SOME FORMULAS FOR APOSTOL-EULER POLYNOMIALS ASSOCIATED WITH HURWITZ ZETA FUNCTION AT RATIONAL ARGUMENTS

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We give some explicit relationships between the Apostol-Euler polynomials and generalized Hurwitz-Lerch Zeta function and obtain some series representations of the Apostol-Euler polynomials of higher order in terms of the generalized Hurwitz-Lerch Zeta function. Several interesting special cases are also shown.

1. INTRODUCTION

Throughout this paper, we always make use of the following notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and Euler polynomials $E_n^{(\alpha)}(x)$ of order α (real or complex) are usually defined by means of the following generating functions (see, for details, [**1**, **5**, **13**, **15**]):

$$(1.1) \qquad \left(\frac{z}{e^z - 1}\right)^\alpha e^{xz} = \sum_{n=0}^\infty B_n^{(\alpha)}(x) \frac{z^n}{n!} \qquad (|z| < 2\pi)$$

and

$$(1.2) \qquad \left(\frac{2}{e^z + 1}\right)^\alpha e^{xz} = \sum_{n=0}^\infty E_n^{(\alpha)}(x) \frac{z^n}{n!} \qquad (|z| < \pi).$$

Obviously, the classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by

$$(1.3) \qquad B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \quad (n \in \mathbb{N}_0),$$

respectively. The classical Bernoulli numbers B_n and Euler numbers E_n are defined by

$$(1.4) \qquad B_n := B_n(0) \quad \text{and} \quad E_n := 2^n E_n\left(\frac{1}{2}\right) \quad (n \in \mathbb{N}_0),$$

respectively.

Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by APOSTOL [**2**, p. 165, Eq. (3.1)] and (more recently) by SRIVASTAVA [**14**, p 83–84]. We begin by recalling Apostol's definitions as follows:

Definition 1.1 (APOSTOL [**2**]; see also SRIVASTAVA [**14**]). *The Apostol-Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$ in x are defined by means of the generating function*:

$$(1.5) \quad \frac{z e^{xz}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{z^n}{n!} \quad (|z| < 2\pi \text{ when } \lambda = 1; \ |z| < |\log \lambda| \text{ when } \lambda \neq 1)$$

with, of course,

$$(1.6) \qquad B_n(x) = \mathcal{B}_n(x; 1) \quad and \quad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0; \lambda),$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers (in fact, it is a function in λ).

Recently, LUO and SRIVASTAVA extended further the Apostol-Bernoulli and Apostol-Euler polynomials and their generalizations as follows:

Definition 1.2 (*cf.* LUO and SRIVASTAVA [**10**, **12**]). *The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x; \lambda)$ of order α are defined by means of the generating function*

$$(1.7) \qquad \left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x; \lambda) \frac{z^n}{n!}$$

$$(|z| < 2\pi \text{ when } \lambda = 1; \ |z| < |\log \lambda| \text{ when } \lambda \neq 1)$$

with, of course,

$$(1.8) \qquad \begin{aligned} B_n^{(\alpha)}(x) &= \mathcal{B}_n^{(\alpha)}(x; 1) & and & \quad \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0; \lambda), \\ \mathcal{B}_n(x; \lambda) &:= \mathcal{B}_n^{(1)}(x; \lambda) & and & \quad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0; \lambda), \end{aligned}$$

where $\mathcal{B}_n(\lambda)$, $\mathcal{B}_n^{(\alpha)}(\lambda)$ and $\mathcal{B}_n(x; \lambda)$ denote the so-called Apostol-Bernoulli numbers, Apostol-Bernoulli numbers of order α and Apostol-Bernoulli polynomials respectively.

Definition 1.3 (*cl.* LUO [**11**]. *The Apostol-Euler polynomials* $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ *of order* α *are defined by means of the generating function*

$$(1.9) \qquad \left(\frac{2}{\lambda e^z + 1}\right)^\alpha e^{xz} = \sum_{n=0}^\infty \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \qquad (|z| < |\log(-\lambda)|),$$

with, of course,

$$(1.10) \qquad \begin{aligned} E_n^{(\alpha)}(x) &= \mathcal{E}_n^{(\alpha)}(x;1) \qquad and \qquad \mathcal{E}_n^{(\alpha)}(\lambda) := 2^n \mathcal{E}_n^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right), \\ \mathcal{E}_n(x;\lambda) &:= \mathcal{E}_n^{(1)}(x;\lambda) \qquad and \qquad \mathcal{E}_n(\lambda) := 2^n \mathcal{E}_n\left(\frac{\alpha}{2};\lambda\right), \end{aligned}$$

where $\mathcal{E}_n(\lambda)$, $\mathcal{E}_n^{(\alpha)}(\lambda)$ *and* $\mathcal{E}_n(x;\lambda)$ *denote the so-called Apostol-Euler numbers, Apostol-Euler numbers of order* α *and Apostol-Euler polynomials respectively.*

The main object of the present paper is to give some explicit relationships between the Apostol-Euler polynomials and generalized Hurwitz-Lerch Zeta function and to investigate some series representations of the Apostol-Euler polynomials in terms of generalized Hurwitz-Lerch Zeta function.

2. SOME EXPLICIT RELATIONSHIPS BETWEEN THE APOSTOL-EULER POLYNOMIALS AND THE GENERALIZED HURWITZ-LERCH ZETA FUNCTION

A family of the Hurwitz-Lerch Zeta function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ defined by (see *e.g.* [**9**, p. 727, Eq. (8)]

$$(2.1) \qquad \Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) := \sum_{n=0}^\infty \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^n}{(n+a)^s}$$

$$(\mu \in \mathbb{C}; \ a,\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-; \ \rho,\sigma \in \mathbb{R}^+; \ \rho < \sigma \quad when \quad s,z \in \mathbb{C};$$
$$\rho = \sigma \text{ and } s \in \mathbb{C} \text{ when } |z| < 1; \ \rho = \sigma \quad and \quad \Re(s-\mu+\nu) > 1 \text{ when } |z| = 1),$$

contains, as its *special* cases, not only the Hurwitz-Lerch Zeta function

$$(2.2) \qquad \Phi_{\nu,\nu}^{(\sigma,\sigma)}(z,s,a) = \Phi_{\mu,\nu}^{(0,0)}(z,s,a) = \Phi(z,s,a) = \sum_{n=0}^\infty \frac{z^n}{(n+a)^s}$$

and the Lipschitz-Lerch Zeta function (*cf.* [**15**, p. 122, Eq. 2.5 (11)]):

$$(2.3) \qquad \phi(\xi,a,s) := \Phi\left(e^{2\pi i\xi},s,a\right) = \sum_{n=0}^\infty \frac{e^{2n\pi i\xi}}{(n+a)^s}$$

$$\left(a \in \mathbb{C}\setminus\mathbb{Z}_0^-; \ \Re(s) > 0 \quad when \quad \xi \in \mathbb{R}\setminus\mathbb{Z}; \ \Re(s) > 1 \quad when \quad \xi \in \mathbb{Z}\right),$$

but also the following generalized Hurwitz-Lerch Zeta functions introduced and studied earlier by GOYAL and LADDHA [**7**, p. 100, Eq. (1.5)]

$$(2.4) \qquad \Phi_{\mu,1}^{(1,1)}(z,s,a) = \Phi_\mu(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s},$$

which, for convenience, are called the *Goyal-Laddha-Hurwitz-Lerch Zeta function.* Here the symbol $(a)_k$ denotes the Pochhammer symbol or the shifted factorial defined, $a \in \mathbb{C}$, by

$$(2.5) \qquad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & (k=0), \\ a(a+1)\cdots(a+k-1) & (k \in \mathbb{N}), \end{cases}$$

where $\Gamma(x)$ is the usual Gamma function.

Recently, GARG *et al.* [**6**] obtained the following interesting formula:

$$(2.6) \quad \mathcal{B}_n^{(\ell)}(a;\lambda) = (-n)_\ell \Phi_\ell(\lambda, \ell-n, a) \quad (n, \ell \in \mathbb{N}; \ n \geqq \ell; \ |\lambda| < 1; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

Clearly, we have

$$(2.7) \qquad \mathcal{B}_n(a;\lambda) = -n\Phi(\lambda, 1-n, a) \quad (n \in \mathbb{N}; \ |\lambda| \leqq 1; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

Below we give the following explicit relationships between the a family of Euler polynomials and a family of Zeta function.

Theorem 2.1. *For $n \in \mathbb{N}$; $-1 < \lambda \leqq 1$; $\alpha \in \mathbb{C}$; $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the following relationship*

$$(2.8) \qquad \mathcal{E}_n^{(\alpha)}(a;\lambda) = 2^\alpha \Phi_\alpha(-\lambda, -n, a)$$

between the Apostol-Euler polynomials of higher order and the Goyal-Laddha-Hurwitz-Lerch Zeta function.

Proof. By (1.9) and the generalized binomial theorem, yields

$$(2.9) \qquad \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(a;\lambda) \frac{z^n}{n!} = \left(\frac{2}{\lambda e^z + 1}\right)^\alpha e^{az} = 2^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}(-\lambda)^k e^{(k+a)z}$$

$$= \sum_{n=0}^{\infty} \left[2^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}(-\lambda)^k (k+a)^n \right] \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[2^\alpha \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{(-\lambda)^k}{(k+a)^{-n}} \right] \frac{z^n}{n!}.$$

Hence, the formula (2.8) follows. $\qquad \square$

Corollary 2.2. *For $n \in \mathbb{N}$; $-1 < \lambda \leqq 1$; $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the following relationship*

$$(2.10) \qquad \mathcal{E}_n(a;\lambda) = 2\Phi(-\lambda, -n, a)$$

holds true between the Apostol-Euler polynomials and the Hurwitz-Lerch Zeta function.

It is well-known that the following relationship between the Bernoulli polynomials and the Hurwitz Zeta function (see Apostol [**3**, p. 264, Theorem 12.13])

$$(2.11) \qquad\qquad B_n(a) = -n\zeta(1-n, a) \qquad (n \in \mathbb{N}),$$

where $\zeta(s, a)$ denotes the Hurwitz Zeta function defined by

$$\zeta(s, a) := \Phi(1, s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (\Re(s) > 1; \; a \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

An alternating series version of the Hurwitz Zeta function is given as follows:

Definition 2.3. *The L-function is defined by*

$$(2.12) \qquad L(s, a) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+a)^s} \qquad \left(\Re(s) > 1; \; a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

In the same method, it is not difficult, we give a quasi formula of (2.11) as follows:

Theorem 2.4. *For $n \in \mathbb{N}$; $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, the following relationship*

$$(2.13) \qquad\qquad E_n(a) = 2L(-n, a)$$

holds true between the Euler polynomials and the L-function.

It is well-known that the following relationship between the Bernoulli numbers and the Riemann Zeta function (see [**3**, p. 266, Theorem 12.16])

$$(2.14) \qquad\qquad B_n = -n\zeta(1-n) \qquad (n \in \mathbb{N}),$$

where $\zeta(s)$ denotes the Riemann Zeta function defined by

$$\zeta(s) := \zeta(s, 1) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

An alternating series version of the Riemann Zeta function is given as follows:

Definition 2.5. *For $\Re(s) > 0$, the ℓ-function is defined by*

$$(2.15) \qquad\qquad \ell(s) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.$$

Similarly, we give an analogue of the formula (2.14) as follows:

Theorem 2.6. *For $n \in \mathbb{N}$, the following relationship*

$$(2.16) \qquad\qquad E_n = 2\ell(-n)$$

holds true between the Euler numbers and the ℓ-function.

3. EXPLICIT SERIES REPRESENTATIONS FOR THE APOSTOL-EULER POLYNOMIALS OF ORDER α

It is not difficult, we make use of the elementary series identity:

$$(3.1) \qquad \sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{q} \sum_{k=0}^{\infty} f(qk+j), \qquad (q \in \mathbb{N}),$$

to the Hurwitz-Lerch Zeta function (2.2), yields that

$$(3.2) \qquad \Phi(z, s, a) = q^{-s} \sum_{j=1}^{q} \Phi\left(z^q, s, \frac{a+j-1}{q}\right) z^{j-1}.$$

Obviously, a special case of (3.2) when

$$z = \exp\left(\frac{2p\pi i}{q}\right) \qquad (p \in \mathbb{Z}, \ q \in \mathbb{N})$$

is the following summation formula for the Lipschitz-Lerch Zeta function $\phi(\xi, a, s)$ defined by (2.3):

$$(3.3) \qquad \Phi\left(\exp\left(\frac{2p\pi i}{q}\right), s, a\right) = \phi\left(\frac{p}{q}, a, s\right)$$
$$= q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{a+j-1}{q}\right) \exp\left(\frac{2(j-1)p\pi i}{q}\right),$$

in terms of the Hurwitz Zeta function $\zeta(s, a)$.

Theorem 3.1. *For $n, q \in \mathbb{N}$; $p \in \mathbb{Z}$; $\xi \in \mathbb{R}$, $\alpha \in \mathbb{C}$, the following formula of the Apostol-Euler polynomials of order α*

$$(3.4) \quad \mathcal{E}_n^{(\alpha)}\left(\frac{p}{q}; e^{2\pi i \xi}\right) = \frac{i\, 2^{\alpha} \cdot n!}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \binom{\alpha-1}{k} \left(k - \frac{p}{q} + 1\right)_{\alpha-k-1}$$

$$\times \sum_{j=0}^{k} \binom{k-1}{j-1} (n+1)_j B_{k-j}^{(k)}$$

$$\times (2\pi q)^{-n-j-1} \Bigg\{ \sum_{r=1}^{q} \zeta\left(n+j+1, \frac{2\xi+2r-1}{2q}\right)$$

$$\times \exp\left[\left(\frac{n+j}{2} - \frac{(2\xi+2r-1)p}{2}\right) \pi i\right] - \sum_{r=1}^{q} \zeta\left(n+j+1, \frac{2r-2\xi+1}{2q}\right)$$

$$\times \exp\left[\left(-\frac{n+j}{2} + \frac{(2r-2\xi+1)p}{2}\right) \pi i\right] \Bigg\},$$

holds true in terms of the Hurwitz Zeta function.

Proof. We now rewrite the result of LIN *et al.* as follows (see [**8**, p. 823, Theorem]):

$$(3.5) \quad \Phi_\mu(z, s, a) = iz^{-a} \Gamma(1-s) \sum_{k=0}^{\infty} \frac{(k-a+1)_{\mu-k-1}}{k! \Gamma(\mu-k)}$$

$$\times \sum_{j=0}^{k} \binom{k-1}{j-1} (1-s)_j B_{k-j}^{(k)} (2\pi)^{s-j-1}$$

$$\times \left[\exp\left(-\frac{1}{2}(s-j)\pi i\right) \Phi\left(e^{-2\pi ai}, 1-s+j, \frac{\log z}{2\pi i}\right) \right.$$

$$\left. - \exp\left[\left(2a + \frac{1}{2}(s-j)\right)\pi i\right] \Phi\left(e^{2\pi ai}, 1-s+j, 1-\frac{\log z}{2\pi i}\right) \right] \quad (\mu \in \mathbb{C}).$$

Setting

$$z = -e^{2\pi i\xi}, \quad a = \frac{p}{q} \quad \text{and} \quad s \mapsto -s, \quad \mu \mapsto \alpha$$

and by applying the series identity (3.3), we find that

$$(3.6) \quad \Phi_\alpha\left(-e^{2\pi i\xi}, -s, \frac{p}{q}\right) = \frac{i\Gamma(s+1)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \binom{\alpha-1}{k} \left(k - \frac{p}{q} + 1\right)_{\alpha-k-1}$$

$$\times \sum_{j=0}^{k} \binom{k-1}{j-1} (s+1)_j B_{k-j}^{(k)}$$

$$\times (2\pi q)^{-s-j-1} \left\{ \sum_{r=1}^{q} \zeta\left(s+j+1, \frac{2\xi+2r-1}{2q}\right) \right.$$

$$\times \exp\left[\left(\frac{s+j}{2} - \frac{(2\xi+2r-1)p}{q}\right)\pi i\right] - \sum_{r=1}^{q} \zeta\left(s+j+1, \frac{2r-2\xi+1}{2q}\right)$$

$$\left. \times \exp\left[\left(-\frac{s+j}{2} + \frac{(2r-2\xi+1)p}{q}\right)\pi i\right] \right\} \quad (\alpha \in \mathbb{C}).$$

Taking $s = n$ in (3.6) and noting that (2.8) of Theorem 2.1, of course, with

$$\lambda = e^{2\pi i\xi} \quad \text{and} \quad a = \frac{p}{q},$$

we obtain the desire (3.4). This proof is complete. □

Theorem 3.2. *For $n, q, \ell \in \mathbb{N}$; $p \in \mathbb{Z}$; $\xi \in \mathbb{C}$, the following formula of the Apostol-Euler polynomials of order ℓ*

$$(3.7) \quad \mathcal{E}_n^{(\ell)}\left(\frac{p}{q}; e^{2\pi i\xi}\right) = \frac{i2^\ell \cdot n!}{(\ell-1)!} \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} \left(k - \frac{p}{q} + 1\right)_{\ell-k-1}$$

$$\times \sum_{j=0}^{k} \binom{k-1}{j-1}(n+1)_j B_{k-j}^{(k)}$$

$$\times (2\pi q)^{-n-j-1}\left\{ \sum_{r=1}^{q}\zeta\left(n+j+1,\frac{2\xi+2r-1}{2q}\right)\right.$$

$$\times \exp\left[\left(\frac{n+j}{2}-\frac{(2\xi+2r-1)p}{2}\right)\pi i\right] - \sum_{r=1}^{q}\zeta\left(n+j+1,\frac{2r-2\xi+1}{2q}\right)$$

$$\times \exp\left[\left(-\frac{n+j}{2}+\frac{(2r-2\xi+1)p}{2}\right)\pi i\right]\Bigg\},$$

holds true in terms of the Hurwitz Zeta function.

Proof. Let $\alpha = \ell$ ($\ell \in \mathbb{N}$) in (3.6) we may obtain the assertion (3.7). $\qquad\square$

Theorem 3.3. *For $n, q, \ell \in \mathbb{N}$; $p \in \mathbb{Z}$; $\xi \in \mathbb{C}$, the following formula of the Apostol-Euler polynomials of order ℓ*

$$(3.8)\quad \mathcal{E}_n^{(\ell)}\left(\frac{p}{q};e^{2\pi i\xi}\right) = -\frac{i(-2)^\ell \cdot n!}{(\ell-1)!}\sum_{k=0}^{\ell-1}\binom{\ell-1}{k}B_{\ell-k-1}^{(\ell)}q^{-n-k-1}$$

$$\times \sum_{j=0}^{k}\binom{k}{j}\binom{-n-1}{j}j!\,p^{k-j}(2\pi)^{-n-j-1}$$

$$\times \left\{\sum_{r=1}^{q}\zeta\left(n+j+1,\frac{2\xi+2r-1}{2q}\right)\exp\left[\left(\frac{n+j}{2}-\frac{(2\xi+2r-1)p}{q}\right)\pi i\right]\right.$$

$$-\sum_{r=1}^{q}\zeta\left(n+j+1,\frac{2r-2\xi+1}{2q}\right)\exp\left[\left(-\frac{n+j}{2}+\frac{(2r-2\xi+1)p}{q}\right)\pi i\right]\Bigg\},$$

holds true in terms of the Hurwitz Zeta function.

Proof. Setting $\mu = m$ ($m \in \mathbb{N}$) in (3.5), we obtain the following transformation formula:

$$(3.9)\qquad \Phi_m(z,s,a) = \frac{iz^{-a}\Gamma(1-s)}{(m-1)!}\sum_{k=0}^{m-1}\binom{m-1}{k}B_{m-k-1}^{(m)}$$

$$\times \sum_{j=0}^{k}(-1)^{m-k+j-1}\binom{k}{j}\binom{s-1}{j}j!(-a)^{k-j}(2\pi)^{s-j-1}$$

$$\times \left[\exp\left(\frac{1}{2}(s-j)\pi i\right)\Phi\left(e^{-2\pi ai},1-s+j,\frac{\log z}{2\pi i}\right)\right.$$

$$-\exp\left[\left(2a+\frac{1}{2}(s-j)\right)\pi i\right]\Phi\left(e^{2\pi ai},1-s+j,1-\frac{\log z}{2\pi i}\right)\Bigg]\quad (m\in\mathbb{N}).$$

Letting

$$z = -e^{2\pi i \xi}, \quad a = \frac{p}{q} \quad \text{and} \quad s \mapsto -s, \quad m \mapsto \ell$$

and by applying the series identity (3.3), we obtain the following consequence

$$(3.10) \quad \Phi_\ell \left(-e^{2\pi i \xi}, -s, \frac{p}{q} \right) = \frac{i(-1)^{\ell-1}\Gamma(s+1)}{(\ell-1)!} \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} B_{\ell-k-1}^{(\ell)} q^{-s-k-1}$$

$$\times \sum_{j=0}^{k} \binom{k}{j}_{-s-1} j j! \, p^{k-j} (2\pi)^{-s-j-1}$$

$$\times \left\{ \sum_{r=1}^{q} \zeta \left(s+j+1, \frac{2\xi+2r-1}{2q} \right) \exp \left[\left(\frac{s+j}{2} - \frac{(2\xi+2r-1)p}{q} \right) \pi i \right] \right.$$

$$\left. - \sum_{r=1}^{q} \zeta \left(s+j+1, \frac{2r-2\xi+1}{2q} \right) \exp \left[\left(-\frac{s+j}{2} + \frac{(2r-2\xi+1)p}{q} \right) \pi i \right] \right\},$$

where $\ell \in \mathbb{N}$. Further taking $s = n$ in (3.10) and noting that (2.8) of Theorem 2.1, of course, with

$$\lambda = e^{2\pi i \xi} \quad \text{and} \quad a = \frac{p}{q} \quad (p \in \mathbb{Z}; \ q \in \mathbb{N}; \ \xi \in \mathbb{R}).$$

Therefore, the formula (3.8) follows. This proof is complete. $\qquad\square$

4. FURTHER OBSERVATIONS AND CONSEQUENCES

Recently, SRIVASTAVA found the following elegant formula for Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ (see [**14**, p. 84, Eq. (4.6)]):

$$(4.1) \quad \mathcal{B}_n \left(\frac{p}{q}; e^{2\pi i \xi} \right)$$

$$= -\frac{n!}{(2q\pi)^n} \left\{ \sum_{j=1}^{q} \zeta \left(n, \frac{\xi+j-1}{q} \right) \exp \left[\left(\frac{n}{2} - \frac{2(\xi+j-1)p}{q} \right) \pi i \right] \right.$$

$$\left. + \sum_{j=1}^{q} \zeta \left(n, \frac{j-\xi}{q} \right) \exp \left[\left(-\frac{n}{2} + \frac{2(j-\xi)p}{q} \right) \pi i \right] \right\}$$

$$(n \in \mathbb{N} \setminus \{1\}; \ p \in \mathbb{Z}; \ q \in \mathbb{N}; \ \xi \in \mathbb{R}).$$

When $\xi \in \mathbb{Z}$ in (4.1), we can deduce a known result given earlier by CVIJOVIĆ and KLINOWSKI [**4**, p. 1529, Theorem A]:

$$(4.2) \quad B_n \left(\frac{p}{q} \right) = -\frac{2 \cdot n!}{(2q\pi)^n} \sum_{j=1}^{q} \zeta \left(n, \frac{j}{q} \right) \cos \left(\frac{2jp\pi}{q} - \frac{n\pi}{2} \right)$$

$$(n \in \mathbb{N} \setminus \{1\}; \ p \in \mathbb{Z}; \ q \in \mathbb{N}).$$

It follows that we set $\alpha = 1$ in (3.4), or $\ell = 1$ in (3.7) and (3.8). Then we obtain the following interesting formula for the Apostol-Euler polynomials $\mathcal{E}_n(x; \lambda)$.

Theorem 4.1 *For $n, q \in \mathbb{N}$; $p \in \mathbb{Z}$; $\xi \in \mathbb{R}$, the following formula of the Apostol-Euler polynomials at rational arguments*

$$(4.3) \quad \mathcal{E}_n\left(\frac{p}{q}; e^{2\pi i \xi}\right) = \frac{2 \cdot n!}{(2q\pi)^{n+1}}$$

$$\times \left\{ \sum_{j=1}^{q} \zeta\left(n+1, \frac{2\xi + 2j - 1}{2q}\right) \exp\left[\left(\frac{n+1}{2} - \frac{(2\xi + 2j - 1)p}{q}\right)\pi i\right]\right.$$

$$\left. + \sum_{j=1}^{q} \zeta\left(n+1, \frac{2j - 2\xi - 1}{2q}\right) \exp\left[\left(-\frac{n+1}{2} + \frac{(2j - 2\xi - 1)p}{q}\right)\pi i\right] \right\}$$

holds true in terms of the Hurwitz Zeta function.

A special case of formula (4.3) when $\xi \in \mathbb{Z}$, is just a known result given earlier by CVIJOVIĆ and KLINOWSKI:

Corollary 4.2 ([**4**, p. 1529, Theorem B]) *For $n, q \in \mathbb{N}$; $p \in \mathbb{Z}$, the following formula of the classical Euler polynomials*

$$(4.4) \quad E_n\left(\frac{p}{q}\right) = \frac{4 \cdot n!}{(2q\pi)^{n+1}} \sum_{j=1}^{q} \zeta\left(n+1, \frac{2j-1}{2q}\right) \sin\left(\frac{(2j-1)p\pi}{q} - \frac{n\pi}{2}\right),$$

holds true in terms of the Hurwitz Zeta function.

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