n(\delta)\, F_n + \big(a_n(\delta)\, F_n - b_n(\delta)\, F_n\big)\,(\xi+\xi^4) \\
&\qquad\qquad = (1+\delta)^n\, a_n\Big(\frac{1}{1+\delta}\Big) + (1+\delta)^n\, b_n\Big(\frac{1}{1+\delta}\Big)\,(\xi+\xi^4),
\end{aligned}$$

and, by applied the reduction rules identities (5.11) and (5.12) follow.

The following identity can be obtained in a similar way:

$$\begin{aligned}
(1+\xi+\xi^4)\,\big(1+\delta\,(\xi^2+\xi^3)\big) &= 1+\delta\,(\xi^2+\xi^3)+\xi+\xi^4-\delta \\
&= 1-2\,\delta+(1-\delta)(\xi+\xi^4) = (1-2\,\delta)\Big(1+\frac{1-\delta}{1-2\delta}(\xi+\xi^4)\Big).
\end{aligned} \tag{5.16}$$

Raising both sides of the above identity to the n-th power and applying (2.1), we obtain:

$$\big(F_{n+1} + F_n\,(\xi+\xi^4)\big)\big(a_n(\delta) + b_n(\delta)\,(\xi^2+\xi^3)\big) = \\ = (1-2\delta)^n\left(a_n\Big(\frac{1-\delta}{1-2\delta}\Big) + b_n\Big(\frac{1-\delta}{1-2\delta}\Big)\,(\xi+\xi^4)\right)$$

which, after some calculations, yields (5.3) and (5.14). □

Theorem 5.6. *We have*:

$$a_n(1-\varepsilon^2) = (1-\varepsilon)^n\Big(a_n(\varepsilon)\,a_n\Big(\frac{1}{1-\varepsilon}\Big) + b_n(\varepsilon)\,b_n\Big(\frac{1}{1-\varepsilon}\Big)\Big), \tag{5.17}$$

$$b_n(1-\varepsilon^2) = (1-\varepsilon)^n\left(a_n(\varepsilon)\,b_n\Big(\frac{1}{1-\varepsilon}\Big) + a_n\Big(\frac{1}{1-\varepsilon}\Big)\,b_n(\varepsilon) - b_n(\varepsilon)\,b_n\Big(\frac{1}{1-\varepsilon}\Big)\right), \tag{5.18}$$

$$(1-\varepsilon-\varepsilon\,\delta)^n\,a_n\Big(\frac{\varepsilon-\delta}{\varepsilon\,\delta+\varepsilon-1}\Big) = a_n(\delta)\,a_n(\varepsilon) - a_n(\delta)\,b_n(\varepsilon) - b_n(\delta)\,b_n(\varepsilon), \tag{5.19}$$

and

$$(1-\varepsilon-\varepsilon\,\delta)^n\,b_n\Big(\frac{\varepsilon-\delta}{\varepsilon\,\delta+\varepsilon-1}\Big) = a_n(\varepsilon)\,b_n(\delta) - a_n(\delta)\,b_n(\varepsilon). \tag{5.20}$$

Proof. Immediately from identity:

$$\big(1 + (1-\varepsilon^2)\,(\xi+\xi^4)\big) = \big(1+\varepsilon\,(\xi+\xi^4)\big)\big(1-\varepsilon+(\xi+\xi^4)\big),$$

we obtain:

$$a_n(1-\varepsilon^2) + b_n(1-\varepsilon^2)\,(\xi+\xi^4) \\ = (1-\varepsilon)^n\,\big(a_n(\varepsilon) + b_n(\varepsilon)\,(\xi+\xi^4)\big)\Big(a_n\Big(\frac{1}{1-\varepsilon}\Big) + b_n\Big(\frac{1}{1-\varepsilon}\Big)\,(\xi+\xi^4)\Big),$$

which, after easy calculations implies (5.17) and (5.18).

Next, we note that

$$\big(1+\mu\,(\xi+\xi^4)\big)^n\,\big(1+\delta\,(\xi^2+\xi^3)\big)^n = \big(1-\delta-\mu\,\delta + (\mu-\delta)\,(\xi+\xi^4)\big)^n \\ = (1-\delta-\mu\,\delta)^n\left(a_n\Big(\frac{\mu-\delta}{1-\delta-\mu\,\delta}\Big) + b_n\Big(\frac{\mu-\delta}{1-\delta-\mu\,\delta}\Big)\,(\xi+\xi^4)\right).$$

On the other hand, we get:

$$\big(1+\mu\,(\xi+\xi^4)\big)^n\,\big(1+\delta\,(\xi^2+\xi^3)\big)^n \\ = \big(a_n(\mu) + b_n(\mu)\,(\xi+\xi^4)\big)^n\,\big(a_n(\delta) + b_n(\delta)\,(\xi^2+\xi^3)\big)^n \\ = a_n(\mu)\,a_n(\delta) - b_n(\mu)\,b_n(\delta) - a_n(\mu)\,b_n(\delta) + \big(a_n(\delta)\,b_n(\mu) - a_n(\mu)\,b_n(\delta)\big)\,(\xi+\xi^4).$$

Comparing both of these decompositions by the reduction rules formulae (5.19) and (5.20) can be deduced. □

6. REDUCTION FORMULAE FOR INDICES

Theorem 6.1 *The following reduction formulae for indices hold*:

a) *Formulae* (3.4) *and* (3.5) *from Section* 3;

b)

$$a_{m-n}(\delta) = \frac{\begin{vmatrix} a_m(\delta)+b_m(\delta) & a_m(\delta) \\ a_n(\delta) & b_n(\delta) \end{vmatrix}}{\begin{vmatrix} a_n(\delta)+b_n(\delta) & a_n(\delta) \\ a_n(\delta) & b_n(\delta) \end{vmatrix}} \tag{6.1}$$

and

$$b_{m-n}(\delta) = \frac{\begin{vmatrix} a_m(\delta) & b_m(\delta) \\ a_n(\delta) & b_n(\delta) \end{vmatrix}}{\begin{vmatrix} a_n(\delta)+b_n(\delta) & a_n(\delta) \\ a_n(\delta) & b_n(\delta) \end{vmatrix}}; \tag{6.2}$$

c)

$$a_{m\cdot n}(\delta) = a_m^n(\delta)\, a_n\left(\frac{b_m(\delta)}{a_m(\delta)}\right), \tag{6.3}$$

$$b_{m\cdot n}(\delta) = a_m^n(\delta)\, b_n\left(\frac{b_m(\delta)}{a_m(\delta)}\right), \tag{6.4}$$

$$a_{m\cdot n\cdot p}(\delta) = a_m^{np}(\delta)\, a_n^p\left(\frac{b_m(\delta)}{a_m(\delta)}\right) a_p\left(\frac{b_n\left(\frac{b_m(\delta)}{a_m(\delta)}\right)}{a_n\left(\frac{b_m(\delta)}{a_m(\delta)}\right)}\right), \tag{6.5}$$

$$b_{m\cdot n\cdot p}(\delta) = a_m^{np}(\delta)\, a_n^p\left(\frac{b_m(\delta)}{a_m(\delta)}\right) b_p\left(\frac{b_n\left(\frac{b_m(\delta)}{a_m(\delta)}\right)}{a_n\left(\frac{b_m(\delta)}{a_m(\delta)}\right)}\right). \tag{6.6}$$

Proof. b) We have:

$$a_{m-n}(\delta) + b_{m-n}(\delta)\,(\xi+\xi^4) = \big(1+\delta\,(\xi+\xi^4)\big)^{m-n} = \\ = \frac{\big(1+\delta\,(\xi+\xi^4)\big)^{m}}{\big(1+\delta\,(\xi+\xi^4)\big)^{n}} = \frac{a_m(\delta)+b_m(\delta)\,(\xi+\xi^4)}{a_n(\delta)+b_n(\delta)\,(\xi+\xi^4)}$$

and formulae b) from Lemma 3.1 e) follows.

c) We have:

$$a_{m\cdot n}(\delta) + b_{m\cdot n}(\delta)\,(\xi+\xi^4) = \big(1+\delta\,(\xi+\xi^4)\big)^{m\cdot n} = \big(a_m(\delta)+b_m(\delta)\,(\xi+\xi^4)\big)^{n} \\ = a_m^n(\delta)\left(a_n\left(\frac{b_m(\delta)}{a_m(\delta)}\right) + b_n\left(\frac{b_m(\delta)}{a_m(\delta)}\right)(\xi+\xi^4)\right),$$

which yields identities (6.3) and (6.4). Similarly, identities (6.5) and (6.6) can be deduced. □

Corollary 6.2. *The following identity holds*:

$$\big(a_n(\delta) - b_n(\delta)\big)\, a_{n+k}(\delta) - b_n(\delta)\, b_{n+k}(\delta) = (1-\delta-\delta^2)^n\, a_k(\delta). \tag{6.7}$$

Proof. The identity follows from (3.4), (3.5) and (4.6). □

Corollary 6.3. *From* (6.3) – (6.6) *and* (3.2) *we get*, *for* $\delta = 1$:

$$F_{mn+1} = F_{m+1}^n\, a_n\left(\frac{F_m}{F_{m+1}}\right) \overset{(3.14)}{\iff} F_{mn+1} = \sum_{k=0}^{n} \binom{n}{k} F_{k-1} \big(-F_m\big)^k F_{m+1}^{n-k}, \tag{6.8}$$

$$F_{mn} = F_m^n\, b_n\left(\frac{F_m}{F_{m+1}}\right) \overset{(3.15)}{\iff} F_{mn} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} F_k\, F_m^k\, F_{m+1}^{n-k}, \tag{6.9}$$

$$F_{mnp+1} = F_{m+1}^{np}\, a_n^p\left(\frac{F_m}{F_{m+1}}\right) a_p\left(\frac{b_n\left(\dfrac{F_m}{F_{m+1}}\right)}{a_n\left(\dfrac{F_m}{F_{m+1}}\right)}\right), \tag{6.10}$$

$$F_{mnp} = F_m^{np}\, a_n^p\left(\frac{F_m}{F_{m+1}}\right) b_p\left(\frac{b_n\left(\dfrac{F_m}{F_{m+1}}\right)}{a_n\left(\dfrac{F_m}{F_{m+1}}\right)}\right), \tag{6.11}$$

and, *next*, *for* $\delta = -1$:

$$F_{2mn-1} = F_{2m-1}^n\, a_n\left(-\frac{F_{2m}}{F_{2m-1}}\right) \overset{(3.14)}{\iff} F_{2mn-1} = \sum_{k=0}^{n} \binom{n}{k} F_{k-1} F_{2m}^k\, F_{2m-1}^{n-k}, \tag{6.12}$$

$$-F_{2mn} = F_{2m-1}^n\, b_n\left(-\frac{F_{2m}}{F_{2m-1}}\right) \overset{(3.15)}{\iff} F_{2mn} = \sum_{k=0}^{n} \binom{n}{k} F_k\, F_{2m}^k\, F_{2m-1}^{n-k}, \tag{6.13}$$

etc.

Remark 6.4. All formulae from Corollary 6.3 are simultaneously generalizations and variations of the known Church and Bicknell's identities [**7**, p. 238].

Remark 6.5. Professor W. Webb asked on the 13th International Conference on Fibonacci Numbers and Their Applications in Patras (july 2008) on closed formulae for the sums

$$\sum_{k=1}^{N} F_{k^r},$$

where r is a fixed positive integer. Admittedly, formulae (6.9), (6,11), etc., do not imply such closed formulae, but allow for the reduction the problem to the sum of the powers of the Fibonacci numbers with indices not greater than N. For example, we have:

$$\sum_{k=1}^{N} F_{k^2} \overset{(6.9)}{=} \sum_{k=1}^{N} F_k^k\, b_k\left(\frac{F_k}{F_{k+1}}\right) = \sum_{k=1}^{N} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell-1} F_l\, F_k^{\ell}\, F_{k+1}^{k-\ell}, \tag{6.14}$$

$$\sum_{k=1}^{N} F_{k^3} \overset{(6.11)}{=} \sum_{k=1}^{N} F_k^{k^2}\, a_k^k\Big(\frac{F_k}{F_{k+1}}\Big) b_k\left(\frac{b_k\Big(\frac{F_k}{F_{k+1}}\Big)}{a_k\Big(\frac{F_k}{F_{k+1}}\Big)}\right). \tag{6.15}$$

REMARK 6.6. From (1.1) we obtain:

$$\big(1+\xi+\xi^4\big)^{n+1} + \big(1+\xi+\xi^4\big)^{n-1} = L_{n+1} + L_n\big(\xi+\xi^4\big),$$

i.e.,

$$\Big(1+2\big(\xi+\xi^4\big)\Big)\big(1+\xi+\xi^4\big)^{n} = L_{n+1} + L_n\big(\xi+\xi^4\big).$$

Hence, by (1.1) and (1.3) we get:

$$\Big(a_k(2)+b_k(2)\,\big(\xi+\xi^4\big)\Big)\Big(F_{kn+1}+F_{kn}\big(\xi+\xi^4\big)\Big) = \\ = L_{n+1}^k\left(a_k\Big(\frac{L_n}{L_{n+1}}\Big) + b_k\Big(\frac{L_n}{L_{n+1}}\Big)\big(\xi+\xi^4\big)\right),$$

which, after some manipulations, gives us two new formulae:

$$a_k(2)\,F_{kn+1} + b_k(2)\,F_{kn} = L_{n+1}^k a_k\Big(\frac{L_n}{L_{n+1}}\Big)$$

and

$$a_k(2)\,F_{kn} + b_k(2)\big(F_{kn+1}-F_{kn}\big) = a_k(2)\,F_{kn} + b_k(2)F_{kn-1} = L_{n+1}^k b_k\Big(\frac{L_n}{L_{n+1}}\Big).$$

Finally, taking into account equalities (3.1) we obtain:

$$5^k\,F_{2kn+1} = L_{n+1}^{2k}\,a_{2k}\Big(\frac{L_n}{L_{n+1}}\Big) = \sum_{r=0}^{2k}\binom{2\,k}{r} F_{r-1}\big(-L_n\big)^r L_{n+1}^{2k-r}, \tag{6.16}$$

$$5^k\,L_{(2k+1)n+1} = L_{n+1}^{2k+1}\,a_{2k+1}\Big(\frac{L_n}{L_{n+1}}\Big) = \sum_{r=0}^{2k+1}\binom{2\,k+1}{r} F_{r-1}\big(-L_n\big)^r L_{n+1}^{2k+1-r}, \tag{6.17}$$

$$5^k\,F_{2kn} = L_{n+1}^{2k}\,b_{2k}\Big(\frac{L_n}{L_{n+1}}\Big) = \sum_{r=0}^{2k}(-1)^{r-1}\binom{2\,k}{r} F_r L_n^r\,L_{n+1}^{2k-r}, \tag{6.18}$$

and

$$5^k\,L_{2kn} = L_{n+1}^{2k+1}\,b_{2k+1}\Big(\frac{L_n}{L_{n+1}}\Big) = \sum_{r=0}^{2k+1}(-1)^{r-1}\binom{2\,k+1}{r} F_r L_n^r\,L_{n+1}^{2k+1-r}. \tag{6.19}$$

7. SOME SUMMATION AND CONVOLUTION TYPE FORMULAE

Theorem 7.1. a) *Let $\mu, \delta \in \mathbb{C}$ and $N \in \mathbb{N}$. Then, we have:*

$$\begin{vmatrix} \mu\,\delta+\mu-1 & \mu-1 \\ \mu-1 & \mu\,\delta \end{vmatrix} \sum_{n=0}^{N-1} \mu^n\,a_n(\delta) \tag{7.1}$$

$$= (\delta^2 + \delta - 1)\,\mu^{N+1}\,a_{N-1}(\delta) + \mu^N\,a_N(\delta) + (\mu - \mu\,\delta - 1)$$

and

$$\begin{vmatrix} \mu\,\delta + \mu - 1 & \mu - 1 \\ \mu - 1 & \mu\,\delta \end{vmatrix} \sum_{n=0}^{N-1} \mu^n\,b_n(\delta) = \begin{vmatrix} \mu^N\,a_N(\delta) - 1 & \mu^N\,b_N(\delta) \\ \mu - 1 & \mu\,\delta \end{vmatrix}. \tag{7.2}$$

In the sequel, we obtain:

$$\delta \sum_{n=0}^{N-1} a_n(\delta) = a_N(\delta) + b_N(\delta) - 1 \tag{7.3}$$

and

$$\delta \sum_{n=0}^{N-1} b_n(\delta) = a_N(\delta) - 1. \tag{7.4}$$

b) *For every* $r, N \in \mathbb{N}$, *we have*:

$$\sum_{n=0}^{N-1} a_{rn}(\delta) = \frac{\begin{vmatrix} a_{rN}(\delta) + b_{rN}(\delta) - 1 & a_{rN}(\delta) - 1 \\ a_r(\delta) - 1 & b_r(\delta) \end{vmatrix}}{\begin{vmatrix} a_r(\delta) + b_r(\delta) - 1 & a_r(\delta) - 1 \\ a_r(\delta) - 1 & b_r(\delta) \end{vmatrix}} \tag{7.5}$$

and

$$\sum_{n=0}^{N-1} b_{rn}(\delta) = \frac{\begin{vmatrix} a_{rN}(\delta) - 1 & b_{rN}(\delta) \\ a_r(\delta) - 1 & b_r(\delta) \end{vmatrix}}{\begin{vmatrix} a_r(\delta) + b_r(\delta) - 1 & a_r(\delta) - 1 \\ a_r(\delta) - 1 & b_r(\delta) \end{vmatrix}}. \tag{7.6}$$

Proof. We have:

$$\sum_{n=0}^{N-1} \Big(1 + \delta\,\big(\xi + \xi^4\big)\Big)^{rn} = \frac{1 - \big(1 + \delta\,(\xi + \xi^4)\big)^{rN}}{1 - \big(1 + \delta\,(\xi + \xi^4)\big)^{r}} \tag{7.7}$$

$$= \frac{1 - a_{rN}(\delta) - b_{rN}(\delta)\,(\xi + \xi^4)}{1 - a_r(\delta) - b_r(\delta)\,(\xi + \xi^4)}$$

$$\overset{(3.11)}{=} \frac{b_r(\delta)\,\big(b_{rN}(\delta) + a_{rN}(\delta) - 1\big) - \big(a_r(\delta) - 1\big)\,\big(a_{rN}(\delta) - 1\big)}{b_r^2(\delta) - \big(1 - a_r(\delta)\big)^2 + b_r(\delta)\,\big(a_r(\delta) - 1\big)}$$

$$+ \frac{b_r(\delta)\,\big(a_{rN}(\delta) - 1\big) - b_{rN}(\delta)\,\big(a_r(\delta) - 1\big)}{b_r^2(\delta) - \big(1 - a_r(\delta)\big)^2 + b_r(\delta)\,\big(a_r(\delta) - 1\big)}\,(\xi + \xi^4).$$

On the other hand, we have:

$$\sum_{n=0}^{N-1}\Big(1+\delta\left(\xi+\xi^4\right)\Big)^{r\,n}=\sum_{n=0}^{N-1}\left(a_{rn}(\delta)+b_{rn}(\delta)\left(\xi+\xi^4\right)\right)$$
$$=\sum_{n=0}^{N-1}a_{rn}(\delta)+(\xi+\xi^4)\sum_{n=0}^{N-1}b_{rn}(\delta),$$

which, by (7.7), implies (7.5) and (7.6). □

Theorem 7.2. *Let* $\mu,\delta,\omega\in\mathbb{C}$ *and* $k,\ell,N\in\mathbb{N}$. *Then, we have*:

a) $$2^N\,a_N\Big(\frac{1}{2}(\mu+\delta)\Big)=\sum_{n=0}^{N}\binom{N}{n}\big(a_n(\mu)\,a_{N-n}(\delta)+b_n(\mu)\,b_{N-n}(\delta)\big),$$

$$2^N\,b_N\Big(\frac{1}{2}(\mu+\delta)\Big)=\sum_{n=0}^{N}\binom{N}{n}\big(a_n(\mu)\,b_{N-n}(\delta)+a_{N-n}(\delta)\,b_n(\mu)-b_n(\mu)\,b_{N-n}(\delta)\,\big);$$

b) $$5\,\delta\sum_{n=0}^{N}a_n(\delta)\,a_{N-n}(\delta)=(N+1)\,\delta\,\big(3\,a_N(\delta)+b_N(\delta)\big)+2\,b_{N+1}(\delta),$$
$$5\,\delta\sum_{n=0}^{N}b_n(\delta)\,b_{N-n}(\delta)=(N+1)\,\delta\,\big(2\,a_N(\delta)-b_N(\delta)\big)-2\,b_{N+1}(\delta),$$
$$5\,\delta\sum_{n=0}^{N}a_n(\delta)\,b_{N-n}(\delta)=(N+1)\,\delta\,\big(a_N(\delta)+2\,b_N(\delta)\big)-b_{N+1}(\delta).$$

c) *In addition, if* $\omega+a_\ell(\delta)\neq 0$,

$$\sum_{n=0}^{N}\binom{N}{n}\omega^{N-n}\,a_{n\,\ell+k}(\delta)=\big(\omega+a_\ell(\delta)\big)^N\left(a_k(\delta)\,a_N\Big(\frac{b_\ell(\delta)}{\omega+a_\ell(\delta)}\Big)+b_k(\delta)\,b_N\Big(\frac{b_\ell(\delta)}{\omega+a_\ell(\delta)}\Big)\right),\tag{7.8}$$

$$\sum_{n=0}^{N}\binom{N}{n}\omega^{N-n}\,b_{n\,\ell+k}(\delta)=\big(\omega+a_\ell(\delta)\big)^N\left[a_k(\delta)\,b_N\Big(\frac{b_\ell(\delta)}{\omega+a_\ell(\delta)}\Big)+b_k(\delta)\left(a_N\Big(\frac{b_\ell(\delta)}{\omega+a_\ell(\delta)}\Big)-b_N\Big(\frac{b_\ell(\delta)}{\omega+a_\ell(\delta)}\Big)\right)\right].\tag{7.9}$$

Proof. c) First, we note that

$$S(\delta;\omega):=\big(1+\delta\,(\xi+\xi^4)\big)^k\,\big(\omega+\big(1+\delta\,(\xi+\xi^4)\big)^\ell\big)^N$$
$$=\big(a_k(\delta)+b_k(\delta)\,(\xi+\xi^4)\big)\,\big(\omega+a_\ell(\delta)+b_\ell(\delta)\,(\xi+\xi^4)\big)^N$$

$$
\begin{aligned}
&= \big(\omega + a_\ell(\delta)\big)^N \bigg[\big(a_k(\delta) + b_k(\delta)\,(\xi+\xi^4)\big)\bigg(a_N\Big(\frac{b_\ell(\delta)}{\omega + a_\ell(\delta)}\Big) \\
&\qquad\qquad + b_N\Big(\frac{b_\ell(\delta)}{\omega + a_\ell(\delta)}\Big)(\xi+\xi^4)\bigg)\bigg] \\
&= \big(\omega + a_\ell(\delta)\big)^N \bigg[a_k(\delta)\, a_N\Big(\frac{b_\ell(\delta)}{\omega + a_\ell(\delta)}\Big) + b_k(\delta)\, b_N\Big(\frac{b_\ell(\delta)}{\omega + a_\ell(\delta)}\Big)\bigg] \\
&\qquad\qquad + \big(\omega + a_\ell(\delta)\big)^N \bigg[a_k(\delta)\, b_N\Big(\frac{b_\ell(\delta)}{\omega + a_\ell(\delta)}\Big) \\
&\qquad\qquad + b_k(\delta)\bigg(a_N\Big(\frac{b_\ell(\delta)}{\omega + a_\ell(\delta)}\Big) - b_N\Big(\frac{b_\ell(\delta)}{\omega + a_\ell(\delta)}\Big)\bigg)\bigg]\,(\xi+\xi^4).
\end{aligned}
$$

On the other hand, by an application of the binomial formula to $S(\delta;\omega)$, we get

$$
\begin{aligned}
S(\delta;\omega) &= \sum_{n=0}^{N}\binom{N}{n}\,\omega^{N-n}\,\big(1+\delta\,(\xi+\xi^4)\big)^{n\,\ell+k} \\
&= \sum_{n=0}^{N}\binom{N}{n}\,\omega^{N-n}\,a_{n\,\ell+k}(\delta) + (\xi+\xi^4)\sum_{n=0}^{N}\binom{N}{n}\,\omega^{N-n}\,b_{n\,\ell+k}(\delta).
\end{aligned}
$$

Comparing two decompositions of $S(\delta;\omega)$ formulae c) follows. □

Remark 7.3. The formulae c) of Lemma 7.2 are, in some sense, the generalizations of the following variations of the de Vries' identities (see [**11**], where only the case of $\omega = \ell = 1$ is discussed):

$$
\sum_{n=0}^{N}\binom{N}{n}\,\omega^{N-n}\,F_{n\ell+i} = \frac{\sqrt{5}}{5}\,\big(\alpha^i\,(\omega+\alpha^\ell)^N - \beta^i\,(\omega+\beta^\ell)^N\big), \tag{7.10}
$$

$$
\sum_{n=0}^{N}\binom{N}{n}\,\omega^{N-n}\,L_{n\ell+i} = \alpha^i\,(\omega+\alpha^\ell)^N + \beta^i\,(\omega+\beta^\ell)^N, \tag{7.11}
$$

where α and β in Lemma 2.1 are defined. The proof of (7.10) and (7.11) can be obtained by an immediate application of the Binet's formula for the Fibonacci and Lucas numbers.

Moreover, from (7.8) and (7.9) both for $\delta = 1$ and from (7.10) and (7.11) we deduce the formulae:

$$
\begin{aligned}
\Big(1+\frac{F_{\ell+1}}{\omega}\Big)^N \bigg(F_i\, a_N\Big(\frac{F_\ell}{\omega+F_{\ell+1}}\Big) + F_{i-1}\, b_N\Big(\frac{F_\ell}{\omega+F_{\ell+1}}\Big)\bigg) \\
= \frac{\sqrt{5}}{5}\,\omega^{-N}\,\big(\alpha^i\,(\omega+\alpha^\ell)^N - \beta^i\,(\omega+\beta^\ell)^N\big),
\end{aligned} \tag{7.12}
$$

$$
\begin{aligned}
\Big(1+\frac{F_{\ell+1}}{\omega}\Big)^N \bigg(L_i\, a_N\Big(\frac{F_\ell}{\omega+F_{\ell+1}}\Big) + L_{i-1}\, b_N\Big(\frac{F_\ell}{\omega+F_{\ell+1}}\Big)\bigg) \\
= \omega^{-N}\,\Big(\alpha^i\,(\omega+\alpha^\ell)^N + \beta^i\,(\omega+\beta^\ell)^N\Big),
\end{aligned} \tag{7.13}
$$

which implies the following relations:

$$(-2)^i \left(\omega + F_{\ell+1}\right)^N a_N\Big(\frac{F_\ell}{\omega + F_{\ell+1}}\Big) = \Big(F_{i-1} - \frac{\sqrt{5}}{5}\,L_{i-1}\Big)\,\alpha^i \left(\omega + \alpha^\ell\right)^N + \Big(F_{i-1} + \frac{\sqrt{5}}{5}\,L_{i-1}\Big)\,\beta^i \left(\omega + \beta^\ell\right)^N, \tag{7.14}$$

$$(-2)^{i-1} \left(\omega + F_{\ell+1}\right)^N b_N\Big(\frac{F_\ell}{\omega + F_{\ell+1}}\Big) = \Big(F_i - \frac{\sqrt{5}}{5}\,L_i\Big)\,\alpha^i \left(\omega + \alpha^\ell\right)^N + \Big(F_i + \frac{\sqrt{5}}{5}\,L_i\Big)\,\beta^i \left(\omega + \beta^\ell\right)^N. \tag{7.15}$$

At the end of this section, two convolution identities will be presented.

Theorem 7.4. *We have*:

$$1 + \sum_{k=0}^{n} \binom{n}{k}\,\delta^{n-k}\,b_k(\delta)\,F_{n-k} = \sum_{k=0}^{n} \binom{n}{k}\,\delta^{n-k}\,a_k(\delta)\,F_{n-k-1}, \tag{7.16}$$

$$\sum_{k=0}^{n} \binom{n}{k}\,\delta^{n-k}\,b_k(\delta)\,F_{n-k+1} = \sum_{k=0}^{n} \binom{n}{k}\,\delta^{n-k}\,a_k(\delta)\,F_{n-k}. \tag{7.17}$$

Final remarks. As noticed by one of the reviewers, identities (1.1) and (1.2) have a corresponding form for a general second order recurrence $u_0 = 0$, $u_1 = 1$, $u_{n+1} = A\,u_n + B\,u_{n-1}$, $n \in \mathbb{N}$. Then we have $\lambda^n = u_n\,\lambda + B\,u_{n-1}$ and $\widehat{\lambda}^n = u_n\,\widehat{\lambda} + B\,u_{n-1}$, where $\lambda = (A + \sqrt{A^2+4B}\,)/2$ and $\widehat{\lambda} = (A - \sqrt{A^2+4B}\,)/2$.

Acknowledgments. The authors wish to express their gratitude to the Reviewers for several helpful comments concerning the first version of the paper.

REFERENCES

1. L. Carlitz: *Some combinatorial identities.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., **461–497** (1974), 87–90.
2. D. M. Cvetković, S. K. Simić: *On enumeration of certain types of sequences.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., **412–460** (1973), 159–164.
3. S. Fairgrieve, H. W. Gould: *Product difference Fibonacci identities of Simson, Gelin-Cesaro, Tagiuri and generalizations.* Fibonacci Quart.,**43** (2005) 137–141.
4. R. Grzymkowski, R. Wituła: *Calculus Methods in Algebra, Part One.* WPKJS, Gliwice, 2000 (in Polish).
5. A. F. Horadam, A. G. Shannon: *Generalization of identities of Catalan and others.* Port. Math., **44** (1987) 137–148.
6. R. Knott: *Fibonacci and golden ratio formulae.* http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibFormulae.pdf.
7. T. Koshy: *Fibonacci and Lucas Numbers with Application.* Wiley, New York, 2001.
8. J. Morgado: *Note on some results of A. F. Horadam and A. G. Shannon concerning a Catalan's identity on Fibonacci numbers.* Port. Math., **44** (1987) 243–252.

9. S. Rabinowitz: *Algorithmic manipulation of Fibonacci identities.* In: Applications of Fibonacci Numbers, Vol. 6, eds. G. E. Bergum et al. (Kluwer, 1996), pp. 389–408.
10. N. J. A. Sloane: *The On-Line Encyclopedia of Integer Sequences.* http://www.research.att.com/~njas/sequences/, 2009.
11. P. C. G. de Vries: Problem 445, Nieuw Arch. Wiskd.
12. R. Wituła, D. Słota, A. Warzyński: *Quasi-Fibonacci numbers of seventh order.* J. Integer Seq., **9** (2006), Article 06.4.3, 1–28.
13. R. Wituła, D. Słota: *New Ramanujan-type formulas and quasi-Fibonacci numbers of order* 7. J. Integer Seq., **10** (2007), Article 07.5.6 1–29.
14. R. Wituła, D. Słota: *Quasi-Fibonacci numbers of order* 11. J. Integer Seq., **10** (2007), Article 07.8.5, 1–29.

Institute of Mathematics,
Silesian University of Technology,
Kaszubska 23,
44-100 Gliwice,
Poland
E–mail: d.slota@polsl.pl

(Received September 14, 2008)
(Revised May 15, 2009)