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CHARACTERISTICS OF $(\gamma, 3)$-CRITICAL GRAPHS

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In this note the $(\gamma, 3)$-critical graphs are fairly classified. We show that a (γ, k)-critical graph is not necessarily a (γ, k')-critical for $k' \neq k$ and $k, k' \in \{1, 2, 3\}$. The $(2, 3)$-critical graphs are definitely characterized. Also the properties of $(\gamma, 3)$-critical graphs are verified once their edge connectivity are 3.

1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. A set $S \subseteq V$ is a *dominating set* if every vertex in V is either in S or is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of minimum cardinality is called a $\gamma(G)$-set. Note that removing a vertex can increase the domination number by more than one, but can decrease it by at most one. We define a graph G to be (γ, k)-*critical*, if $\gamma(G - S) < \gamma(G)$ for any set S of k vertices [**1**]. Obviously, a (γ, k)-critical graph G has $\gamma(G) \geq 2$, unless in trivial case $k = |V(G)|$ that $\gamma = 1$ (for more, we refer to [**1–5**]).

The open neighborhood of a vertex $v \in V$ is $N(v) = \{x \in V | vx \in E\}$ while $N[v] = N(v) \cup \{v\}$ is the closed neighborhood. So a set $S \subseteq V$ is a *dominating set*, if $V = \bigcup_{s\in S} N[s]$. The connectivity of G, written $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is k-connected if its connectivity is at least k. A graph is k-edge-connected if every disconnecting set has at least k edges. The edge connectivity of G, written $\lambda(G)$, is the minimum size of a disconnecting set. We denote the distance between two vertices x and y in G by $d_G(x, y)$ and the minimum degree of G with $\delta(G)$, the pendant vertex is a vertex of degree 1 and the support vertex is a vertex adjacent

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to a pendant vertex. A graph G is vertex-transitive if for every pair $u,\ v \in V(G)$ there is an automorphism that maps u to v. In a graph $G = (V, E)$, if $|V| = n$ then we say G is of order n. Let P_n, C_n and K_n be a path, a cycle and a complete graph of order n respectively (for more, we refer to [**6**]).

In this note, the characteristics of $(\gamma, 3)$-critical graphs are studied and we shall know the individualities of $(\gamma, 3)$-critical graphs.

The following result is useful.

Observation A. ([**2**], Observation 5) *If G is any graph and $x, y \in V(G)$ such that $\gamma(G - \{x, y\}) = \gamma(G) - 2$, then $d_G(x, y) \geq 3$.*

2. EXAMPLES OF $(\gamma, 3)$-CRITICAL GRAPHS

In this section, we present two examples of $(\gamma, 3)$-critical graphs: circulant graph $C_{12}\langle 1, 4\rangle$ and the Cartesian product $K_t \Box K_t$. In general, the circulant graph $C_{n+1}\langle 1, m\rangle$ is a graph with vertex set $\{v_0, v_1, \ \dots, v_n\}$ and edge set $\{v_i v_{i+j \pmod{n+1}} |\ i \in \{0, 1, \dots, n\}$ and $j \in \{1, m\}\}$.

The graph $G_t = K_t \Box K_t$ is the Cartesian product of complete graph K_t by itself. The graph G_t has t disjoint copies of K_t in rows and t disjoint copies of K_t in columns. For ease of discussion, we will use the words row and column to mean a "copy of K_t". The vertices of i^{th} row are $v_{i1}, v_{i2}, \cdots, v_{it}$ and the vertices of j^{th} column are $v_{1j}, v_{2j}, \cdots, v_{tj}$ for $1 \leq i, j \leq t$.

Proposition 1. *The circulant $C_{12}\langle 1, 4\rangle$ is $(4, 1)$-critical, $(4, 2)$-critical and $(4, 3)$-critical.*

Proof. Let $G = C_{12}\langle 1, 4\rangle$. It has domination number 4, and $\{v_0, v_3, v_6, v_9\}$ is a minimum dominating set for $C_{12}\langle 1, 4\rangle$. The set vertices $\{v_3, v_6, v_9\}$ dominates $G - \{v_0\}$. Since G is vertex transitive, then G is $(4, 1)$-critical. In what follows, $S \succ G$ means that the set S dominates G.

For $(4, 2)$-criticality, $\{v_3, v_6, v_9\} \succ G - \{v_0, v_1\}$ and $G - \{v_0, v_5\}$, $\{v_3, v_7, v_9\} \succ G - \{v_0, v_2\}$, $\{v_5, v_7, v_{10}\} \succ G - \{v_0, v_3\}$, $\{v_2, v_7, v_9\} \succ G - \{v_0, v_4\}$ and $\{v_3, v_9\} \succ G - \{v_0, v_6\}$, now the vertex transitivity of G prove that G is $(4, 2)$-critical.

For $(4, 3)$-criticality, $\{v_5, v_7, v_{10}\} \succ G - \{v_0, v_1, v_2\}$, $\{v_6, v_8, v_{11}\} \succ G - \{v_0, v_1, v_3\}$, $\{v_6, v_7, v_9\}, \succ G - \{v_0, v_1, v_4\}$, $\{v_3, v_8, v_{10}\} \succ G - \{v_0, v_1, v_5\}$, $\{v_3, v_9\} \succ G - \{v_0, v_1, v_6\}$, $\{v_7, v_9, v_{10}\}, \succ G - \{v_0, v_2, v_4\}$, $\{v_4, v_7, v_9\}, \succ G - \{v_0, v_2, v_5\}$, $\{v_3, v_9\} \succ G - \{v_0, v_2, v_6\}$, $\{v_4, v_5, v_{10}\} \succ G - \{v_0, v_2, v_7\}$, $\{v_8, v_9, v_{10}\} \succ G - \{v_0, v_3, v_6\}$, $\{v_5, v_8, v_{10}\} \succ G - \{v_0, v_3, v_7\}$, $\{v_2, v_7, v_9\} \succ G - \{v_0, v_4, v_6\}$, $\{v_2, v_6, v_{10}\} \succ G - \{v_0, v_4, v_8\}$, $\{v_3, v_9\} \succ G - \{v_0, v_5, v_6\}$, here the vertex transitivity of G implies that G is $(4, 3)$-critical too. □

Proposition 2. *The graph $G_t = K_t \Box K_t$ for $t \geq 3$ is $(t, 3)$-critical.*

Proof. By removing three vertices v_{ij}, v_{sr}, v_{kl} from G_t, there are three cases: suppose $i = s = k$. Without loss of generality, let these three vertices be v_{11}, v_{12}, v_{13}.

Then $\{v_{ss}|\ 2 \leq s \leq t\}$ is a dominating set of cardinality $t-1$. Suppose $s = i$ and $i \neq k$. Without loss of generality, let the vertices be v_{11}, v_{12}, v_{33}, then $\{v_{23}, v_{32}\} \cup \{v_{ss}|\ 4 \leq s \leq t\}$ is a dominating set of cardinality $t-1$. Suppose s, i and k are mutually distinct. Without loss of generality let the vertices be v_{11}, v_{22}, v_{33}, then $\{v_{32}, v_{23}\} \cup \{v_{ss}|\ 4 \leq s \leq t\}$ is a dominating set of cardinality $t-1$. Thus, for three vertices u, v and w of G_t, $\gamma(G_t - \{u, v, w\}) \leq t-1$ implying G_t is $(t, 3)$-critical. □

3. (γ, k) AND (γ, k')-CRITICALITY FOR $1 \leq k \neq k' \leq 3$

In the following examples we show (γ, k)-critical graphs are not necessarily (γ, k')-critical graphs for $1 \leq k \neq k' \leq 3$.

(1) The cycle C_{3n+1} is a $(n+1, 1)$-critical graph but is not $(n+1, k)$-critical for $k \in \{2, 3\}$.

(2) Let $G = (V, E)$, $x \in V$ and $G_{[x]}$ be a graph with vertex set $V \cup \{x'\}$ and edge set $E \cup \{x'y : y \in N_G[x]\}$, thus $G_{[x]}$ is obtained from G by adding a new vertex x' that has the same closed neighborhood as x. Let G be the circulant graph $C_8\langle 1, 4\rangle$ with vertex set $\{v_0, v_1, \ldots, v_7\}$, then $G_{[v_0]}$ is $(3, 2)$-critical but is not $(3, k)$-critical for $k \in \{1, 3\}$ (Figure 1 (a)).

(3) Let H be a graph constructed from the Cartesian product $K_3 \Box K_3$ by adding a new vertex x adjacent to v_{11}, v_{12}, v_{23} and v_{33}. Let $H_{[x]}$ be a graph constructed from H using same method in (2). It is easy to see that $H_{[x]}$ (Figure 1 (b)) is $(3, 3)$-critical but is not $(3, k)$-critical for $k \in \{1, 2\}$. Also the path P_4 is a $(2, 3)$-critical but is not $(2, k)$-critical for $k \in \{1, 2\}$.

(4) The circulant $C_8\langle 1, 4\rangle$ is $(3, k)$-critical for $k \in \{1, 2\}$ but is not $(3, 3)$-critical.

(5) Let $G = K_{2n} - M$ where M is a perfect matchings of K_{2n}. Graph G is $(2, k)$-critical for $k \in \{1, 3\}$ but is not $(2, 2)$-critical (see Proposition 9). For $n = 5$ see the Figure 1(c) $K_{10} - M$ where $M = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}\}$ which is not $(2, 2)$-critical, because of $\gamma(G - \{v_1, v_2\}) = 2$.

(6) The graph H (see (3), Figure 1 (b) once x' is omitted) is $(3, k)$-critical for $k \in \{2, 3\}$ but is not $(3, 1)$-critical (because of $\gamma(H - \{x\}) = 3$).

(7) The Harary graph $H_{2m,n(2m+1)+2m}$ $(m \geq 2)$ by making each vertex adjacent to the nearest m vertices in each direction around the circle (see Figure 1 (d) for $m = 2 = n$ and more generally, we refer to [**6**]), is not (γ, k)-critical, for $k \in \{1, 2, 3\}$ because of its domination number is $n+1$ and each vertex just dominates $2m+1$ vertices [**5**]. But $\gamma(H_{2m,n(2m+1)+2m} - \{v_1\}) = \gamma(H_{2m,n(2m+1)+2m} - \{v_1, v_2\}) = \gamma(H_{2m,n(2m+1)+2m} - \{v_1, v_2, v_3\}) = n+1 = \gamma(H_{2m,n(2m+1)+2m})$.

(8) $C_{12}\langle 1, 4\rangle$ and $G_t = K_t \Box K_t$ are (γ, k)-critical for $k \in \{1, 2, 3\}$. (See Propositions 1, 2 and also Proposition 2 of [**2**]).

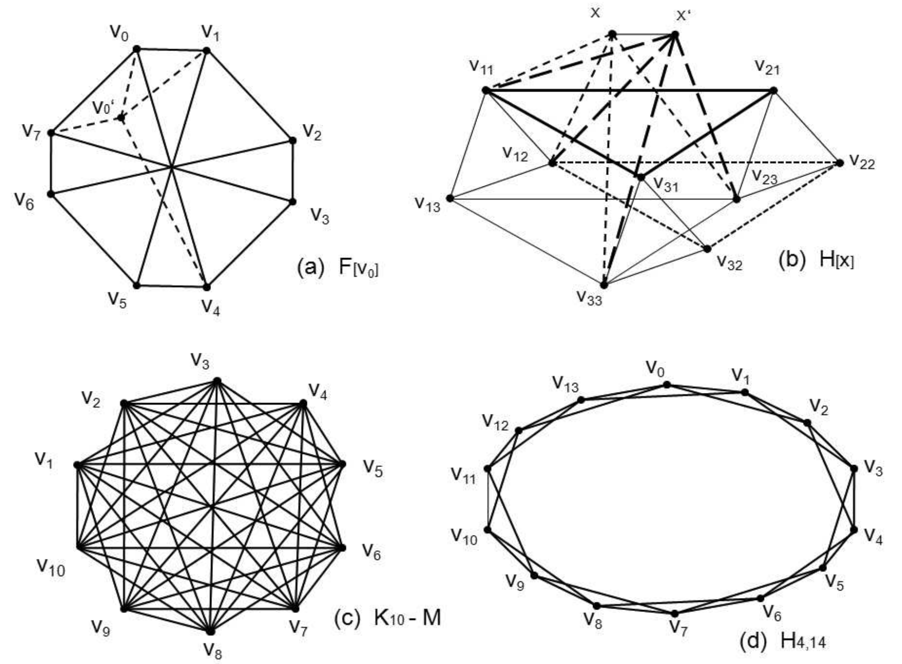

Figure 1.

4. CHARACTERISTICS OF $(\gamma, 3)$-CRITICAL GRAPHS

By noting that removing three vertices can decrease the domination number, we can prove some useful results.

Observation 3. *For a $(\gamma,3)$-critical graph G and $x, y, z \in V(G), \gamma(G) - 3 \leq \gamma(G - \{x, y, z\}) \leq \gamma(G) - 1$.*

Observation 4. *Let G be any graph and $x_1, x_2, x_3 \in V(G)$. If $\gamma(G - \{x_1, x_2, x_3\}) = \gamma(G) - 3$. Then $d_G(x_i, x_j) \geq 3$ for $i \neq j$.*

Proof. On the contrary, suppose, without loss of generality, that D is a $\gamma(G - \{x_1, x_2, x_3\})$-set and that $d_G(x_1, x_2) \leq 2$. Let y be a common adjacent vertex (if the distance is 2) or be x_1 (if the distance is 1). Then $D \cup \{y, x_3\}$ dominates G and so $\gamma(G - \{x_1, x_2, x_3\}) < 3$, which is a contradiction. □

As an immediate result we have:

Observation 5. *If G is a connected $(\gamma, 3)$-critical graph such that* $\operatorname{diam}(G) = 2$, *then $\forall\ x, y, z \in V(G),\ \gamma(G - \{x, y, z\}) \geq \gamma(G) - 2$.*

Observation 4 implies that, if $\gamma(G - \{x, y, z\}) = \gamma(G) - 3$ for any three distinct vertices x, y and z, then G has no edge.

The proof of the result below has been given for any k in [**4**].

Observation 6. *The* $(\gamma, 3)$*-critical graph does not have a vertex of degree* 3.

Let $V(G) = V^0 \cup V^+ \cup V^-$ where $V^0 = \{v \in V | \gamma(G - v) = \gamma(G)\}$, $V^+ = \{v \in V | \gamma(G - v) > \gamma(G)\}$, $V^- = \{v \in V | \gamma(G - v) < \gamma(G)\}$.

Proposition 7. *If* $G \neq P_3$ *is a connected* $(\gamma, 3)$*-critical graph, then* $V = V^- \cup V^0$*, that is,* $V^+ = \emptyset$*. Furthermore,* (1) *either* G *is* $(\gamma, 1)$*-critical, or* $G - v$ *is* $(\gamma, 2)$*-critical for all* $v \in V^0$ *and* (2) *either* G *is* $(\gamma, 2)$*-critical or* $G - \{v, w\}$ *is* $(\gamma, 1)$*-critical for every* $\{v, w\}$ *such that* $\gamma(G - \{v, w\}) = \gamma(G)$.

Proof. Suppose that $V^+ \neq \emptyset$ and $a \in V^+(G)$, then $\gamma(G - \{a\}) \geq \gamma(G) + 1$. Since G is $(\gamma, 3)$-critical, then for all $a, b, c \in V(G)$, $\gamma(G - \{a, b, c\}) \leq \gamma(G) - 1$. Furthermore $\gamma(G - \{a\}) = \gamma(G) + 1$, because, if $\gamma(G - \{a\}) > \gamma(G) + 1$, then it is impossible for the removal of two vertices in $V(G) - a$ to decrease the domination number 3. So $\gamma(G - \{a\}) = \gamma(G) + 1$. Now $\gamma((G - \{a\}) - \{b, c\}) = \gamma(G) - 1$. So $d_{G-\{a\}}(b, c) \geq 3$ (Observation A), that follows $G - \{a\}$ has no edge, hence G is a star. We claim that G has no edge. Since $G \neq P_3$ is $(\gamma, 3)$-critical and $a \in V^+$ so G is not a star of center a with degree at least 2. Now let ax be an edge and $G = ax$, therefore $\gamma(G - \{a\}) = \gamma(G)$, then $a \notin V^+(G)$, a contradiction. So $V^+ = \emptyset$. Next parts of proposition has straightforward proof and dispense with it. □

$(2, 3)$***-critical graphs are characterized***.

Observation 8. *There is no* $(2, 3)$*-critical graph of order* 5.

Proof. Let G be a $(2, 3)$-critical graph and x, y, z be any three vertices of G. Then $G - \{x, y, z\} = K_2$. It shows that $G = K_5$, a contradiction. □

Proposition 9. *A graph* G *is* $(2, 3)$*-critical if and only if* $G = P_2 \cup P_1$, P_4, $2P_2$, $P_3 \cup P_1$, $C_3 \cup P_1$ *or* $G = K_{2n} - M$ *where* $n \geq 2$ *and* M *is a perfect matching of* K_{2n}.

Proof. Let $G = P_2 \cup P_1$, P_4, $2P_2$, $P_3 \cup P_1$, $C_3 \cup P_1$. It is clear that G is $(2, 3)$-critical. Let $G = K_{2n} - M$ where $n \geq 2$ and M is a perfect matching of K_{2n}. Then each vertex is adjacent to $2n - 2$ vertices and since $n \geq 2$, deleting any 3 vertices of G implies that there exists a vertex with degree $2n - 4$. Thus $\gamma(G - \{x, y, z\}) = 1$.

Conversely, let G be a $(2, 3)$-critical graph. The $(2, 3)$-critical graphs with 4 vertices are $P_2 \cup P_1$, $C_4 = K_4 - M$, P_4, $2P_2$, $P_3 \cup P_1$ and $C_3 \cup P_1$. Observation 8 implies that, there is not $(2, 3)$-critical graph of order 5. Let $V(G) \geq 6$. We show that G is $(\gamma, 1)$-critical. If G is not $(\gamma, 1)$-critical, then there is a $v \in V^0$ such that $G - v$ is $(2, 2)$-critical. If $G - v$ is connected, then by Proposition 13 of [**2**] $\gamma(G - v) \geq 3$ a contradiction. If $G - v$ is disconnected, then there are two vertices x, y in $G - v$ such that $G - \{v, x, y\}$ has two nonempty components. So $\gamma(G - \{v, x, y\}) \geq 2$ a contradiction. Thus G is $(2, 1)$-critical and the results of [**1**] imply that $G = K_{2n} - M$. □

As an immediate result of Proposition 9 we have:

Corollary 10. *If G is a connected $(\gamma, 3)$-critical graph with $|V(G)| \geq 6$ and $G \neq K_{2n} - M$ where M is a perfect matching, then $\gamma(G) \geq 3$.*

Remark 1.

1. Let G be a $(\gamma, 3)$-critical graph with a pendant vertex x and support vertex y, then $\deg(y) = 2$. Let u and w be the pendant vertex and support vertex respectively with $\deg(w) \geq 3$. Vertices x, y are two neighborhoods of w other than u, then $\gamma(G-\{w,x,y\}) = \gamma(G) - 1$, because u is an isolated vertex in $(G - \{w, x, y\})$, and then it belongs to a $\gamma(G-\{w,x,y\})$-set, D. Now $(D-\{u\}) \cup \{w\}$ is a $\gamma(G)$-set with the cardinality of $\gamma(G)-1$, a contradiction.

2. Let G be any graph with a pendant vertex, then G is not $(\gamma, 1)$-critical and $(\gamma, 2)$-critical. Because the support vertex w belongs to V^0 and $\gamma(G - \{w, v\}) = \gamma(G)$ where v is an adjacent vertex of w other than pendant vertex.

3. If $G \neq P_4, C_4$ is a $(\gamma, 3)$-critical graph and is not $(\gamma, 1)$-critical , then G has at most one vertex of degree 2. Suppose $v \in V^0$ and G has at least two vertices of degree 2 such as u, w with neighborhoods $\{x, y\}$ and $\{z, t\}$ respectively. Suppose that $v \notin \{x, y\}$, then $\gamma(G - \{x, y, v\}) = \gamma(G) - 1$. Since the vertex u is an isolated vertex and belongs to every $\gamma(G - \{x, y, v\})$-set, then $\gamma(G) - 1$ vertices dominate $G - \{v\}$. Hence $v \in V^-$, a contradiction.

4. Any $(\gamma, 3)$-critical graph G other than P_3 and P_4 has at most one pendant vertex. Because more than one pendant vertex in G leads to at least 2 support vertices of degree 2.

Proposition 11. *Let G be a connected $(\gamma, 3)$-critical graph. If G is a graph other than P_3, C_3, P_4 and C_4, then G has at most one vertex of degree 1, one vertex of degree 2 and the other vertices of degree at least 4.*

Proof. By Observation 6 and Remark 1 the result holds. □

Remark 2. By Remark 1 and Proposition 11, one can say that, almost all $(\gamma, 3)$-critical graphs have $\delta(G) \geq 4$.

The below result has been proved in [**4**]. For seeing result, the below definition is added.

Definition. *A block of a graph G is a maximal connected subgraph of G that has no cut-vertex.*

Corollary 12. *A graph G is $(\gamma, 1)$-critical, $(\gamma, 2)$-critical and $(\gamma, 3)$-critical if and only if each block of G is $(\gamma, 1)$-critical, $(\gamma, 2)$-critical and $(\gamma, 3)$-critical. Further, if G is $(\gamma, 1)$-critical, $(\gamma, 2)$-critical and $(\gamma, 3)$-critical with blocks $G_1, G_2, \ldots, G_k$, then $\gamma(G) = \sum_{i=1}^{k} \gamma(G_i) - k + c(G)$, where $c(G)$ is the number of components of G.*

Now we find a $(\gamma, 1)$-critical, $(\gamma, 2)$-critical and $(\gamma, 3)$-critical graph G_γ with given $\gamma \geq 3$ and diameter $\gamma - 1$.

Proposition 13. *For every integer $\gamma \geq 3$, there exists a connected graph G_γ that is $(\gamma, 1)$-critical, $(\gamma, 2)$-critical and $(\gamma, 3)$-critical satisfying $\gamma(G_\gamma) = \gamma$ and* $\mathrm{diam}(G_\gamma) = \gamma - 1$.

Proof. Let H be the Cartesian product $K_3 \square K_3$. Then $\mathrm{diam}(H) = 2$ and by Proposition 2, H is $(3, 1)$-critical, $(3, 2)$-critical and $(3, 3)$-critical. Let F be the circulant $C_{12}\langle 1, 4\rangle$ then $\mathrm{diam}(F)=3$ and, by Proposition 1, F is $(4, 1)$-critical, $(4, 2)$-critical and $(4, 3)$-critical. If $\gamma = 3$ or $\gamma = 4$, then we can take $G_\gamma = H$ or $G_\gamma = F$, respectively. Hence we may assume that $\gamma \geq 5$. We consider two possibilities, depending on whether γ is odd or even.

Suppose $\gamma = 2k+1$, where $k \geq 2$. Let u and w be any two nonadjacent vertices of H. Let $B_1, B_2, \ldots, B_k$ be k disjoint copies of H. For $i = 1, 2 \ldots, k$, let u_i and w_i denote the vertices of B_i corresponding to u and w, respectively in H. Let G_γ be obtained by identifying w_i and u_{i+1} for $i = 1, 2, \ldots, k-1$. Then $B_1, B_2, \ldots, B_k$ are blocks of G_γ. Since each B_i is $(\gamma, 1)$-critical, $(\gamma, 2)$-critical and $(\gamma, 3)$-critical with $\gamma(B_i) = 3$, we know from Corollary 12, that G_γ is $(\gamma, 1)$-critical, $(\gamma, 2)$-critical and $(\gamma, 3)$-critical with $\gamma(G_\gamma) = 2k + 1 = \gamma$. Furthermore, $\mathrm{diam}(G_\gamma) = 2k = \gamma - 1$.

Suppose $\gamma = 2k$, where $k \geq 3$. In the construction of G_γ in the preceding paragraph, replace B_{k-1} and B_k with a copy L of F. Then $B_1, B_2, \ldots, B_{k-2}, L$ are blocks of G_γ. By Corollary 12, G_γ is $(\gamma, 1)$-critical, $(\gamma, 2)$-critical and $(\gamma, 3)$-critical with $\gamma(G_\gamma) = 2k = \gamma$. Furthermore, $\mathrm{diam}(G_\gamma) = 2k - 1 = \gamma - 1$. □

5. EDGE CONNECTIVITY

As it has been seen in the previous section, there exist connected $(\gamma, 3)$-critical graphs that contain cut-vertices. In this section we study the edge connectivity $\lambda(G)$ of $(\gamma, 3)$-critical graphs.

In any graph a vertex of degree 1 leads to $\lambda \leq 1$ and a vertex of degree 2 leads to $\lambda \leq 2$. So by Observation 6 we have:

Observation 14. *If G is a $(\gamma, 3)$-critical graph and $\lambda(G) \geq 3$, then $\delta(G) \geq 4$.*

Theorem 15. *Suppose that G is a connected $(\gamma, 3)$-critical graph with $\lambda(G) = 3$ and an edge cut $\{ab, cd, ef\}$. Let G_1 and G_2 be two components of $G - ab - cd - ef$, with $a, c, e \in V(G_1)$, $b, d, f \in V(G_2)$ and a, c, e are distinct. Then the following must all be true.*

(i) *It is not the case that $b = d = f$. Hereafter b, d, f are distinct or $(b = d) \neq f$*

(ii) $\gamma(G) = \gamma(G_1) + \gamma(G_2)$, *if $(b = d) \neq f$.*

(iii) $a, c, e \notin V^+(G_1)$, *if $(b = d) \neq f$.*

(iv) *If $\{a, c, e\} \subseteq V^-(G_1)$, then none of b, d, f is in $\gamma(G_2)$-set.*

(v) *If b, d, f are distinct vertices, then $\gamma(G_2 - \{b, d, f\}) = \gamma_2 - 1$ and a $\gamma(G_2 - \{b, d, f\})$-set simultaneously dominates none of two of b, d, f.*

(vi) *If* b, d, f *are distinct and belong to* $V^0(G_2)$*, then there is a* $\gamma(G_2 - \{b, d\})$*-set containing* f*, a* $\gamma(G_2 - \{b\})$*-set containing* d *or a* $\gamma(G_2 - \{d\})$*-set containing* b*.*

(vii) *If* b, d, f *are distinct. Then there is no* $\gamma(G_1)$*-set containing* $\{a, c, e\}$*.*

(viii) *Let* b, d, f *are distinct and* $\{a, c, e\} \subseteq V^-(G_1)$*. There is no* $\gamma(G_1 - \{x\})$*-set containing* $\{y, z\}$*, where* $\{x, y, z\} = \{a, c, e\}$*.*

Proof. Let $\gamma(G) = \gamma$. For $i = 1, 2$, let $V_i = V(G_i)$ and let $\gamma_i = \gamma(G_i)$.

(i) Let $b = d = f$. We show that, there is a $\gamma(G_2)$-set containing b. $\lambda(G) = 3$ implies that $\deg(b) \geq 6$, in other word $\deg_{G_2}(b) \geq 3$. Thus there are at least two vertices x, y in $V(G_2)$ such that $b \in N(x) \cap N(y)$, so $\gamma(G_2 - \{x, b, y\}) = \gamma(G_2) - 1$. Let D_2 be a $\gamma(G_2)$-set to include b. Now $\gamma - 1 \geq \gamma(G - \{a, c, e\}) = \gamma(G_1 - \{a, c, e\}) + \gamma_2$, and so $\gamma(G_1 - \{a, c, e\}) \leq \gamma_1 - 1$. Let D_1 be a $\gamma(G_1 - \{a, c, e\})$-set. Then $D_1 \cup D_2$ is a dominating set for G of cardinality $|D_1 \cup D_2| \leq \gamma - 1$, a contradiction. Therefore $f \neq (b = d)$, $b \neq (d = f)$, $d \neq (b = f)$ or b, d, f are distinct.

(ii) Clearly, $\gamma \leq \gamma_1 + \gamma_2$. It suffices to show that $\gamma \geq \gamma_1 + \gamma_2$. Since $(b = d) \neq f$ and $\delta(G) \geq 4$, there is a vertex $x \neq f$ such that $x \in V(G_2) \cap N(b)$. It is clear $\gamma_2(G_2 - \{b, x, f\}) \leq \gamma_2 - 2$. Suppose that $\gamma_2(G_2 - \{b, x, f\}) = \gamma_2 - 2$, there is a γ_2-set D_2 for G_2 includes b and f. Now, $\gamma - 1 \geq \gamma(G - \{a, c, e\}) = \gamma(G_1 - \{a, c, e\}) + \gamma_2$. Let D_1 be $\gamma(G_1 - \{a, c, e\})$-set, hence $|D_1 \leq \gamma_1 - 1$ and then $D = D_1 \cup D_2$ is a γ-set with cardinality $|D| \leq \gamma - 1$, a contradiction. Thus $\gamma_2(G_2 - \{b, x, f\}) = \gamma_2 - 1$ and $\gamma - 1 \geq \gamma(G - \{b, x, f\}) = \gamma(G_1) + \gamma_2(G_2 - \{b, x, f\}) = \gamma_1 + \gamma_2 - 1$. Therefore $\gamma \geq \gamma_1 + \gamma_2$.

(iii) Suppose $(b = d) \neq f$ and $a \in V^+(G_1)$. It is well known $\gamma_2 - 2 \leq \gamma(G_2 - \{b, f\}) \leq \gamma_2$.

First, let $\gamma(G_2 - \{b, f\}) = \gamma_2 - 2$. There is a γ_2-set D_2 for G_2 includes b and f. Now, $\gamma - 1 \geq \gamma(G - \{a, c, e\}) = \gamma(G_1 - \{a, c, e\}) + \gamma_2$. Let D_1 be $\gamma(G_1 - \{a, c, e\})$-set, hence $|D_1 \leq \gamma_1 - 1$ and then $D = D_1 \cup D_2$ is a γ-set of cardinality $|D| \leq \gamma - 1$ a contradiction.

Second, let $\gamma_2 - 1 \leq \gamma(G_2 - \{b, f\}) \leq \gamma_2$. Then $\gamma(G_2 - \{b, f\}) \geq \gamma_2 - 1$ and $\gamma - 1 \geq \gamma(G - \{a, b, f\}) = \gamma(G_1 - \{a\}) + \gamma(G_2 - \{b, f\}) =\geq \gamma_1 + 1 + \gamma_2 - 1 = \gamma$, a contradiction. Same proof can be used for c and e.

(iv) Suppose b is in $\gamma(G_2) - set = D_2$. Let D_1 be a $\gamma(G_1 - \{a\})$-set. Since $a \in V^-(G_1)$, $|D_1| = \gamma_1 - 1$. Now $D = D_1 \cup D_2$ dominates G and $|D| = \gamma_1 - 1 + \gamma_2 = \gamma - 1$, a contradiction. Hence b does not belong to any $\gamma(G_2) - set$. The result for d and f, follows from an identical argument.

(v) Since G is $(\gamma, 3)$-critical $\gamma - 1 \geq \gamma(G - \{b, d, f\}) = \gamma(G_1) + \gamma(G_2 - \{b, d, f\})$ and $\gamma(G_2 - \{b, d, f\}) \leq \gamma_2 - 1$. Let D_2 be a $\gamma(G_2 - \{b, d, f\})$-set, if $|D_2| = \gamma_2 - 3$, then $D_2 \cup \{b, d, f\}$ is a $\gamma(G_2)$-set, a contradiction with (iv). If $|D_2| = \gamma_2 - 2$, then there are two vertices x, y that dominate b, d, f. If x dominates b, then $D_2 \cup \{b, y\}$ is a dominating set of G_2 a contradiction, thus $|D_2| = \gamma_2 - 1$. Now if D_2 dominates b and d, then $D_2 \cup \{f\}$ is a $\gamma(G_2)$-set that also a contradiction.

(vi) The part (v) implies $\gamma(G_2 - \{b, d, f\}) = \gamma_2 - 1$. $\{b, d, f\} \subseteq V^0(G_2)$ implies $\gamma_2 = \gamma(G_2 - \{b\}) = \gamma(G_2 - \{d\}) = \gamma(G_2 - \{f\})$. There are two cases.

1. Let $\gamma(G_2 - \{b, d\}) = \gamma_2$. Then $\gamma(G_2 - \{b, d, f\}) = \gamma_2 - 1$. Suppose that $D_2 = \gamma(G_2 - \{b, d, f\})$-set, so $D_2 \cup \{f\}$ is a $\gamma(G_2 - \{b, d\})$-set.

2. Let $\gamma(G_2 - \{b, d\}) = \gamma_2 - 1$ and let $D = \gamma(G_2 - \{b, d\})$-set. Then $D \cup \{d\}$ is a $\gamma(G_2 - \{b\}$-set and $D \cup \{b\}$ is a $\gamma(G_2 - \{d\}$.

(vii) Suppose there is a $\gamma(G_1)$-set D_1 containing a, c and e. Let D_2 be a $\gamma(G_2 - \{b, d, f\})$-set. By (vi) $|D_2| \leq \gamma_2 - 1$, and so $D_1 \cup D_2$ is a dominating set for G of cardinality $\gamma_1 + \gamma_2 - 1$, a contradiction.

(viii) Suppose there is a $\gamma(G_1 - \{a\})$-set D_1 containing c, e, then $D_1 \cup \{a\} = \gamma(G_1)$-set and there is a $\gamma(G_1)$-set containing a, c, e contradicting (vii). Identical arguments show there is no a $\gamma(G_1 - \{c\})$-set containing a, e, and there is no a $\gamma(G_1 - \{e\})$-set containing a, c. □

Theorem 16. *Let G be a connected graph. If G is $(3, 3)$-critical or $(4, 3)$-critical, $\gamma_1 \neq \gamma_2$ and $\lambda(G) \neq 1, 2$ then $\lambda(G) \geq 4$.*

Proof. $\lambda(G) \neq 1, 2$ and $(\gamma, 3)$-criticality imply that $\delta(G) \geq 4$. Let $\lambda(G) = 3$ and $\{ab, cd, ef\}$ be an edge cut. Let G_1 and G_2 be two components of $G - ab - cd - ef$, with $a, c, e \in V(G_1)$, $b, d, f \in V(G_2)$. By (ii) of Theorem 15, $\gamma(G) = \gamma(G_1) + \gamma(G_2)$. Let $\gamma(G) = 3$. Without loss of generality, suppose $2 \leq \gamma(G_1) \leq 3$ and $0 \leq \gamma(G_2) \leq 1$. Let $\gamma(G_1) \geq 2$ and $\gamma(G_2) = 1$. There are at least 5 vertices in G_2, because of $\delta(G) \geq 4$. Let $\{x, y, z\} \subseteq V(G_2)$ such that contains $\{b, d, f\}$, then $\gamma(G_2 - \{x, y, z\}) = 1 - 1 = 0$, a contradiction.

Let $\gamma(G_1) = 3$ and $\gamma(G_2) = 0$, so b, d, f all vertices of $V(G_2)$ should be dominated by a, c, e. Since at least one of b, d and f of degree 2 or 3, in G_2 that also a contradiction. Therefore $\lambda(G) \geq 4$.

Now let $\gamma(G) = 4$. Without loss of generality, suppose $3 \leq \gamma(G_1) \leq 4$ and $0 \leq \gamma(G_2) \leq 1$. These are proved by using manner of proof once $\gamma(G) = 3$, $2 \leq \gamma(G_1) \leq 3$ and $0 \leq \gamma(G_2) \leq 1$. □

We close with some open questions.

Questions

1. Characterize the $(\gamma, 3)$-critical graphs.

2. Is it true that if G is a connected $(\gamma, 3)$-critical graph of order at least 6, then $\lambda(G) \geq 3$ and $\delta(G) \geq 4$? Though, we know that for $(\gamma, 3)$-critical graphs, if $\lambda(G) = 3$, then $\delta(G) \geq 4$.

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