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MIRRORING AND INTERLEAVING IN THE PAPERFOLDING SEQUENCE

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Three equivalent methods of generating the paperfolding sequence are presented as well as a categorisation of runs of identical terms. We find all repeated subsequences, the largest repeated subsequences and the spacing of singles, doubles and triples throughout the sequence. The paperfolding sequence is shown to have links to the Binary Reflected Gray Code and the Stern-Brocot tree.

1. INTRODUCTION

Take a sheet of paper and fold it, right over left, n times. When the paper is unfolded we see a sequence of $2^n - 1$ creases, some downward and some upward. An analysis of this sequence first appeared in Davis and Knuth [**1**], who labeled these creases D and U. We will label them 1 and 0 as do Dekking et al [**8**]. Prodinger and Urbanek [**14**] label them 0 and 1 while Allouche and Bousquet-Mélou [**4**] allow both 1 and 0 and 0 and 1.

This sequence of $2^n - 1$ 1s and 0s we call S_n as do Davis and Knuth [**1**] with their sequence of $2^n - 1$ Ds and Us. The middle element of S_n is always a 1 (a D for Davis and Knuth); S_{n-1} appears to the left of this 1 and $\overline{S^R_{n-1}}$ (obtained from S_{n-1} by reversing the order and swapping 0s and 1s – a notation adopted by Lothaire [**10**, p. 526], Mendès France and Shallit [**12**] and Prodinger and Urbanek [**14**]) appears to the right.

So as in Davis and Knuth [**1**].

Theorem 1. $S_n = S_{n-1}\ 1\ \overline{S^R_{n-1}}$.

We use S for $\lim\limits_{i\to\infty} S_i$ and call this the paperfolding sequence.

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Mendès France and Shallit [**12**] give four different methods for representing the sequence. One of their representations, called the Dragon Curve in Davis and Knuth [**1**], is a sequence of lattice points obtained by unfolding the paper so that all the folds are 90^0 and then looking at the edge of the paper. Their third representation has R for 1 and L for 0.

An important concept, similar to perfect shuffling, is useful in understanding paperfolding. It is called interleaving.

Definition 1. (Interleave Operator). *The interleave operator # acting on the two sequences* $U = u_1, u_2, \dots, u_k$ *and* $V = v_1, v_2, \dots, v_n$ *where* $k > n$, *generates the following interleaved sequence:*

$$U\#V = u_1, \dots, u_p, v_1, u_{p+1}, \dots, u_{2p}, v_2, u_{2p+1}, \dots, u_{np}, v_n, u_{np+1}, \dots, u_k$$

where $p = \left\lfloor \frac{k}{n+1} \right\rfloor$.

Example 1. Let $U = u_1, u_2, \dots, u_{n+1}$ and $V = v_1, v_2, \dots, v_n$. Then

$$U\#V = u_1, v_1, u_2, v_2, \dots, u_n, v_n, u_{n+1}.$$

Note that Definition 1 requires the two sequences to have different parity. Accordingly, # does not define a perfect shuffle although there are obvious similarities.

Davis and Knuth [**1**] and Prodinger and Urbanek [**14**] have yet another method for constructing S_n and S. This can be expressed through interleaving, as follows.

Theorem 2. *Let* $A_{2k} = (10)^k$ *that is,* $1010 \cdots 10$ *with* k 10*s, then*

$$\begin{aligned} S_n &= A_{2^{n-1}} \# A_{2^{n-2}} \# \cdots \# A_2 \# 1, \\ &= A_{2^{n-1}} \# S_{n-1}. \end{aligned}$$

Allouche and Bacher [**2**] use Toeplitz transformations to construct S_n in essentially the same way. Dekking et al. [**8**] have a similar result.

The following theorem appears in various forms in the literature, notably Davis and Knuth [**1**] and Dekking et al. [**8**]. Our variant arises from using 1s and 0s in S. In what follows f_i is the ith element in S.

Theorem 3. *For* $i \geq 1$,
i) $f_{2i} = f_i$ and
ii) $f_{2i-1} = \frac{1-(-1)^i}{2}$.

Proof. By Theorem 2,
i) the sequence of successive even entries in S_n is S_{n-1}, establishing i),
ii) the sequence of successive odd entries in S_n is $A_{2^{n-1}}$, establishing ii). □

The following result, though similar to other formulations (for example, Allouche and Bousquet-Mélou [**3**]), is compact and yields an interesting corollary.

Theorem 4. *For $h, k \geq 0$,*

$$f_i = \begin{cases} 1, & \text{if } i = 2^k(4h+1) \\ 0, & \text{if } i = 2^k(4h+3)\,. \end{cases}$$

Proof. We prove this by induction on n.

We have $S_1 = 1$, so $f_1 = 1 = 2^k(4h+1)$ where $k = h = 0$.

Suppose the theorem is true for S_{n-1}. By Theorem 2,

$$S_{n-1} = f_2 f_4 \ldots f_{2^n-2}.$$

That is, the i^{th} entry in S_{n-1} becomes the $2i^{\text{th}}$ entry in S_n. Also

$$\begin{aligned} A_{2^{n-1}} &= 1010 \cdots 10, \\ &= f_1 f_3 f_5 \ldots f_{2^n-1}. \end{aligned}$$

That is, for $0 \leq h \leq 2^{n-2} - 1$, $f_{4h+1} = 1$ and $f_{4h+3} = 0$. □

Corollary 1. *For $k \geq 0, r \geq 0$, $f_i = 1 + r \bmod 2$ where $i = 2^k(2r+1)$.*

Proof. From Theorem 4, for $k \geq 0$, $h \geq 0$,
i) $f_i = 1$, if $i = 2^k(4h+1) = 2^k(2r+1)$ where r is even. That is, $f_i = 1 + r \bmod 2$, for r even.
ii) $f_i = 0$, if $i = 2^k(4h+3) = 2^k(2\,(2h+1)+1) = 2^k(2r+1)$ where r is odd. That is, $f_i = 1 + r \bmod 2$, for r odd. □

Davis and Knuth [**1**] prove the following result to which we provide an interesting corollary.

Theorem 5. $S_{n+1} = S_n\ 1\ \overline{S_n^R}$ *and* $\overline{S_{n+1}^R} = S_n\ 0\ \overline{S_n^R}$ *where* $S_1 = 1$.

Corollary 2.

$$f_{2^i+k} = \begin{cases} f_k, & \text{for } 1 \leq k < 2^i \text{ and } k \neq 2^{i-1} \\ 0, & \text{for } k = 2^{i-1}. \end{cases}$$

Proof. From Theorem 5, S_i and $\overline{S_i^R}$ are identical except in their respective $\left(2^{i-1}\right)^{\text{th}}$ terms. Since

$$\begin{aligned} S_{i+1} &= S_i 1 \overline{S_i^R}, \\ &= f_1 f_2 \ldots f_k \ldots f_{2^i-1}(f_{2^i} = 1) f_{2^i+1} \ldots f_{2^i+k} \ldots f_{2^{i+1}-2} f_{2^{i+1}-1} \end{aligned}$$

we have

- $f_{2^i+k} = f_k$, for $1 \leq k < 2^i$ and $k \neq 2^{i-1}$,
- $f_{2^i+k} = f_{3(2^{i-1})} = 0$ for $k = 2^{i-1}$. □

We note that Theorem 5 offers the simplest way to generate S_{i+1}. It requires only S_i.

In this paper we use a generalised version of A_{2k} to produce yet another way of generating S_n. We find the number and position of single 1s and 0s (that is, instances of 010 and 101) as well as those of double 11s and 00s, and triple 111s and 000s, in S. We link the Paperfolding and Stern-Brocot sequences, examine functions related to S and show that one of these has properties similar to those of the Gray code function of Bunder et al [**7**].

2. THE GENERAL ALTERNATING PAPERFOLDING SEQUENCE

We define a generalisation of A_{2k} which utilises $S_{i-n}\ \overline{S^R_{i-n}}$ instead of 10.

Definition 2. *For $n < i$,*

$$\mathcal{A}_{i,n} = \left(S_{i-n}\overline{S^R_{i-n}}\right)^{2^{n-1}}.$$

Note that $\mathcal{A}_{i,n}\#S_n = S_{i-n}f_1\overline{S^R_{i-n}}f_2S_{i-n}f_3\overline{S^R_{i-n}}\cdots S_{i-n}f_{2^n-1}\overline{S^R_{i-n}}$.

Theorem 6.

$$S_i = \mathcal{A}_{i,n} \# S_n \text{ and}$$
$$\overline{S^R_i} = \mathcal{A}_{i,n} \# \overline{S^R_n}.$$

Proof. We prove the first result by induction on n.

We have $S_i = \mathcal{A}_{i,1}\#S_1$.

Let $S_i = \mathcal{A}_{i,t}\#S_t$, where $t < n$. Therefore by Theorem 2,

$$\begin{aligned} S_i &= S_{i-t}f_1\overline{S^R_{i-t}}f_2S_{i-t}f_3\overline{S^R_{i-t}}\cdots S_{i-t}f_{2^t-1}\overline{S^R_{i-t}}, \\ &= \left(S_{i-(t+1)}\ 1\ \overline{S^R_{i-(t+1)}}\right)f_1\ \left(S_{i-(t+1)}\ 0\ \overline{S^R_{i-(t+1)}}\right)f_2\left(S_{i-(t+1)}\ 1\ \overline{S^R_{i-(t+1)}}\right)f_3\ \cdots \\ &\quad \cdots\left(S_{i-(t+1)}\ 1\ \overline{S^R_{i-(t+1)}}\right)f_{2^t-1}\ \left(S_{i-(t+1)}\ 0\ \overline{S^R_{i-(t+1)}}\right), \\ &= \mathcal{A}_{i,t+1}\#A_{2^t}\#S_t, \\ &= \mathcal{A}_{i,t+1}\#S_{t+1}. \end{aligned}$$

Therefore our result is true for all $1 \leq n < i$. A similar argument holds for $\overline{S^R_i}$. □

Allouche and Bacher [**2**] generate S using a particular Toeplitz transform. They commence with the sequence

$$\begin{aligned} B_0 &= 1\omega0\omega1\omega0\omega\ldots, \\ &= (1\omega0\omega)^\infty. \end{aligned}$$

To obtain B_{k+1} from B_k, for $i = 1, 2, \ldots$, the i^{th} ω in B_k is replaced by the i^{th} term in B_0. Thus

$$\begin{aligned} B_1 &= 110\omega 100\omega 110\omega \ldots, \\ B_2 &= 1101100\omega 1100100\omega \ldots, \\ &\vdots \end{aligned}$$

Continuing in this way, we have $S = \lim_{k\to\infty} B_k$.

This transformation can be restated through interleaving. Let $W_n = (\omega)^{2^{n-1}-1}$. Then

$$\begin{aligned} B_0 &= S_1\ \omega\ \overline{S_1^R}\ \omega\ S_1\ \omega\ \overline{S_1^R}\ \omega \ldots, \\ &= \left(S_1\ \omega\ \overline{S_1^R}\ \omega\right)^{\infty}, \\ &= \lim_{n\to\infty} \mathcal{A}_{n+1,n} \# W_n. \end{aligned}$$

B_1 is obtained from B_0 by replacing the i^{th} ω in B_0 by the i^{th} term in B_0. That is,

$$\begin{aligned} B_1 &= \left(S_1\ 1\ \overline{S_1^R}\ \omega\ S_1\ 0\ \overline{S_1^R}\ \omega\right)^{\infty}, \\ &= \left(S_2\ \omega\ \overline{S_2^R}\ \omega\right)^{\infty}, \\ &= \lim_{n\to\infty} \mathcal{A}_{n+2,n} \# W_n. \end{aligned}$$

Continuing in this way, $B_k = \lim_{n\to\infty} \mathcal{A}_{n+k+1,n} \# W_n$. It follows that, $S = \lim_{k\to\infty} B_k$, as before.

Summary 1. *There are three equivalent representations of the paperfolding sequence, S_i and $\overline{S_i^R}$.*

i) *Mirroring:* $S_i = S_{i-1}\ 1\ \overline{S_{i-1}^R}$ *and* $\overline{S_i^R} = S_{i-1}\ 0\ \overline{S_{i-1}^R}$,

ii) *Interleaving:* $S_i = \mathcal{A}_{i,n} \# S_n$ *and* $\overline{S_i^R} = \mathcal{A}_{i,n} \# \overline{S_n^R}$ *where* $0 < n < i$.

In particular, $S_i = A_{2^{i-1}} \# S_{i-1}$ *and* $\overline{S_i^R} = A_{2^{i-1}} \# \overline{S_{i-1}^R}$; *and*

iii) *Alternation:* $S_i = A_{2^{i-1}} \# A_{2^{i-2}} \# \cdots \# A_2 \# 1$ *and* $\overline{S_i^R} = A_{2^{i-1}} \# A_{2^{i-2}} \# \cdots \# A_2 \# 0$.

3. SOME FUNCTIONS RELATED TO THE PAPERFOLDING SEQUENCE

Davis and Knuth [**1**] define the following two functions, which are in our notation:

Definition 3.

$$d(n) = \begin{cases} 1 & \text{if } f_n = 1 \\ -1 & \text{if } f_n = 0, \end{cases}$$

$$g(n) = \sum_{i=1}^{n-1} d(i).$$

It follows that $g(n)$ is the excess of 1s over 0s in $f_1 \dots f_{n-1}$. Clearly:

Theorem 7. i) $d(n) = 2f_n - 1$ *and*

ii) $g(n) = 1 - n + 2\sum_{i=1}^{n-1} f_i$.

Davis and Knuth [**1**] prove the following interesting results, which have obvious f_n counterparts.

Theorem 8. i) $d\left(2^{n+1} - m\right) = -d(m)$ *if* $0 < m < 2^n$,

ii) $g\left(2^{n+1}\right) = 1$,

iii) $g\left(2^{n+1} - m + 1\right) = 1 + g(m)$ *if* $1 \le m \le 2^n$ *and*

iv) $\min_n \ (g(n) = k) = \left\lceil \frac{2^{k+1}}{3} \right\rceil$.

Prodinger and Urbanek [**14**] have a function $n_1(k)$, which represents the number of 1s in the first k places of their S, which has 1s and 0s swapped from our S. We will use:

Definition 4. $N_1(k)$ *is the number of* 1*s in* $f_1 f_2 \dots f_k$.

Thus $N_1(k) = k - n_1(k)$ and

Theorem 9. $N_1(k) = \sum_{i=1}^{k} f_i$.

Prodinger and Urbanek [**14**] also introduce $v(k)$, the number of changes of consecutive digits in the binary expansion of k, where the leftmost 1 counts as a change. So, for example, $v(7) = 1$ as 7 is 111 in binary form and $v(10) = 4$ as 10 is 1010 in binary form. They adopt the following definition:

Definition 5. $v(0) = 0$ *and* $v(2j + i) = v(j) + \delta$ *where* $i, \delta \in \{0, 1\}$ *and* $\delta = i + j \pmod 2$.

They give the following interesting connection with the function $N_1(k)$:

Theorem 10.

$$n_1(k) = \frac{1}{2}\big(k - v(k)\big),$$

$$\textit{that is, } N_1(k) = \frac{1}{2}\big(k + v(k)\big)$$

$$\textit{and } v(k) = 2N_1(k) - k.$$

Theorem 11. $v(k) = \sum_{i=1}^{k}(2f_i - 1) = g(k+1)$.

Prodinger and Urbanek [**14**] also have the interesting result (converted to $N_1(k)$):

Theorem 12. $N_1(k) = k - \sum_{i \geq 0} \left\lfloor \frac{k+2^i}{2^{i+2}} \right\rfloor.$

The obvious counterpart via Theorem 11 for $v(k)$ to Theorem 8 iii) is:

Theorem 13. *If* $2^n \leq k < 2^{n+1}$, *then* $v(k) = v\left(2^{n+1} - k - 1\right) - 1.$

This relation is very similar to the recurrence relation for the Gray code function $b(k)$ found in Bunder et al [**7**].

Definition 6. *If* $2^n \leq k < 2^{n+1}$, *then* $b(k) = b\left(2^{n+1} - k - 1\right) + 2^n.$

Not surprisingly, some of the results in [**7**] for $b(k)$ can also be proved for $v(k)$.

Theorem 14. i) $v(4k+1) = v(2k) + 1,$
ii) $v(2^r(4k+1)) = v(2k) + 2$ *if* $r > 0$ *and*
iii) $v(2^r(4k+3)) = v(2k+1)$ *if* $r \geq 0.$

Proof. i) By Definition 5.

ii)
$$\begin{aligned} v\big(2^r(4k+1)\big) &= v\big(2^{r-1}(4k+1)\big) \text{ if } r > 1, \\ &\ \ \vdots \\ &= v\big(2(4k+1)\big) \text{ if } r > 0, \\ &= v(4k+1) + 1, \\ &= v(2k) + 2 \text{ by i)}. \end{aligned}$$

iii)
$$\begin{aligned} v\big(2^r(4k+3)\big) &= v\big(2^{r-1}(4k+3)\big) \text{ if } r > 1, \\ &\ \ \vdots \\ &= v\big(2(4k+3)\big), \\ &= v(4k+3) \text{ if } r > 0, \\ &= v(2k+1). \end{aligned}$$
□

Theorem 15. i) $v(2k+1) - v(2k) = (-1)^{k+1}$

ii) $v(2^r(4k+3)) - v(2^s(4k+1)) = \begin{cases} (-1)^{k+1} - 2, & \text{if } s > 0 \\ (-1)^{k+1} - 1, & \text{if } s = 0 \end{cases}$

i) By Theorems 3 and 11,

$$\begin{aligned} v(2k+1) - v(2k) &= 2f_{2k+1} - 1, \\ &= (-1)^{k+1}. \end{aligned}$$

ii) By Theorem 14,

$$\begin{aligned} v(2^r(4k+3)) - v(2^s(4k+1)) &= v(2k+1) - v(2k) - 2 \text{ if } s > 0, \\ &= v(2k+1) - v(2k) - 1 \text{ if } s = 0 \end{aligned}$$

and the result follows by i). □

4. TRIPLES, DOUBLES AND SINGLES

We now consider consecutive runs of identical terms.

Definition 7. (n-tuple). *An n-tuple is an instance of n, and only n, consecutive identical values in a binary sequence.*

EXAMPLE 2. $S_4 = 110110011100100$ contains:

- one instance of the triple, 111;
- two instances of the double, 11, and three instances of the double, 00;
- one instance each of the single 1 and the single 0.

Theorem 16. *For $n \geq 4$, S_n contains:*
i) 2^{n-4} *instances of the triple,* 111, *and* $2^{n-4} - 1$ *instances of the triple,* 000*;*
ii) 2^{n-3} *instances of the double,* 11, *and* $2^{n-3} + 1$ *instances of the double,* 00*;*
iii) 2^{n-4} *instances each of the single,* 1, *and the single,* 0.

Proof. We prove i) by induction. Similar proofs exist for ii) and iii). The theorem holds for $n = 4$. Suppose for some k, where $k \geq 4$, S_k contains 2^{k-4} instances of the triple, 111, and $2^{k-4} - 1$ instances of the triple, 000.

Consider $S_{k+1} = S_k \; 1 \; \overline{S_k^R}$. Since S_k contains $2^{k-4} - 1$ instances of the triple, 000, $\overline{S_k^R}$ contains $2^{k-4} - 1$ instances of the triple, 111 (they are mirrors of each other). Similarly, since S_k contains 2^{k-4} instances of the triple, 111, $\overline{S_k^R}$ contains 2^{k-4} instances of the triple, 000. Now because S_k is a mirror sequence that begins with 110, it concludes with 100. Therefore $\overline{S_k^R}$ begins with 110. It follows that the middle term of $S_{k+1} = S_k \; 1 \; \overline{S_k^R}$ generates a triple of 111 in addition to those already found in S_k and $\overline{S_k^R}$.

Thus S_{k+1} contains $2^{k-4} + 2^{k-4} - 1 + 1 = 2^{k-3}$ instances of the triple, 111. S_{k+1} also contains $2^{k-4} + 2^{k-4} - 1 = 2^{k-3} - 1$ instances of the triple, 000. It follows that our result holds for all $k \geq 4$. □

Corollary 3. *The middle element of S_n is located at the beginning of the* $(2^{n-5} + 1)^{\text{th}}$ *instance of the triple,* 111.

Proof. By Theorem 16, S_{n-1} contains 2^{n-5} instances of the triple, 111. Since $S_n = S_{n-1} \; 1 \; \overline{S_{n-1}^R}$ and $1 \; \overline{S_{n-1}^R}$ begins with an instance of the triple, 111, the result follows. □

Theorem 17. *S_n contains only singles, doubles or triples.*

Proof. Since $S_i = A_{2^{i-1}} \# S_{i-1}$, every odd term in S_i is different to adjacent odd terms. Therefore the maximum run of like terms is three before a change occurs. That is, only singles, doubles and triples can exist within S_i. □

We discover that singles are regularly spaced in the paperfolding sequence unlike doubles and triples. The following theorem shows that, beginning at f_{13}, single instances of 1 repeat every 16 spaces in S, and beginning at f_3, single instances of 0 also repeat every 16 spaces in S.

Theorem 18. *In S:*
i) *The set of all single instances of* 1 *is given by* $\{f_{16h+13} : h = 0, 1, 2, \ldots\}$.
ii) *The set of all single instances of 0 is given by* $\{f_{16h+3} : h = 0, 1, 2, \ldots\}$.

Proof. From Theorem 6, for $i = n + 3$ and letting $n \to \infty$, we have

$$S = S_3 1 \ \overline{S_3^R} \ 1 \ S_3 \ 0 \ \overline{S_3^R} \ldots \tag{4.1}$$

where the bracketed entries in (4.1) are consecutive terms in S. Each triple is formed at these bracketed entries and nowhere else. This is because S_3 and $\overline{S_3^R}$ contain no triples and both begin with 11 and end with 00. Singles can only then be found within the S_3 and $\overline{S_3^R}$ components of (4.1) without the first two and last two terms of each component. Consider the following typical subsequence in (4.1) beginning with the component S_3 and deleting the middle term

$$S_3 \ (1 \text{ or } 0) \ \overline{S_3^R} = 1101100 \ (1 \text{ or } 0) \ 1100100. \tag{4.2}$$

Note that in (4.2) the third term is the single 0 and the thirteenth term is the single 1. There are no other singles in (4.2) . Since the bracketed term does not affect the incidence of singles (which is the reason for us not including it in (4.2)), it follows that single 0s occur every third term in repeats of S_3 (1 or 0) $\overline{S_3^R}$, that is, at a spacing of 16 terms. Similarly, single 1s occur every thirteenth term in repeats of S_3 (1 or 0) $\overline{S_3^R}$. The result follows. □

The following theorem shows that triples are not regularly spaced, but nonetheless can be identified exactly within the paperfolding sequence.

Theorem 19. *In S :*
i) *The set of all* ***first*** *terms of all triples* 111 *is given by*

$$\left\{f_{2^k(4h+1)} : k = 3, 4, 5, \ldots; \ h = 0, 1, 2, \ldots\right\}.$$

ii) *The set of all* ***last*** *terms of all triples* 000 *is given by*

$$\left\{f_{2^k(4h+3)} : k = 3, 4, 5, \ldots; \ h = 0, 1, 2, \ldots\right\}.$$

Proof. From Theorem 6, for $i = n + 3$ and letting $n \to \infty$, we have

$$S = S_3 \ 1 \ \overline{S_3^R} \ 1 \ S_3 \ 0 \ \overline{S_3^R} \ldots \tag{4.3}$$

where the bracketed entries in (4.3) are consecutive terms in S. The bracketed entries in (4.3) are at $f_8, f_{16}, f_{24}, \ldots$ and by Theorem 4, bracketed entries with value 1 are of the form $f_{2^k(4h+1)}$. Thus each triple 111 starts at an f_n with n of the form $n = 2^k(4h+1)$ and $n \mid 8$, establishing i). Similarly, by Theorem 4, bracketed entries in (4.3) with value 0 are of the form $f_{2^k(4h+3)}$. Accordingly, each triple 000 ends at an f_n with n of the form $n = 2^k(4h+3)$ and $n \mid 8$, establishing ii). □

We now give attention to doubles which have a semi-regular pattern of spacings.

Theorem 20. *In S:*

i) *The set of all first terms of double instances of 11 is the union of the sets*

- $\{f_1\}$,
- $\left\{f_{2^k(4h+3)+1} : k = 3, 4, 5, \ldots;\ h = 0, 1, 2, \ldots\right\}$ *and*
- $\{f_{16h+4} : h = 0, 1, 2, \ldots\}$.

ii) *The set of all first terms of double instances of 00 is the union of the sets*

- $\left\{f_{2^k(4h+1)-2} : k = 3, 4, 5, \ldots;\ h = 0, 1, 2, \ldots\right\}$ *and*
- $\{f_{16h+11} : h = 0, 1, 2, \ldots\}$.

Proof. i) Since $S = 110\ldots$, f_1 begins a double 11. In (4.3) where a 0 appears, it is always succeeded by a double 11. From Theorem 19, $\{f_{2^k(4h+3)+1} : k = 3, 4, 5, \ldots;$ $h = 0, 1, 2, \ldots\}$ must therefore be the set of first terms of doubles 11 following triples 000. From (4.2), a double 11 always begins at the fourth term of each S_3 component of (4.3). These are all the possibilities for doubles 11 and so the result follows.

ii) In (4.3) where a 1 appears, it is always preceded by a double 00. From Theorem 19, $\left\{f_{2^k(4h+3)-2} : k = 3, 4, 5, \ldots;\ h = 0, 1, 2, \ldots\right\}$ must therefore be the set of first terms of doubles 00 preceding triples 111. From (4.2), a double 00 always begins at the third term of each $\overline{S_3^R}$ component of (4.3). These are all the possibilities for doubles 00 and so the result follows. □

5. REPEATED SUBSEQUENCS OF THE PAPERFOLDING SEQUENCE

We now consider repeated subsequences of S_n. Our aim is to discover all the repeated subsequences of S_n and their size. By so doing, we discover the largest repeated subsequences of S_n.

Definition 8. (Repeated Subsequences). *Let $S_n = f_1 f_2 \ldots f_{2^n-1}$. Let also $U_{k,s} = f_k f_{k+1} \ldots f_{k+s}$ and $V_{j,s} = f_j f_{j+1} \ldots f_{j+s}$ be subsequences of S_n. If for $j \neq k$,*

i) *$U_{k,s} = V_{j,s}$ and*

ii) *one of $j = 1, k = 1,$ or $f_{j-1} \neq f_{k-1}$ is true, and*

iii) *one of* $j+s=2^n-1$, $k+s=2^n-1$, *or* $f_{j+s+1} \neq f_{k+s+1}$ *is true, then* $U_{k,s}$ *and* $V_{j,s}$ *are called repeated subsequences (rsss) of* S_n.

Consider S_n for $n > 4$. We have

$$S_n = S_{n-1} 1\ \overline{S_{n-1}^R}, \tag{5.1}$$

$$= S_{n-2}\ 1\ \overline{S_{n-2}^R}\ 1\ S_{n-2}\ 0\ \overline{S_{n-2}^R}, \tag{5.2}$$

$$= S_{n-3}\ 1\ \overline{S_{n-3}^R}\ 1\ S_{n-3}\ 0\ \overline{S_{n-3}^R}\ 1\ S_{n-3}\ 1\ \overline{S_{n-3}^R}\ 0\ S_{n-3}\ 0\ \overline{S_{n-3}^R}, \tag{5.3}$$

$$\vdots$$

$$= S_{i+1}\ 1\ \overline{S_{i+1}^R}\ 1\ \cdots\ 0\ \overline{S_{i+1}^R}\ 1\ S_{i+1}\ 1\ \cdots\ 0\ S_{i+1}\ 0\ \overline{S_{i+1}^R}, \tag{5.4}$$

$$= S_i\ 1\ \overline{S_i^R}\ 1\ \cdots\ 0\ \overline{S_i^R}\ 1\ S_i\ 1\ \cdots\ 0\ S_i\ 0\ \overline{S_i^R}, \tag{5.5}$$

$$\vdots$$

$$= S_5\ 1\ \overline{S_5^R}\ 1\ \cdots\ 0\ \overline{S_5^R}\ 1\ S_5\ 1\ \cdots\ 0\ S_5\ 0\ \overline{S_5^R}, \tag{5.6}$$

$$= S_4\ 1\ \overline{S_4^R}\ 1\ \cdots\ 0\ \overline{S_4^R}\ 1\ S_4\ 1\ \cdots\ 0\ S_4\ 0\ \overline{S_4^R}, \tag{5.7}$$

$$= S_3\ 1\ \overline{S_3^R}\ 1\ S_3\ 0\ \overline{S_3^R} \cdots,$$

$$\cdots\ 0\ \overline{S_3^R}\ 1\ S_3\ 1\ \cdots S_3\ 1\ \overline{S_3^R}\ 0\ S_3\ 0\ \overline{S_3^R}, \tag{5.8}$$

$$= S_2\ 1\ \overline{S_2^R}\ 1\ S_2\ 0\ \overline{S_2^R} \cdots,$$

$$\cdots\ 0\ \overline{S_2^R}\ 1\ S_2\ 1\ \cdots S_2\ 1\ \overline{S_2^R}\ 0\ S_2\ 0\ \overline{S_2^R}. \tag{5.9}$$

Lemma 1. *If* $n > i > 2$, *there is no occurrence of* S_i *or* $\overline{S_i^R}$ *in* S_n *or* $\overline{S_n^R}$ *other than those shown explicitly in* (5.5) *or its counterpart for* $\overline{S_n^R}$.

Proof. (By induction) We prove both the S_n and $\overline{S_n^R}$ cases by induction on i.

Consider $i = 3$.

If there were an occurrence of S_3 (or $\overline{S_3^R}$) other than one explicitly shown in (5.5) or its counterpart for $\overline{S_n^R}$, S_3 (or $\overline{S_3^R}$) would have to appear as a part of $S_3\ 1\ \overline{S_3^R}$, $\overline{S_3^R}\ 1\ S_3$, $S_3\ 0\ \overline{S_3^R}$ or $\overline{S_3^R}\ 0\ S_3$. This can be shown, by inspection, to not be the case.

We now assume the result for i.

Consider $i + 1$.

If S_{i+1} (or $\overline{S_{i+1}^R}$) appears in (5.4) or its counterpart for $\overline{S_n^R}$, in other than the explicitly shown positions, it must appear in a part of $S_{i+1}\ 1\ \overline{S_{i+1}^R}$, $\overline{S_{i+1}^R}\ 1\ S_{i+1}$, $S_{i+1}\ 0\ \overline{S_{i+1}^R}$ or $\overline{S_{i+1}^R}\ 0\ S_{i+1}$ that includes the 1 or 0 . But then S_i (or $\overline{S_i^R}$) appears in $S_i\ 1\ \overline{S_i^R}$, $S_i\ 0\ \overline{S_i^R}$, $\overline{S_i^R}\ 1\ S_i$ or $\overline{S_i^R}\ 0\ S_i$ other than as an explicit S_i (or $\overline{S_i^R}$). This is impossible by the induction hypothesis. □

Proof. (By contradiction) Suppose our result is false, that is, there exists an S_i

or $\overline{S_i^R}$ not explicitly identified in (5.5) or its counterpart for $\overline{S_n^R}$. Since

$$S_i \text{ or } \overline{S_i^R} = S_3\ 1\ \overline{S_3^R} \cdots \ (1 \text{ or } 0)\ \cdots S_3\ 0\ \overline{S_3^R} \tag{5.10}$$

and (5.8) contains every instance of S_3 or $\overline{S_3^R}$ in S_n and $\overline{S_n^R}$, the right hand side of (5.10) must be explicitly identified in (5.8) or its $\overline{S_n^R}$ counterpart. But each S_3 and $\overline{S_3^R}$ entry in (5.8) is used to form only the S_i and $\overline{S_i^R}$ entries of (5.5), a contradiction. □

Lemma 2. *Every repeated subsequence (rss) of S_n $(n > 5)$ of length*
i) *greater than 7, must start with S_3 or $\overline{S_3^R}$.*
ii) *7, is a) S_3 or $\overline{S_3^R}$, or b) $\overline{S_2^R}$ (1 or 0) S_2.*
iii) *less than 7, does not contain triples and can be discovered within the following two subsequences of S_n : $001000S_3111001$ and $011000\overline{S_3^R}111011$.*

Proof. There are two cases to consider:

1.) Repeated subsequences containing triples.

Triples only form at 1 or 0 in (5.8) since each 1 or 0 is preceded by 00 and is followed by 11. Consider the first triple in the rss. From (5.8) any 1 or 0 is preceded by a 100 from S_3 or $\overline{S_3^R}$. Accordingly, the smallest possible rsss containing a triple is $\overline{S_2^R}$ (1 or 0) S_2, establishing ii)b). If the rss is any longer and contains a preceding 1 from (5.8) it must contain a whole S_3 and, as it doesn't contain an earlier triple, must start with S_3. Alternatively, if it contains a preceding 0 from (5.8), it must start with $\overline{S_3^R}$. Thus i) is established.

2.) Repeated subsequences that do not contain triples.

The largest sequences that do not contain triples in S_n, both of length 11, are

$$0\ 0\ \left(S_3 \text{ or } \overline{S_3^R}\right)\ 1\ 1 \tag{5.11}$$

where the subsequences $0\ \left(S_3 \text{ or } \overline{S_3^R}\right)\ 1$ are entries in (5.8).

Any rss of length 7 or greater, without containing a triple, must contain an S_3 or $\overline{S_3^R}$ from (5.11), or the initial or final $(6/7)^{\text{th}}$ or $(5/7)^{\text{th}}$ of S_3 or $\overline{S_3^R}$ from (5.11). Now S_3 and $\overline{S_3^R}$ are rsss, but any other rss in (5.11) of length 7 or greater will include an explicit 0 or 1 from (5.8). Any such 0 will be preceded by 00 and any such 1 will be followed by 11, which have to be part of the rss. This is impossible, as then a triple is included. So there are no rsss of length 7 to 10, without triples, other than S_3 or $\overline{S_3^R}$, establishing ii)a).

From 1.), the smallest possible rss containing a triple has length 7. Accordingly, all rsss of length less than 7 do not contain triples and are discoverable, by observation, from $001000S_3111001$ and $011000\overline{S_3^R}111011$, establishing iii). □

Lemma 3. *If $i > 2$ and $\overline{S_i^R}$ starts a repeated subsequence (rss) of S_n then for*
i) *$n = i + 2$, the rss is $\overline{S_i^R}$.*

ii) $n = i + 3$, *the rsss are* $\overline{S_i^R}$ *and* $\overline{S_i^R}\ 1\ S_i$.
iii) $n > i + 3$, *the rsss are* $\overline{S_i^R}$, $\overline{S_i^R}\ 1\ S_i$ *and* $\overline{S_i^R}\ 0\ S_i$.

Proof. i) By Lemma 1 and (5.2), $\overline{S_i^R}$ is the only rss of S_{i+2} starting with $\overline{S_i^R}$.

ii) By Lemma 1 and (5.3), $\overline{S_i^R}$, $\overline{S_i^R}\ 1\ S_i$ are the only such rsss of S_{i+3}.

iii) By Lemma 1 and (5.5), if $n > i + 3$, $\overline{S_i^R}$, $\overline{S_i^R}\ 1\ S_i$ and $\overline{S_i^R}\ 0\ S_i$ are such rsss of S_n.

If there are other such rsss starting with a particular instance of $\overline{S_i^R}$, by Lemma 1 and (5.5), these start with

a) $\overline{S_i^R}\ 0\ S_i\ 0\ \overline{S_i^R} = \overline{S_i^R}\ 0\ \overline{S_{i+1}^R}$
b) $\overline{S_i^R}\ 0\ S_i\ 1\ \overline{S_i^R} = \overline{S_i^R}\ 0\ S_{i+1}$
c) $\overline{S_i^R}\ 1\ S_i\ 0\ \overline{S_i^R} = \overline{S_i^R}\ 1\ \overline{S_{i+1}^R}$
d) $\overline{S_i^R}\ 1\ S_i\ 1\ \overline{S_i^R} = \overline{S_i^R}\ 1\ S_{i+1}$.

In (5.4) it is clear that each $\overline{S_i^R}$ is part of an $\overline{S_{i+1}^R}$ or S_{i+1}, and that this is the case for all occurrences of the rss in S_n. Thus the rss has $\overline{S_i^R}$ as part of S_{i+1} or $\overline{S_{i+1}^R}$ and so does not start with $\overline{S_i^R}$. This is a contradiction, so the given rsss starting with $\overline{S_i^R}$ are the only ones. □

Theorem 21. *The repeated subsequences (rsss) of* S_n, *by length, are:*

Length	***Repeated Subsequences of:***				
	S_2	S_3	S_4	S_5	S_n, $n > 5$
1	1	1, 0	1, 0	1, 0	1, 0
2			11, 10, 01	11, 10, 01, 00	11, 10, 01, 00
3		110	$S_2, \overline{S_2^R}$, 011	$S_2, \overline{S_2^R}$, 011, 001	$S_2, \overline{S_2^R}$, 011, 001
4				1100, 1001	1100, 1001, 0011, 0110
5				11001, 01100	11001, 01100
6				011001	011001
7				$S_3, \overline{S_3^R}$, $\overline{S_2^R}\ 1\ S_2$	$S_3, \overline{S_3^R}$, $\overline{S_2^R}\ 1\ S_2$, $\overline{S_2^R}\ 0\ S_2$
15					$S_4, \overline{S_4^R}$, $\overline{S_3^R}\ 1\ S_3$, $\overline{S_3^R}\ 0\ S_3$
⋮					⋮
$2^{n-4} - 1$					$S_{n-4}, \overline{S_{n-4}^R}$, $\overline{S_{n-5}^R}\ 1\ S_{n-5}$, $\overline{S_{n-5}^R}\ 0\ S_{n-5}$

$2^{n-3}-1$					$S_{n-3}, \overline{S^R_{n-3}}$, $\overline{S^R_{n-4}}\ 1\ S_{n-4}$, $\overline{S^R_{n-4}}\ 0\ S_{n-4}$
$2^{n-2}-1$					$S_{n-2}, \overline{S^R_{n-2}}$, $\overline{S^R_{n-3}}\ 1\ S_{n-3}$

where the S_is are those shown explicitly in (5.2) *to* (5.9)*.*

Proof. There are two cases:
i) Lengths 1 to 7: For $n \leq 5$, by observation. For $n > 5$, use Lemma 2 ii) and iii).
ii) Lengths greater than 7: By Lemma 2 part i), any rss of S_n must start with an S_i or $\overline{S_i^R}$ for $i \geq 3$. Lemma 3 provides all the rsss starting with $\overline{S_i^R}$.

If an rss starts with S_i, it can be just S_i for $3 \leq i \leq n-2$, or it can, by (5.5), continue as

a) $S_i\ 0\ \overline{S_i^R} = \overline{S_{i+1}^R}$, or

b) $S_i\ 1\ \overline{S_i^R} = S_{i+1}$.

In case a), by Lemma 3, we have one of the above rsss.

In case b), the rss is S_{i+1} or, applying the above again, starts with $\overline{S_{i+2}^R}$ or S_{i+2}, so the same rsss are obtained. □

Corollary 4. *The longest rsss of S_n for $n > 4$, are S_{n-2}, $\overline{S_{n-2}^R}$ and $\overline{S_{n-3}^R}\ 1\ S_{n-3}$.*

Proof. The case for $n = 5$ is proven by observation. The case for $n > 5$ follows from Theorem 21. □

Corollary 5. *Let $\mathcal{L}$ be the length of the largest repeated subsequence of S_n. Then*

$$\mathcal{L} = \begin{cases} 2^{n-1}-1 & \text{for } n < 4 \\ 5 & \text{for } n = 4 \\ 2^{n-2}-1 & \text{for } n > 4. \end{cases}$$

Proof. The case for $n \leq 4$ is proven by observation. The case for $n > 4$ follows from Corollary 4. □

6. POWER SERIES AND THE PAPERFOLDING SEQUENCE

We conclude with the generalisation of a result mentioned in Dekking et al [**8**].

Theorem 22. *Let $S = f_1 f_2 f_3 \ldots$ and $F(x) = \sum_{n=1}^{\infty} f_n x^n$ where $|x| < 1$. Then for $m = 1, 2, \ldots$,*

$$F(x) - F\left(x^{2^m}\right) = \sum_{i=0}^{m-1} \frac{x^{2^i}}{1 - x^{2^{i+2}}}.$$

Proof. By Theorem 4, for $|x| < 1$,

$$\begin{aligned} F(x) - F\left(x^{2^m}\right) &= \sum_{i=0}^{\infty}\sum_{h=0}^{\infty} x^{2^i(4h+1)} - \sum_{i=0}^{\infty}\sum_{h=0}^{\infty} x^{2^{i+m}(4h+1)}, \\ &= \sum_{i=0}^{m-1}\sum_{h=0}^{\infty} x^{2^i(4h+1)}, \\ &= \sum_{i=0}^{m-1} \frac{x^{2^i}}{1 - x^{2^{i+2}}}. \end{aligned}$$

□

The result in Dekking et al [**8**] can be represented as a corollary to Theorem 22.

Corollary 6. *Let* $S = f_1 f_2 f_3 \ldots$ *and* $F(x) = \sum_{n=1}^{\infty} f_n x^n$ *where* $|x| < 1$. *Then* $F(x) - F\left(x^2\right) = \frac{x}{1 - x^4}$.

Proof. This is the case $m = 1$ in Theorem 22. □

7. LINKING PAPERFOLDING AND THE STERN-BROCOT TREE

An excellent presentation of the Stern-Brocot tree can be found in Graham et al [**9**]. Links between the Paperfolding and Stern-Brocot sequences using continued fractions have been identified previously (See for example, Mendès France et al. [**13**]). We now continue this linkage. Firstly, we give some background definitions.

Definition 9. (Mediants) *We define the mediant of* $\frac{m}{n}$ *and* $\frac{r}{s}$ *as* $\frac{m}{n} \oplus \frac{r}{s} = \frac{m+r}{n+s}$.

The Stern-Brocot tree is made up of successive levels of mediants with the 0^{th} level consisting of $\frac{0}{1}$ and $\frac{1}{0}$. The mediants in any one level are generated by forming mediants from terms in previous levels. Thus the mediant of $\frac{1}{2}$ (found in level 2) and $\frac{2}{3}$ (found in level 3) is $\frac{1+2}{2+3} = \frac{3}{5}$ which is found in level 4. The Stern-Brocot sequence is related to these mediants according to the following definition:

Definition 10. (Stern-Brocot Sequence) *Let* $H_0 = \left\langle \frac{0}{1}, \frac{1}{0} \right\rangle$ *and for* $k \geq 1$,

$$H_k = H_{k-1} \# \operatorname{med} H_{k-1}$$

where $\operatorname{med} H_{k-1}$ *denotes the increasing sequence of mediants that are generated from consecutive terms in* H_{k-1}. *That is, if*

$$H_{k-1} = \left\langle h_{k-1,1}, h_{k-1,2}, \ldots, h_{k-1,2^k+1} \right\rangle,$$

then

$$\operatorname{med} H_{k-1} = \left\langle \left(h_{k-1,1} \oplus h_{k-1,2}\right), \left(h_{k-1,2} \oplus h_{k-1,3}\right), \cdots, \left(h_{k-1,2^k} \oplus h_{k-1,2^k+1}\right) \right\rangle.$$

H_k represents the increasing sequence containing both the first k generations of mediants based on H_0, and the terms of H_0 itself. It is called the Stern-Brocot sequence.

We now explore the presence of a paperfolding pattern in the Stern-Brocot tree.

Definition 11. (Left and Right Mediants) *A left mediant is the mediant formed in level $k+1$ that is smaller than its parent found in level k. A right mediant is the mediant formed in level $k+1$ that is greater than its parent found in level k.*

From Definition 11, if m is a term in the i^{th} level of the Stern-Brocot tree, the left mediant of m is the mediant that has m as its parent and appears in the $(i+1)^{\text{th}}$ level and to the left of m. Similarly the right mediant of m is the mediant that has m as its parent and appears in the $(i+1)^{\text{th}}$ level and to the right of m.

Example 3. The left mediant of $\frac{3}{8}$ is $\frac{4}{11}$. The right mediant of $\frac{3}{8}$ is $\frac{5}{13}$. As expected these mediants appear in the 6^{th} level because $\frac{3}{8}$ is in the 5^{th} level.

Since every term possesses a left and right mediant (except $\frac{0}{1}$ and $\frac{1}{0}$) and from level 2 onwards there are exactly twice the number of terms in any level to that level immediately above it, all terms in any level are simply a succession of left and right mediants of terms in the preceding level. Moreover since in any one level (except the 0^{th} and 1^{st}), the left mediant of a term is immediately followed by the right mediant of the same term, each level after the first consists of an alternating sequence of left and right mediants.

Bates et al. [**5**] have shown that when expressed in the shortest form of their simple continued fractions:

i) Left mediants possess continued fractions of the form $[a_0, a_1, \ldots, a_k]$ where k is odd.

ii) Right mediants possess continued fractions of the form $[a_0, a_1, \ldots, a_k]$ where k is even.

Accordingly, if we designate any entry in the Stern-Brocot tree with short form continued fraction $[a_0, a_1, \ldots, a_k]$ as 1 for k odd and 0 for k even, then every level of the Stern-Brocot tree represents an alternating sequence. Under this mapping, if we displace vertically downwards levels 1 to $j-1$ onto level j, (that is, if we generate the Stern-Brocot sequence H_j and delete the terms $\frac{0}{1}$ and $\frac{1}{0}$ and then redesignate all terms as either 0 or 1 according to this mapping) we have, by Theorem 2, the paperfolding sequence of size $2^j - 1$. That is, for j unbounded, we have established the following result:

Theorem 23. *Delete $\frac{0}{1}$ and $\frac{1}{1}$ from the Stern-Brocot sequence and represent every other term, except* 1, *in its short form continued fraction* $[a_0, a_1, \ldots, a_k]$. *If k is odd, replace the continued fraction with* 1; *if k is even, replace the continued fraction with* 0. *The resulting sequence is the paperfolding sequence.*

Theorem 23 tells us that the parity of the length of the short form continued fraction for $\frac{a(n)}{b(n)}$, where $a(n)$ is Sloane's sequence A007305 and $b(n)$ is Sloane's sequence A047679, is the paperfolding sequence. (See SLOANE [**15**, A007305, A047679]). Equivalently, Theorem 23 tells us that if we designate all left mediants as 1 and all right mediants as 0 in the Stern-Brocot tree, and set the term in level 1 as 1, then for $n > 1$, level n converts to the alternating sequence. By then moving every term vertically downward from level 1 to $n-1$ onto level n we obtain S_n.

8. LINKING PAPERFOLDING AND THE BINARY REFLECTED GRAY CODE

Our attention now turns to the Binary Reflected Gray Code (BRGC) as yet another example of the way in which the paperfolding sequence, and interleave operators in general, are embedded in many constructions.

Gray Codes owe their name to Frank Gray, a research physicist at the Bell Telephone Laboratory. Though they were first used in telegraphy by Emile Baudot (1845–1903), Gray used these codes to minimise errors that arose in signals transmitted by pulse code modulation. The codes are still used for checking errors in communications systems. There are several types of code with that property. We confine ourselves to the Binary Reflected Gray Code (BRGC).

Definition 12. (Binary Reflected Gray Code, BRGC) *Let $n = 2^k + j$, where n, j and k are non-negative integers such that $2^k \leq n < 2^{k+1}$, $k \geq 0$ and so $0 \leq j < 2^k$.*

Then the Binary Reflected Gray Code of n is defined by

$$\begin{aligned} G(n) &= n - j + G(n - (2j+1)), \\ &= 2^k + G\left(2^k - (j+1)\right) \end{aligned}$$

where $G(0) = 0$.

We designate $G(n)$ to base 2 as $G(n)_2$ and define the First Forward Difference Function of G, ΔG, as follows:

Definition 13. (First Forward Difference of G) *The First Forward Difference Function of G, ΔG, is defined as $\Delta G(n) = G(n+1) - G(n)$.*

Also we define the Binary First Forward Difference of G sequence, $\mathcal{P}_i$, as follows:

Definition 14. (Binary First Forward Difference of G Sequence) *The Binary First Forward Difference of G Sequence, $\mathcal{P}_i$, is defined as $\mathcal{P}_i = \left\langle p(0), p(1), \ldots, p\left(2^i - 2\right)\right\rangle$, where for $n = 0, 1, 2, \ldots$*

$$p(n) = \begin{cases} 1 & \text{if } \Delta G(n) > 0 \\ 0 & \text{if } \Delta G(n) < 0. \end{cases}$$

Table 1 shows $G(n)$ for $n = 0$ to 32 with corresponding values for $G(n)$, $G(n)_2$, $\Delta G(n)$ and $p(n)$. Notice that each successive value of $G(n)$ differs from its previous value in one bit change. For example, for $n = 9, 10, 11$, $G(n)_2 = 1101$, 1111, 1110. It is this feature that makes Gray Codes attractive as a tool for checking errors in data transmission. Notice also in Table 1 that we have grouped successive integers into blocks. That is, the k^{th} block, designated as B_k, consists of all integers $2^k \le n < 2^{k+1}$.

n	$G(n)$	$G(n)_2$	$\Delta G(n)$	$p(n)$
0	0	0	1	1
1	1	1	2	1
2	3	11	−1	0
3	2	10	4	1
4	6	110	1	1
5	7	111	−2	0
6	5	101	−1	0
7	4	100	8	1
8	12	1100	1	1
9	13	1101	2	1
10	15	1111	−1	0
11	14	1110	−4	0
12	10	1010	1	1
13	11	1011	−2	0
14	9	1001	−1	0
15	8	1000	16	1
16	24	11000	1	1
17	25	11001	2	1
18	27	11011	−1	0
19	26	11010	4	1
20	30	11110	1	1
21	31	11111	−2	0
22	29	11101	−1	0
23	28	11100	−8	0
24	20	10100	1	1
25	21	10101	2	1
26	23	10111	−1	0
27	22	10110	−4	0
28	18	10010	1	1
29	19	10011	−2	0
30	17	10001	−1	0
31	16	10000	32	1
32	48	110000	1	1

Table 1: The Binary Reflected Gray Code

Lemma 4. *In the column $\Delta G(n)$ in Table 1, for $k \ge 0$ and $0 < t < 2^{k+1}$,*
i) $\Delta G\left(2^k - 1\right) = 2^k$, *and*
ii) $\Delta G\left(2^{k+1} - 1 + t\right) = -\Delta G\left(2^{k+1} - 1 - t\right)$.

Proof. i) From Definition 13, $\Delta G\left(2^k-1\right)=G\left(2^k\right)-G\left(2^k-1\right)$ and from Definition 12, $G\left(2^k\right)=2^k+G\left(2^k-1\right)$. Thus $\Delta G\left(2^k-1\right)=2^k$.

ii) By Definitions 13 and 12,

$$\begin{aligned}\Delta G\left(2^{k+1}-1+t\right) &= \Delta G\left(2^{k+1}+(t-1)\right),\\ &= G\left(2^{k+1}+t\right)-G\left(2^{k+1}+(t-1)\right),\\ &= \left[2^{k+1}+G\left(2^{k+1}-t-1\right)\right]-\left[2^{k+1}+G\left(2^{k+1}-t\right)\right],\\ &= G\left(2^{k+1}-t-1\right)-G\left(2^{k+1}-t\right),\\ &= -\Delta G\left(2^{k+1}-1-t\right). \end{aligned}$$

□

Lemma 4 ii) tells us that in the column $\Delta G(n)$ in Table 1, for $k>0$, entries that are equally spaced either side of 2^k have the same magnitude but are opposite in sign.

Theorem 24. *$\mathcal{P}_i=S_i$, the paperfolding sequence of length 2^i-1.*

Proof. We prove this by induction. Since $\Delta G(0)=1$, $\mathcal{P}_1=1=S_1$. Suppose $\mathcal{P}_k=S_k$. From Lemma 4,

i) $\Delta G\left(2^k-1\right)=2^k$, and so $p\left(2^k-1\right)=1$.

ii) All entries $\Delta G\left(2^k+t\right)=-\Delta G\left(2^k-t-2\right)$ for $t=0,1,2,\ldots,2^k-2$.

But this means that $\mathcal{P}_{k+1}=S_k\ 1\ \overline{S_k^R}=S_{k+1}$. □

Note that Theorem 24 can also be proven from Theorem 4 and Lemma 11 in Bunder et al. [**7**].

We conclude with an interesting aspect of the code whereby a variant of the paperfolding sequence appears through examining the inverse of the code.

Definition 15. (Inverse Binary First Forward Difference of G Sequence). *Let $\Delta G^{-1}(n)=G^{-1}(n+1)-G^{-1}(n)$ whereby*

$$q(n)=\begin{cases}1 & \text{if } \Delta G^{-1}(n)>0\\ 0 & \text{if } \Delta G^{-1}(n)<0.\end{cases}$$

The Inverse Binary First Forward Difference of G Sequence, $\mathcal{Q}_i$, is given by:

$$\mathcal{Q}_i=\left\langle q(0),q(1),\ldots,q\left(2^i-2\right)\right\rangle.$$

Table 2 shows $\Delta G^{-1}(n)$ and $q(n)$ for $n=0$ to 32. As with the paperfolding sequence, $\mathcal{Q}_i$, can be expressed in three forms, namely,

i) Mirroring: $\mathcal{Q}_{i+1}=\mathcal{Q}_i\ 1\ \overline{\mathcal{Q}_i}$, where $\mathcal{Q}_1=1$ and $\overline{\mathcal{Q}_i}$ is defined as the sequence formed when each 1 in $\mathcal{Q}_i$ is replaced by 0 and each 0 in $\mathcal{Q}_i$ is replaced by 1.

ii) Interleaving: $\mathcal{Q}_{i+1}=D_{2^i}\#\mathcal{Q}_i$, where $D_{2^i}=D_{2^{i-1}}\overline{D_{2^{i-1}}}$ for which $D_2=A_2=10$, $\mathcal{Q}_1=1$.

iii) Alternation: $\mathcal{Q}_{i+1}=D_{2^i}\#D_{2^{i-1}}\#\cdots\#D_2\#1$.

n	$G^{-1}(n)$	$G^{-1}(n)_2$	$\Delta G^{-1}(n)$	$q(n)$
0	0	0	1	1
1	1	1	2	1
2	3	11	−1	0
3	2	10	5	1
4	7	111	−1	0
5	6	110	−2	0
6	4	100	1	1
7	5	101	10	1
8	15	1111	−1	0
9	14	1110	−2	0
10	12	1100	1	1
11	13	1101	−5	0
12	8	1000	1	1
13	9	1001	2	1
14	11	1011	−1	0
15	10	1010	21	1
16	31	11111	−1	0
17	30	11110	−2	0
18	28	11100	1	1
19	29	11101	−5	0
20	24	11000	1	1
21	25	11001	2	1
22	27	11011	−1	0
23	26	11010	−10	0
24	16	10000	1	1
25	17	10001	2	1
26	19	10011	−1	0
27	18	10010	5	1
28	23	10111	−1	0
29	22	10110	−2	0
30	20	10100	1	1
31	21	10101	42	1
32	63	111111	−1	0

Table 2: The Inverse Binary Reflected Gray Code

9. NUMBERED PAPERFOLDING SEQUENCE

We introduce a numbering of the entries in the paperfolding sequence to generate the numbered paperfolding sequence.

Definition 16. (Numbered Paperfolding sequence) *Let $\mathbb{S}_n$ represent the numbered paperfolding sequence after n folds commencing with $\mathbb{S}_1 = 1$. That is, $\mathbb{S}_n$ is formed by the interleaving of the numbered creases $\langle 2^{n-1}, 2^{n-1}+1, \cdots, 2^n - 1\rangle$ with $\mathbb{S}_{n-1}$.*

Note that the numbered paperfolding sequence is simply the stickbreaking sequence without its first term 0. Details on the stickbreaking sequence can be found at [**16**].

Example 4. $\mathbb{S}_5 = \left\langle \begin{array}{c} 16,8,17,4,18,9,19,2,20,10,21,5,22,11,23,1, \\ 24,12,25,6,26,13,27,3,28,14,29,7,30,15,31 \end{array} \right\rangle$

Lemma 5. *Every even integer can be expressed in the form* $2^{k-1} + a2^k$ *where* $k = 2,3,\ldots$ *and* $a = 0,1,2,\ldots$

Proof. For $p > 1$, every even integer u can be expressed in one of two ways. That is, $u = 2^p\,(4h+1)$ or $u = 2^p\,(4h+3)\,, h = 0,1,2,\ldots$

Consider each case:

i) $u = 2^p\,(4h+1):$

$$\begin{aligned} u &= 2^p\,(4h+1)\,, \\ &= 2^p\,(2\cdot 2h+1)\,, \\ &= 2^{p+1}\cdot 2h + 2^p. \end{aligned} \tag{9.1}$$

ii) $u = 2^p\,(4h+3):$

$$\begin{aligned} u &= 2^p\,(4h+3)\,, \\ &= 2^p\,(4h+2+1)\,, \\ &= 2^{p+1}\,(2h+1) + 2^p. \end{aligned} \tag{9.2}$$

□

Theorem 25. *Let* $f_{t,n}$ *denote the* t^{th} *entry in* $\mathbb{S}_n$. *Then,*

i) *for odd entries in* $\mathbb{S}_n$, $f_{2k-1,n} = 2^{n-1} + k - 1$, where $k = 1,2,\cdots,2^{n-1}$.

ii) *for even entries in* $\mathbb{S}_n$, $f_{2^{k-1}+a2^k,n} = 2^{n-k} + a$, where $k = 2,3,\cdots,n$ *and* $a = 0,1,2,\cdots,2^{n-k}-1$.

Proof. We prove by induction on n.

$\mathbb{S}_1 = \langle f_{1,1}\rangle = \langle 2^0 + 1 - 1\rangle = \langle 1\rangle\,.$

$\mathbb{S}_2 = \langle f_{1,2}, f_{2,2}, f_{3,2}\rangle = \langle 2^1 + 1 - 1, 2^{2-2}, 2^1 + 2 - 1\rangle = \langle 2,1,3\rangle\,.$

Suppose our inductive hypothesis is true for all n up to some value m. Since $\mathbb{S}_{m+1}$ is formed by interleaving $\langle 2^m, 2^m+1,\cdots,2^{m+1}-1\rangle$ with $\mathbb{S}_m$, we have

i) For odd entries in $\mathbb{S}_{m+1}$,

$$\begin{aligned} f_{2k-1,m+1} &= f_{2k-1,m} + 2^{m-1}, \\ &= 2^{m-1} + k - 1 + 2^{m-1}, \\ &= 2^m + k - 1. \end{aligned}$$

The first part of our result follows.

ii) For even entries in $\mathbb{S}_{m+1}$, consecutive even entries in $\mathbb{S}_{m+1}$ are consecutive entries in $\mathbb{S}_m$. There are two subcases:

a) If $f_{2t,m+1} \mid t$ odd, then these are the odd entries in $\mathbb{S}_m$ which are the entries spaced 4-apart, commencing with $f_{2,m+1}$, in $\mathbb{S}_{m+1}$. That is, they are all terms $f_{2^{k-1}+a2^k,m+1} \mid a = 0,1,2,\cdots,2^{m+1-k}-1$ and $k = 2$. Hence from our induction hypothesis

$$\begin{aligned} f_{2t,m+1} &= f_{2+a2^2,m+1} \mid a = 0,1,2,\cdots,2^{m-1}-1, \\ &= 2^{m-1}+r-1 \mid r = 1,2,\cdots,2^{m-1}, \\ &= 2^{m-1}+a \mid a = 0,1,2,\cdots,2^{m-1}-1, \\ &= 2^{m+1-2}+a \mid a = 0,1,2,\cdots,2^{m+1-2}-1. \end{aligned}$$

And so this subcase proves the second part of our result for $k = 2$.

b) If $f_{2t,m+1} \mid t$ even, then these are the even entries in $\mathbb{S}_m$ which are the entries spaced 4-apart, commencing with $f_{4,m+1}$, in $\mathbb{S}_{m+1}$. From Lemma , after excluding the case $k = 2$ in a), they are all terms $f_{2^{k-1}+a2^k,m+1} \mid a = 0,1,2,\cdots,2^{m+1-k}-1$ and $k = 3,4,\cdots,m+1$. Hence from our induction hypothesis,

$$\begin{aligned} f_{2t,m+1} &= f_{2^{k-1}+a2^k,m+1} \mid a = 0,1,2,\cdots,2^{m+1-k}-1 \text{ and } k = 3,4,\cdots,m+1, \\ &= f_{2^{j-1}+a2^j,m} \mid a = 0,1,2,\cdots,2^{m-j}-1 \text{ and } j = 2,3,4,\cdots,m, \\ &= 2^{m-j}+a \mid a = 0,1,2,\cdots,2^{m-j}-1 \text{ and } j = 2,3,4,\cdots,m, \\ &= 2^{m+1-k}+a \mid a = 0,1,2,\cdots,2^{m+1-k}-1 \text{ and } k = 3,4,\cdots,m+1. \end{aligned}$$

And so, this subcase proves the second part of our result for $k = 3,4,\cdots,m+1$. Combining a) and b) the second part of our result follows. Combining i) and ii) our result follows. □

The following corollary restates Theorem 25 in a longer form that is easier to work with for even entries of $\mathbb{S}_n$.

Corollary 7. *Let $f_{t,n}$ denote the t^{th} entry in $\mathbb{S}_n$. Then,*

i) *for odd entries in $\mathbb{S}_n$,* $f_{2k-1,n} = 2^{n-1}+k-1$,
where $k = 1,2,\cdots,2^{n-1}$.

ii) *for even entries in $\mathbb{S}_n$,*
a) $f_{2^k(4a+1),n} = 2^{n-k-1}+2a$ and
b) $f_{2^k(4a+3),n} = 2^{n-k-1}+2a+1$
where $k = 1,2,\cdots,n-1$ and $a = 0,1,2,\cdots,2^{n-k-1}-1$.

Proof. i) is directly taken from Theorem 25 i).
ii) a) follows from (9.1) and Theorem 25 ii).
ii) b) follows from (9.2) and Theorem 25 ii). □

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