SOME OSCILLATION CRITERIA FOR SECOND-ORDER DELAY DYNAMIC EQUATIONS

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We investigate the oscillation of second-order delay dynamic equations. Our results extend and improve known results for oscillation of second-order differential equations that have been established by ERBE [Canad. Math. Bull. **16** (1973), 49–56]. We apply results from the theory of upper and lower solutions and give some examples to illustrate the main results.

1. INTRODUCTION

Since 1998 much attention has been given to dynamic equations on time sales, and we refer the reader to the landmark paper of HILGER [**15**] for a comprehensive treatment of the subject. Since its introduction, many authors have expounded on various aspects of this new theory, and we refer specifically to the paper by AGARWAL et al. [**2**] and the references cited therein. A book on the subject of time scales by BOHNER and PETERSON [**5**] summarizes and organizes much of time scale calculus.

In recent years there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on a time scale (i.e., a closed subset of the real line $\mathbb{R}$). This has lead to many attempts to harmonize the oscillation theory for the continuous and the discrete cases, to include them in one comprehensive theory, and to extend the results to more general time scales. We refer the reader to the papers [**1**], [**3**], [**7, 8, 9, 10**], [**17**], [**18**], [**20**], and the references cited therein

Since we are interested in the oscillatory behavior of solutions near infinity, we assume throughout this paper that our time scale is unbounded above. We

assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval $[t_0, \infty)_\mathbb{T}$ by

$$[t_0, \infty)_\mathbb{T} := [t_0, \infty) \cap \mathbb{T}.$$

The purpose of this paper is to extend the oscillation criteria to the nonlinear second-order delay dynamic equation

$$(1.1) \qquad \left[p(t) y^\Delta \right]^\Delta + f\left(t, y^\sigma(t), y(\tau_1(t)), \ldots, y(\tau_n(t))\right) = 0, \quad t \in [t_0, \infty)_\mathbb{T}$$

where $n \in \mathbb{N}$, $f \in C(\mathbb{T} \times \mathbb{R}^{n+1}, \mathbb{R})$, and $p \in C_{rd}([t_0, \infty)_\mathbb{T}, (0, \infty)_\mathbb{T})$ satisfies

$$\int_{t_0}^{\infty} \frac{1}{p(t)} \Delta t = \infty, \quad t \in [t_0, \infty)_\mathbb{T}.$$

We shall assume the following conditions hold:
(A_0) $f(t, u, v_1, \ldots, v_n) = -f(t, -u, -v_1, \ldots, -v_n)$.
(A_1) $f(t, u, v_1, \ldots, v_n) > 0$ if $u, v_1, \ldots, v_n > 0$, $t \in \mathbb{T}$.
(A_2) for each fixed $t \in \mathbb{T}$ and $u > 0$, f is nondecreasing in v_i for $v_i > 0, 1 \leq i \leq n$.
(A_3) for each fixed $t \in \mathbb{T}$ and $v_i > 0, 1 \leq i \leq n$, f is nondecreasing in u for $u > 0$.
We also assume that the delay functions $\tau_i : \mathbb{T} \to \mathbb{T}$ are right-dense continuous and satisfy

$$\tau_i(t) \leq t \leq \sigma(t) \text{ for all } t \in \mathbb{T} \text{ and } \lim_{t \to \infty} \tau_i(t) = \infty$$

for all $1 \leq i \leq n$.

Our attention is restricted to those solutions $y(t)$ of (1.1) which exist on some half-line $[t_y, \infty)_\mathbb{T}$ and satisfy $\sup\{|y(t)| : t > t_0\} > 0$ for any $t_0 \geq t_y$. A solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

2. SOME PRELIMINARIES

In this section we establish fundamental results needed to prove our main results. We begin by introducing the auxiliary functions

$$(2.1) \qquad P(t, a) = \int_a^t \frac{\Delta s}{p(s)} \quad \text{and} \quad \eta_i(t, a) = \frac{P(\tau_i(t), a)}{P(\sigma(t), a)}, \ 1 \leq i \leq n$$

where $a \in [t_0, \infty)_\mathbb{T}$. Following the technique of [**12**, Lemma 1.2], we have

Lemma 2.1. *Let $y(t)$ be a solution of (1.1) which satisfies*

$$y(t) > 0, \quad y^\Delta(t) > 0, \quad and \quad \left(p(t) y^\Delta(t)\right)^\Delta \leq 0$$

for all $t \geq \tau_i(t) \geq T \geq t_0$. Then for each $1 \leq i \leq n$, we have

$$y(\tau_i(t)) \geq \eta_i(t, T) y^\sigma(t), \quad t \geq \tau_i(t) \geq T.$$

Proof. For $t > T \geq t_0$ and $1 \leq i \leq n$, we have

$$y^\sigma(t) - y(\tau_i(t)) = \int_{\tau_i(t)}^{\sigma(t)} \frac{1}{p(s)} p(s) y^\Delta(s) \Delta s \leq p(\tau_i(t)) y^\Delta(\tau_i(t)) \int_{\tau_i(t)}^{\sigma(t)} \frac{1}{p(s)} \Delta s$$

which yields

$$y^\sigma(t) \leq y(\tau_i(t)) + p(\tau_i(t)) y^\Delta(t) P(\sigma(t), \tau_i(t)).$$

Dividing both sides of this inequality by $y(\tau_i(t))$ we obtain

(2.2)
$$\frac{y^\sigma(t)}{y(\tau_i(t))} \leq 1 + \frac{p(\tau_i(t)) y^\Delta(\tau_i(t))}{y(\tau_i(t))} P(\sigma(t), \tau_i(t)).$$

Also we have

$$y(\tau_i(t)) - y(T) = \int_T^{\tau_i(t)} \frac{1}{p(s)} p(s) y^\Delta(s) \Delta s \geq p(\tau_i(t)) y^\Delta(\tau_i(t)) \int_T^{\tau_i(t)} \frac{1}{p(s)} \Delta s$$

and hence

$$y(\tau_i(t)) \geq p(\tau_i(t)) y^\Delta(\tau_i(t)) P(\tau_i(t), T).$$

Therefore we have

(2.3)
$$\frac{p(\tau(t)) y^\Delta(\tau(t))}{y(\tau_i(t))} \leq \frac{1}{P(\tau_i(t), T)} .$$

Therefore, (2.2) and (2.3) imply

$$\frac{y^\sigma(t)}{y(\tau_i(t))} \leq 1 + \frac{p(\tau_i(t)) y^\Delta(\tau_i(t))}{y(\tau_i(t))} P(\sigma(t), \tau_i(t)) \leq \frac{P(\sigma(t), T)}{P(\tau_i(t), T)} .$$

This gives us the desired result

$$y(\tau_i(t)) > \eta_i(t, T) y^\sigma(t), \quad 1 \leq i \leq n. \qquad \square$$

In addition to the above lemma, we need a method of studying separated boundary value problems (SBVPs) to prove our main results. Namely, we will define functions called upper and lower solutions that, not only imply the existence of a solution of a SBVP, but also provide bounds on the location of the solution. Consider the SBVP

(2.4)
$$-(p(t) y^\Delta)^\Delta + q(t) y^\sigma = f(t, y^\sigma), \quad t \in [a, b]^{\kappa^2}$$
(2.5)
$$y(a) = A, \qquad y(b) = B$$

where the functions $f \in C([a, b]^{\kappa^2} \times \mathbb{R}, \mathbb{R})$ and $p, q \in C_{rd}([a, b]^{\kappa^2})$ are such that $p(t) > 0$ and $q(t) \geq 0$ on $[a, b]^{\kappa^2}$. We define the set

$$\mathbb{D}_1 := \{y \in \mathbb{X} : y^\Delta \text{ is continuous and } (py^\Delta)^\Delta \text{ is rd-continuous on } [a, b]^{\kappa^2}\},$$

where the Banach space $\mathbb{X} = C([a, b])$ is equipped with the norm $\|\cdot\|$ defined by

$$\|y\| := \max_{t \in [a,b]_{\mathbb{T}}} |y(t)| \quad \text{for all} \quad y \in \mathbb{X}.$$

A function y is called a solution of the equation $-(p(t)y^\Delta)^\Delta + q(t)y^\sigma = 0$ on $[a, b]^{\kappa^2}$ if $y \in \mathbb{D}_1$ and the equation $-(p(t)y^\Delta)^\Delta + q(t)y^\sigma = 0$ holds for all $t \in [a, b]^{\kappa^2}$. Next we define for any $u, v \in \mathbb{D}_1$ the sector $[u, v]_1$ by

$$[u, v]_1 := \{w \in \mathbb{D}_1 : u \leq w \leq v\}.$$

Definition 2.2. [**6**, Definition 6.1] *We call* $\alpha \in \mathbb{D}_1$ *a lower solution of the* SBVP (2.4)-(2.5) *on* $[a, b]$ *provided*

$$-(p\alpha^\Delta)^\Delta(t) + q(t)\alpha^\sigma(t) \leq f(t, \alpha^\sigma(t)) \quad \text{for all} \quad t \in [a, b]^{\kappa^2}$$

and

$$\alpha(a) \leq A, \quad \alpha(b) \leq B.$$

Similarly, $\beta \in \mathbb{D}_1$ *is called an upper solution of the* SBVP (2.4)-(2.5) *on* $[a, b]$ *provided*

$$-(p\beta^\Delta)^\Delta(t) + q(t)\beta^\sigma(t) \geq f(t, \beta^\sigma(t)) \quad \text{for all} \quad t \in [a, b]^{\kappa^2}$$

and

$$\beta(a) \geq A, \quad \beta(b) \geq B.$$

The following is an extension of [**6**, Theorem 6.5] to $[a, \infty)_{\mathbb{T}}$.

Theorem 2.3. [**14**, Theorem 1.5] *Assume that there exists a lower solution* α *and an upper solution* β *of* (2.4) *with* $\alpha(t) \leq \beta(t)$ *for all* $t \in [a, \infty)_{\mathbb{T}}$. *Then*

$$(2.6) \qquad\qquad -(p(t)y^\Delta)^\Delta + q(t)y^\sigma = f(t, y^\sigma)$$

has a solution y *with* $y(a) = A$ *and* $y \in [\alpha, \beta]_1$ *on* $[a, \infty)_{\mathbb{T}}$.

Our next preliminary result is a generalization of [**19**, Theorem 3].

Theorem 2.4. *Let* $f(t, y)$ *be a continuous function of the variables* $t \geq t_0$ *and* $|y| < \infty$. *Assume that for all* $t > 0$ *and* $y \neq 0$, $yf(t, y) > 0$, *and for each fixed* t, $f(t, y)$ *is nondecreasing in* y *for* $y > 0$. *Then a necessary condition for*

$$(2.7) \qquad\qquad \left(p(t)y^\Delta\right)^\Delta + f(t, y^\sigma) = 0, \quad t \geq t_0 > 0$$

to have a bounded nonoscillatory solution is that

$$\int^\infty P(t, a)f(t, c)\Delta t < \infty$$

for any fixed $a \in [t_0, \infty)_{\mathbb{T}}$ *and for some constant* $c > 0$.

Proof. Suppose $y(t)$ is a bounded eventually positive solution of (2.7). So there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$ for $t \geq T$. As $f(t, y) > 0$ for all $y > 0$, $\left(p(t)y^{\Delta}\right)^{\Delta}$ is eventually negative. So $p(t)y^{\Delta}(t)$ is decreasing and tends to a limit L that is positive, zero, negative, or $-\infty$. If $L < 0$ or $L = -\infty$, $p(t)y(t)$ would be eventually negative. This contradicts y being eventually positive. Hence $\lim_{t \to \infty} P(t)y^{\Delta}(t) = L$ with $0 \leq L < \infty$.

Integrating (2.7) from s to T_1, we obtain

$$p(T_1)y^{\Delta}(T_1) - p(s)y^{\Delta}(s) + \int_s^{T_1} f(r, y^{\sigma}(r))\Delta r = 0.$$

It follows that

$$y^{\Delta}(s) \geq \frac{1}{p(s)} \int_s^{\infty} f(r, y^{\sigma}(r))\Delta r.$$

Integrating again for $T < t_1 < t$, we obtain

$$y(t) - y(t_1) \geq \int_{t_1}^t \frac{1}{p(s)} \int_s^{\infty} f(r, y^{\sigma}(r))\Delta r \Delta s.$$

If we let

$$I_1(t) := \int_{t_1}^t \frac{1}{p(s)} \int_s^{\infty} f(r, y^{\sigma}(r))\Delta r \Delta s$$

and

$$I_2(t) := \int_{t_1}^t P(r, t_1)f(r, y^{\sigma}(r))\Delta r + \int_t^{\infty} P(t, t_1)f(r, y^{\sigma}(r))\Delta r,$$

we obtain $I_1(t) \geq I_2(t)$. Consequently, for $t \geq t_1 \geq T$, we have

$$\int_{t_1}^t y^{\Delta}(s)\Delta s \geq I_1(t) \geq \int_{t_1}^t P(r, t_1)f(r, y^{\sigma}(r))\Delta r,$$

and so

$$y(t) > \int_{t_1}^t P(r, t_1)f(r, y^{\sigma}(r))\Delta r.$$

Since $y(t) \leq M$ for some $M > 0$ and $\int_{t_1}^t P(r, t_1)f(r, y^{\sigma}(r))\Delta r$ is an increasing function of t, we have

$$\int_{t_1}^{\infty} P(r, t_1)f(r, y^{\sigma}(r))\Delta r < \infty.$$

By the monotonicity of f, we have

$$\int^{\infty} P(r, t_1)f(r, y(T))\Delta r < \infty.$$

By letting $c = y(T)$, we obtain the desired result. This completes the proof. $\square$

We end this section with time scale version of the Arzela-Ascoli theorem and the Schauder fixed-point theorem. These will be used in the proof of Theorem 3.3.

Lemma 2.5. [**16**, Lemma 2.8] *Let Y be a subset of $C_0([a,\infty)_{\mathbb{T}})$ having the following properties:*

  (i) *Y is bounded;*

 (ii) *on every compact subinterval J of $[a,\infty)_{\mathbb{T}}$, there exists, for any $\epsilon > 0, \delta > 0$ such that $t_1,\ t_2 \in J,\ |t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \epsilon$ for all $f \in Y$;*

(iii) *for every $\epsilon > 0$, there exists $b \in [a,\infty)_{\mathbb{T}}$ such that $t_1,\ t_2 \in [b,\infty)_{\mathbb{T}}$ implies $|f(t_1) - f(t_2)| < \epsilon$ for all $f \in Y$.*

Then Y is relatively compact.

Lemma 2.6. (Schauder fixed-point theorem, [**16**, Proposition 2.7]) *Let $\mathcal{N}$ be a normed space and Y be a nonempty, closed, convex subset of $\mathcal{N}$. If $\mathcal{T}$ is a continuous mapping such that $\mathcal{T}(X) \subseteq X$ and $\mathcal{T}(X)$ is relatively compact, then $\mathcal{T}$ has a fixed point in X.*

3. MAIN RESULTS

In this section we establish several results results for

$$(1.1) \qquad \left[p(t)y^{\Delta}\right]^{\Delta} + f\left(t, y^{\sigma}(t), y(\tau_1(t)), \ldots, y(\tau_n(t))\right) = 0.$$

Theorem 3.1. *Assume conditions $(A_0) - (A_3)$ hold and let $M > 0$. Then any bounded solution $y(t)$ of (1.1) is oscillatory in case*

$$(3.1) \qquad \int^{\infty} |P(t,a)f\left(t, \alpha, \alpha\eta_1(t,a), \ldots, \alpha\eta_n(t,a)\right)|\, \Delta t = \infty$$

for all $\alpha \neq 0$ where $\eta_i(t,a)$, $1 \leq i \leq n$, is given in (2.1).

Proof. Assume not and let $u(t)$ be a bounded nonoscillatory solution of (1.1) which we may assume satisfies

$$u(t) > 0,\ u(\tau_i(t)) > 0, \quad t \geq T \geq t_0, \quad 1 \leq i \leq n.$$

Consequently,

$$[p(t)u^{\Delta}(t)]^{\Delta} = -f\left(t, u^{\sigma}(t), u(\tau_1(t)), \ldots, u(\tau_n(t))\right) < 0$$

for $t \geq T$ and so $p(t)u^{\Delta}(t)$ is decreases for $t \geq T$.

We claim that $p(t)u^{\Delta}(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. If not, there is a $t_1 \geq T$ such that $p(t_1)u^{\Delta}(t_1) < 0$. Then

$$p(t)u^{\Delta}(t) \leq p(t_1)u^{\Delta}(t_1), \quad t \in [t_1,\infty)_{\mathbb{T}},$$

and therefore

$$u^{\Delta}(t) \leq \frac{p(t_1)u^{\Delta}(t_1)}{p(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Integrating, we obtain

$$u(t) - u(t_1) = \int_{t_1}^{t} u^{\Delta}(s)\Delta s \leq p(t_1)u^{\Delta}(t_1) \int_{t_1}^{t} \frac{\Delta s}{p(s)} \to -\infty$$

as $t \to \infty$, contradicting that $u(t)$ is eventually positive. Hence, we conclude that for all $t \geq T$ and $1 \leq i \leq n$,

$$u(t) > 0, \quad u(\tau_i(t)) > 0, \quad u^{\Delta}(t) > 0, \quad \left(p(t)u^{\Delta}(t)\right)^{\Delta} < 0.$$

From Lemma 2.1, we have

$$u(\tau_i(t)) \geq \eta_i(t,T)u^{\sigma}(t) \quad t \geq \tau_i(t) \geq T.$$

Then by the monotonicity of f, we have

$$(3.2) \qquad 0 = [p(t)u^{\Delta}(t)]^{\Delta} + f\left(t, u^{\sigma}(t), u(\tau_i(t)), \ldots, u(\tau_n(t))\right)$$
$$\geq [p(t)u^{\Delta}(t)]^{\Delta} + f\left(t, u^{\sigma}(t), \eta_1(t,T)u^{\sigma}(t), \ldots, \eta_n(t,T)u^{\sigma}(t)\right)$$

for $t \geq T$. Define

$$F(t,w) := f\left(t, w, \eta_1(t,T)w, \ldots, \eta_n(t,T)w\right).$$

Immediately, we see that

$$F(t, u^{\sigma}(t)) = f\left(t, u^{\sigma}(t), \eta_1(t,T)u^{\sigma}(t), \ldots, \eta_n(t,T)u^{\sigma}(t)\right).$$

Applying Theorem 2.3 with $\alpha(t) \equiv u(T) \leq u(t) \equiv \beta(t)$, we obtain the existence of a solution $y(t)$ of

$$[p(t)y^{\Delta}]^{\Delta} + F(t, y^{\sigma}(t)) = 0, \quad y(T) = u(T)$$

with $u(T) \leq y(t) \leq u(t)$ on $[T, \infty)_{\mathbb{T}}$. However, by Theorem 2.4, it follows that

$$\int^{\infty} P(t,a)F(t,c) < \infty$$

for some $c > 0$, which contradicts (3.1). This completes the proof. $\qquad\square$

The next theorem shows that the converse of Theorem 3.1 is true under an additional assumption.

Theorem 3.2. *Assume f satisfies conditions $(A_0) - (A_3)$ and that for each $i \in \{1, \ldots, n\}$, there exists $\rho_i > 0$ such that*

$$(3.3) \qquad\qquad \liminf_{t \to \infty} \eta_i(t,a) \geq \rho_i \quad for \quad a \in \mathbb{T}.$$

Also, let $M > 0$ and assume that $P(\sigma(t), a)/P(t, a)$ is bounded. Then, if $y(t)$ is a nonoscillatory solution of

$$(1.1) \qquad \left[p(t)y^\Delta\right]^\Delta + f\left(t, y^\sigma(t), y(\tau_1(t)), \ldots, y(\tau_n(t))\right) = 0$$

with $|y(t)| \le M$,

$$(3.4) \qquad \left|\int^\infty P(\sigma(t), a)f(t, \alpha, \alpha, \ldots, \alpha)\Delta t\right| < \infty$$

for all $\alpha > 0$.

Proof. Note that for any β

$$\int^\infty |P(\sigma(t), a)f(t, \beta, \ldots, \beta)| \ \Delta t < \infty$$

if, and only if,

$$\int^\infty |P(t, a)f(t, \beta, \ldots, \beta)| \ \Delta t < \infty$$

since $P(\sigma(t), a)/P(t, a)$ is bounded on $\mathbb{T}$. Furthermore, observe that by (3.3), given any $\epsilon > 0$ with $\epsilon < \dfrac{1}{2} \min\{\rho_i | 1 \le i \le n\}$, there exists $T_i \ge t_0$ such that $\eta_i(t, a) \ge \rho_i - \epsilon =: \tilde{\rho}_i$ for $t \ge T_i$ and $1 \le i \le n$.

Assume (1.1) has a bounded nonoscillatory solution. Then by Theorem 3.1

$$\int^\infty |P(t, a)f\left(t, \alpha, \alpha\eta_1(t, a), \ldots, \alpha\eta_n(t, a)\right)| \ \Delta t < \infty$$

for some $\alpha \ne 0$. Let $\tilde{\rho} := \min\{\tilde{\rho}_i | 1 \le i \le n\}$. Observe that $\eta_i(t, a) \le 1$ implies that $0 < \tilde{\rho}_i \le 1$ for $t \ge T_i$ and for all i. Consequently, $\alpha\tilde{\rho} \le \alpha\eta_i(t, a) \le \alpha$ for all $1 \le i \le n$, and so by the monotonicity f, we have

$$\int^\infty |P(t, a)f\left(t, \alpha\tilde{\rho}, \alpha\tilde{\rho}, \ldots, \alpha\tilde{\rho}\right)| \ \Delta t < \infty.$$

With $\upsilon = \alpha\tilde{\rho}$, we obtain (3.4) as desired.   □

The previous result says the condition (3.3) is sufficient in order replace the auxiliary functions $\eta_i(t, a), 1 \le i \le n$ with upper bounds. Our next result gives a sufficient condition for

$$(1.1) \qquad \left[p(t)y^\Delta\right]^\Delta + f\left(t, y^\sigma(t), y(\tau_1(t)), \ldots, y(\tau_n(t))\right) = 0$$

to have bounded nonoscillatory solutions.

Theorem 3.3. *Assume f satisfies conditions $(A_0) - (A_3)$. If*

$$\left| \int^{\infty} P(\sigma(t), a) f(t, \alpha, \alpha, \ldots, \alpha) \Delta t \right| < \infty \tag{3.4}$$

for all $\alpha > 0$ and there exists $K > 0$ such that

$$\frac{1}{p(t)} \int_t^{\infty} f(s, \alpha, \ldots, \alpha) \Delta s \leq K \tag{3.5}$$

for all $t \geq a$, then (1.1) has a bounded nonoscillatory solution.

Proof. Assume (3.4) holds and let $0 < \beta < \alpha$. Choose $T \geq t_1 \in \mathbb{T}$ such that $\tau_i(t) \geq t_1$ for $t \geq T$ and all $i = 1, \ldots, n$ and such that

$$\int_T^{\infty} P(\sigma(t), a) f(t, \beta, \beta, \ldots, \beta) \, \Delta t < \frac{\beta}{2} \,.$$

Define $Y := \{ y \in \mathbb{X} : \frac{\beta}{2} \leq y(t) \leq \beta$ for $t \geq T\}$ and the operator $T : Y \to \mathbb{X}$ by

$$T(y)(t) = \beta - \int_t^{\infty} [P(\sigma(s), a) - P(t, a)] f(s, y^{\sigma}(s), y(\tau_1(s)), \ldots, y(\tau_n(s))) \, \Delta s,$$

where the Banach space $\mathbb{X} = C([T, \infty)_{\mathbb{T}}, \mathbb{R})$ is equipped with the norm $\| \cdot \|$ defined by

$$\|y\| := \sup_{t \in [T, \infty)_{\mathbb{T}}} |y(t)| < \infty \quad \text{for all} \quad y \in \mathbb{X}.$$

The operator T is well defined and one can show that Y is closed and convex. For the sake of convenience, let $F(s) := f(s, y^{\sigma}(s), y(\tau_1(s)), \ldots, y(\tau_n(s)))$.

We first show that T maps Y onto itself. Suppose $y \in Y$. Then

$$T(y)(t) = \beta - \int_t^{\infty} [P(\sigma(s), a) - P(t, a)] F(s) \Delta s$$

$$> \beta - \int_t^{\infty} P(\sigma(s), a) f(s, \beta, \beta, \ldots, \beta) \, \Delta s > \frac{\beta}{2} \,.$$

Furthermore, since $s \geq T$, we have $y^{\sigma}(s), y(\tau_1(s)), \ldots, y(\tau_n(s))$ are all positive. Hence, by condition (A_1),

$$[P(\sigma(s), a) - P(t, a)] F(s) \geq 0$$

for $s \geq t \geq T$. Consequently, $y(t) \leq \beta$ for $t \geq T$. Hence $T(Y) \subseteq Y$.

Next we show that $T(Y)$ is relatively compact. The fact that $T(Y) \subseteq Y$ implies the boundedness of $T(Y)$. To prove the equicontinuity of the elements of

$T(Y)$, we show that $(Ty)^{\Delta}(t)$ is uniformly bounded on $[T, \infty)_{\mathbb{T}}$. To that end, consider

$$\left|(Ty)^{\Delta}(t)\right| = \left|\int_t^{\infty} -P^{\Delta}(t,a)F(s)\Delta s - [P(\sigma(t),a) - P(\sigma(t),a)]F(s)\right|$$

$$= \left|\frac{1}{p(t)}\int_t^{\infty} F(s)\Delta s\right| \leq \left|\frac{1}{p(t)}\int_t^{\infty} f(s,\alpha,\alpha,\ldots,\alpha)\Delta s\right| < \infty.$$

Finally, we verify that condition (iii) of 2.5 holds for $T(Y)$. Let $\epsilon > 0$ be given. We have to show that there exists $T_0 \in [T, \infty)_{\mathbb{T}}$ such that for any $t_2, t_3 \in [T, \infty)_{\mathbb{T}}$, it holds that $|(Ty)(t_2) - (Ty)(t_3)| < \epsilon$ for any $y \in Y$. Without loss of generality, suppose $t_2 < t_3$. Using the triangle inequality and the fact that $P(t,a)$ is an increasing function of t, it follows that

$$(3.6) \quad \left|(Ty)^{\Delta}(t_2) - (Ty)^{\Delta}(t_3)\right|$$

$$= \left|\int_{t_2}^{\infty}[P(\sigma(s),a)-P(t_2,a)]F(s)\Delta s - \int_{t_3}^{\infty}[P(\sigma(s),a)-P(t_3,a)]F(s)\Delta s\right|$$

$$\leq 2\left|\int_{t_2}^{\infty}P(\sigma(s),a)F(s)\Delta s\right| + 2\left|\int_{t_3}^{\infty}P(\sigma(s),a)F(s)\Delta s\right|$$

$$\leq 2\left|\int_{t_2}^{\infty}P(\sigma(s),a)f(s,\alpha,\alpha,\ldots,\alpha)\Delta s\right|$$

$$+ 2\left|\int_{t_3}^{\infty}P(\sigma(s),a)f(s,\alpha,\alpha,\ldots,\alpha)\Delta s\right|.$$

Since the integrals in the last line (3.6) are convergent, for any $\epsilon > 0$, one can find $T_0 \in [T, \infty)_{\mathbb{T}}$ such that

$$\left|\int_t^{\infty}P(\sigma(s),a)f(s,\alpha,\alpha,\ldots,\alpha)\Delta s\right| < \frac{\epsilon}{4}, \quad t = t_2, t_3,$$

whenever $t_3 > t_2 \geq T_0$. From here and (3.6), we obtain the desired inequality. Hence, by 2.5, $T(Y)$ is relatively compact.

The last hypothesis to be verified is the continuity of T on Y. Let $\{y_m\}, m \in \mathbb{N}$, be a sequence in Y which converges uniformly on every compact subinterval of $[T, \infty)_{\mathbb{T}}$ to $\overline{y} \in Y$. Since $T(Y)$ is relatively compact, the sequence $\{T(y_m)\}$ admits a subsequence $\{T(y_{m_k})\}$ converging in the topology of $\mathbb{X}$ to $\overline{z}$. Since T maps Y onto itself, we have $|T(y_{m_k})| \leq \beta$ for all k where β is integrable on every compact subinterval of $[T, \infty)_{\mathbb{T}}$. Hence, by the Lebesgue dominated convergence theorem on time scales, see [4], the sequence $\{T(y_{m_k})\}$ converges to $T(\overline{y})$. In view of the uniqueness of the limit, $T(\overline{y}) = \overline{z}$ is the only limit point of the sequence $\{T(y_m)\}$. Hence, T is continuous on Y.

Therefore, by the Schauder fixed-point theorem, there is an element $y \in Y$ such that $T(y) = y$. It follows that

$$y(t) = \beta - \int_t^{\infty}[P(\sigma(s),a) - P(t,a)]f(s,y^{\sigma}(s),y(\tau_1(s)),\ldots,y(\tau_n(s)))\Delta s$$

for $t \geq T$, and hence $\left[p(t)y^{\Delta}\right]^{\Delta} + f\left(t, y^{\sigma}(t), y(\tau_1(t)), \ldots, y(\tau_n(t))\right) = 0$. This proves the theorem. $\square$

To extend Theorems 3.1 and 3.2 to unbounded solutions, we introduce the class Φ of functions ϕ such that $\phi(u)$ is a nondecreasing continuous function of u satisfying $u\phi(u) > 0$, $u \neq 0$ with

$$\int_{\pm 1}^{\pm \infty} \frac{\mathrm{du}}{\phi(u)} < \infty.$$

We will say that $f(t, u, v_1, \ldots, v_n)$ satisfies condition (C) provided for some $\phi \in \Phi$ there exists $c \neq 0$ and $0 < \alpha < 1$ such that for all $t \geq T$

$$\inf_{|u| \to \infty} \frac{f(t, u, \alpha\eta_1(t, T), \ldots, \alpha\eta_n(t, T))}{\phi(u)} \geq k|f(t, c, \alpha\eta_1(t, T)c, \ldots, \alpha\eta_n(t, T)c)|$$

for some positive constant k. We continue with a generalization of Theorem 4 of [**19**].

Theorem 3.4. *Suppose $\phi \in \Phi$. Let $f(t, y)$ be a continuous function of the variables $t \geq t_0$ and $|y| < \infty$ such that for all $t > 0$ $yf(t, y) > 0$, $y \neq 0$ and satisfies with respect to $\phi(y)$ the following conditions: there is a $c \neq 0$ such that*

$$(3.7) \qquad\qquad \liminf_{|y| \to \infty} \frac{f(t, y)}{\phi(y)} \geq k|f(t, c)|$$

for some positive constant k and for all $t \geq T$, and that

$$(3.8) \qquad\qquad \lim_{|y| \to \infty} \left| \int^{y} \frac{1}{\phi(u)} \mathrm{du} \right| < \infty.$$

If $P(\sigma(t), a)/P(t, a)$ is bounded on $\mathbb{T}$, then a necessary and sufficient condition for the second-order dynamic equation

$$(2.7) \qquad\qquad \left(p(t)y^{\Delta}\right)^{\Delta} + f(t, y^{\sigma}) = 0$$

to be oscillatory is that

$$(3.9) \qquad\qquad \int^{\infty} P(\sigma(t), a)f(t, c)\Delta t = \infty$$

for all $c \neq 0$.

Proof. Note that (3.9) holds if and only if $\left| \int^{\infty} P(t, a)f(t, c)\Delta t \right| = \infty$ for all $c \neq 0$.

Assume (2.7) is oscillatory and $\left| \int^{\infty} P(t, a)f(t, c)\Delta t \right| < \infty$ for some $c \neq 0$. It follows that (2.7) has a bounded nonoscillatory solution from Theorem 3.2. This contradiction shows that (3.9) is necessary.

Conversely, assume (3.9) holds and let $y(t)$ be a an eventually positive solution of (2.7). It follows from Theorem 2.4 that $y(t)$ cannot be bounded. So we assume that $\lim_{t\to\infty} y(t) = \infty$. Also, as in the proof of Theorem 2.4,

$$(3.10) \qquad \int_T^t y^\Delta(s)\Delta s \geq \int_T^t P(r,T)f(r,y^\sigma(r))\Delta r$$

for sufficiently large T.

We next define the continuously differentiable real-valued function

$$G(u) := \int_{u_0}^u \frac{\mathrm{ds}}{\phi(s)}.$$

Observe that $G'(u) = 1/\phi(u)$. By the Pötzsche Chain Rule [**5**, Theorem 1.90],

$$(G(y(t))^\Delta = \left(\int_0^1 \frac{\mathrm{dh}}{\phi(y_h(t))}\right) y^\Delta(t) \geq \left(\int_0^1 \frac{\mathrm{dh}}{\phi(y^\sigma(t))}\right) y^\Delta(t) = \frac{y^\Delta(t)}{\phi(y^\sigma(t))},$$

where $y_h(t) := y(t) + h\mu(t)y^\Delta(t) \leq y^\sigma(t)$. Since ϕ is nondecreasing, we have

$$[G(y(t))]^\Delta \geq \frac{y^\Delta(t)}{\phi(y^\sigma(t))}.$$

Now multiplying (3.10) by $[\phi(y^\sigma(s))]^{-1}$, we obtain

$$\int_T^t \frac{y^\Delta(s)}{\phi(y^\sigma(s))}\Delta s \geq \int_T^t P(r,T)\frac{f(r,y^\sigma(r))}{\phi(y^\sigma(r))}\Delta r \geq \int_T^t kP(r,T)f(r,c)\Delta r$$

for sufficiently large T by (3.7) where $c := u(T) > 0$. Since $\lim_{t\to\infty} y(t) = \infty$, we have

$$\lim_{t\to\infty} G(y(t)) = \lim_{t\to\infty} \int_T^{y(t)} \frac{\mathrm{du}}{\phi(u)} = \int_T^\infty \frac{\mathrm{du}}{\phi(u)} < \infty$$

by assumption. Therefore,

$$\int_T^t [G(y(t))]^\Delta \Delta s \geq \int_T^t \frac{y^\Delta(s)}{\phi(y^\sigma(s))}\Delta s \geq \int_T^t kP(r,T)f(r,c)\Delta r.$$

However, by letting $t \to \infty$ in the above, the left side is bounded whereas the right side is unbounded by assumption (3.9). This contradiction shows that (3.9) is sufficient for all solutions of (2.7) to be oscillatory. $\square$

We may now prove our last main result.

Theorem 3.5. *Assume f satisfies conditions* $(A_0) - (A_3)$ *and condition* (C)*. Further assume that* $P(\sigma(t), a)/P(t, a)$ *is bounded on* $\mathbb{T}$*. Then all solutions of*

$$(1.1) \qquad \left[p(t)y^\Delta\right]^\Delta + f\left(t, y^\sigma(t), y(\tau_1(t)), \ldots, y(\tau_n(t))\right) = 0$$

are oscillatory in case

$$(3.11) \qquad \left| \int^{\infty} P(\sigma(t), a) f\left(t, \alpha, \alpha\eta_1(t, a), \ldots, \alpha\eta_n(t, a)\right) \Delta t \right| = \infty$$

holds for all $\alpha \neq 0$. In addition, if inequalities (3.3) and (3.5) hold, then (3.11) is also necessary.

Proof. Assume (3.11) holds for all $\alpha \neq 0$ and let $u(t)$ be a nonoscillatory solution of (1.1) which we may assume satisfies

$$u(t), \; u(\tau_i(t)), \; u^{\Delta}(t) > 0, \quad \left(p(t)u^{\Delta}(t)\right)^{\Delta} \leq 0, \quad t \geq T \geq t_0, \quad 1 \leq i \leq n.$$

From Lemma 2.1, we have

$$u(\tau_i(t)) \geq \eta_i(t, T)u^{\sigma}(t) \quad t \geq \tau_i(t) \geq T.$$

Then by the monotonicity of f, we have

$$[p(t)u^{\Delta}(t)]^{\Delta} + f\left(t, u^{\sigma}(t), \eta_1(t, T)u^{\sigma}(t), \ldots, \eta_n(t, T)u^{\sigma}(t)\right) \leq 0.$$

As in the proof of Theorem 3.1, we obtain the existence of a solution $y(t)$ of $[p(t)u^{\Delta}]^{\Delta} + F(t, y^{\sigma}(t)) = 0$ with $0 < u(T) \leq y(t) \leq u(t)$, $t \geq T$. Now by Theorem 3.4, it follows that

$$\left| \int^{\infty} P(\sigma(t), a) f(t, c, c\eta_1(t, T), \ldots, c\eta_n(t, T))\Delta t \right| < \infty,$$

for some $c \neq 0$, which is a contradiction.

Conversely, assume (3.3) and (3.5) hold and (3.11) does not for some $\alpha \neq 0$. It follows that for any $\epsilon > 0$ with $\epsilon < \frac{1}{2}\min\{\tilde{\rho}_i | 1 \leq i \leq n\}$ there exists $T_i \geq t_0$ such that
$\eta_i(t, a) \geq \rho_i - \epsilon =: \tilde{\rho}_i$ for $t \geq T_i$ and $1 \leq i \leq n$. Let $\tilde{\rho} := \min\{\tilde{\rho}_i | 1 \leq i \leq n\}$. Then

$$\alpha\eta_i(t, a) \geq \alpha\tilde{\rho}$$

for $t \geq T_i$. Then by the monotonicity of f and the fact that $\eta_i \leq 1$ for $t \geq T$, we have

$$\left| \int^{\infty} P(\sigma(t), a) f(t, \alpha\tilde{\rho}, \ldots, \alpha\tilde{\rho})\Delta t \right| < \infty,$$

which gives (3.4). Therefore, by Theorem 3.3,

$$\left[p(t)y^{\Delta}\right]^{\Delta} + f(t, y^{\sigma}(t), y(\tau_1(t)), \ldots, y(\tau_n(t))) = 0$$

has a bounded nonoscillatory solution.   □

4. EXAMPLES

In this section we give two examples of the main results applied to

$$(4.1) \qquad y^{\Delta\Delta}(t) + q(t)(y(\tau(t)))^\gamma = 0$$

which is a special case of (1.1). We assume that $q(t)$ is continuous and eventually negative on $[t, \infty)_{\mathbb{T}}$ and $\gamma > 1$ is the quotient of odd positive integers. We begin with the following corollary which extends a result of GOLLWITZER [13].

Corollary 4.1. *All solutions of* (4.1) *are oscillatory provided*

$$(4.2) \qquad \int^\infty t^{1-\gamma} q(t)(\tau(t))^\gamma \Delta t = \infty$$

and $\mu(t)/t$ is bounded.

Proof. Assume (4.2) holds. Define $\phi(u) := u^\gamma$. Then $u\phi(u) > 0$ for $u \neq 0$ and by Theorem 2.6 of ERBE and HILGER [11], $\displaystyle\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty$ and $u\phi(u) > 0$ for $u \neq 0$. Let $f(t, u, v) := q(t)v^\gamma(t)$ and let $c = 1$ and $0 < \alpha < 1$. Then

$$\frac{f(t, u, \alpha\eta(t)u)}{\phi(u)} = \frac{\alpha^\gamma q(t)\left(\frac{\tau(t)}{t}\right)^\gamma u^\gamma}{u^\gamma} = k|f(t, c, \alpha\eta(t)c)|$$

for $k = 1$ and all $t \geq T$. Furthermore

$$\left|\int^\infty t(f, t, \alpha, \alpha\eta(t))\Delta t\right| = \left|\int^\infty t^{1-\gamma} q(t)\alpha^\gamma(\tau(t))^\gamma \Delta t\right| = \infty.$$

Hence, by Theorem 3.5, equation (4.1) is oscillatory. $\qquad\square$

REMARK 4.2. Theorem 3.1 shows that

$$(4.3) \qquad \int^\infty tq(t)\Delta t = \infty$$

is a necessary condition for all solutions of (4.1) to oscillate, in case $\gamma > 1$, with just the assumptions that $0 < \tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$. However, (4.3) is no longer sufficient as the following examples demonstrate.

EXAMPLE 4.3. Let $\mathbb{T} = [1, \infty)_{\mathbb{R}}$. Consider equation (4.1) with

$$q(t) = \beta(1 - \beta)t^\alpha \quad \text{and} \quad \tau(t) = t^\delta,$$

where $\alpha = \beta(1 - \gamma\delta) - 2$ with $0 < \beta, \delta < 1$ and $\gamma\delta < 1$, and γ is the quotient of odd integers. For this example, $y(t) = t^\beta$ is a nonoscillatory solution but $\int^\infty tq(t)\, dt = \infty$.

We have

$$y''(t) + q(t)(y(\tau(t)))^\gamma = \beta(\beta - 1)\left[t^{\beta-2} - t^{\alpha+\beta\gamma\delta}\right] = 0.$$

However,

$$\int^\infty tq(t)\mathrm{dt} = \int_T^\infty t\beta(1-\beta)t^\alpha \mathrm{dt}, \quad \text{for T sufficiently large}$$

$$= \beta(1-\beta) \lim_{R\to\infty} \int_T^R t^{1+\alpha}\mathrm{dt} = \frac{\beta(1-\beta)}{2+\alpha} \lim_{R\to\infty} \left[R^{2+\alpha} - T^{2+\alpha}\right] = \infty$$

since $2 + \alpha > 0$.

EXAMPLE 4.4. For $q > 1$, let $\mathbb{T} = q^{\mathbb{N}_0}$. We want to find a function $Q : \mathbb{T} \to \mathbb{R}$ and a function $y : \mathbb{T} \to \mathbb{R}$ such that $y(t)$ is a nonoscillatory solution $y^{\Delta\Delta} + Q(t)y^\gamma(\tau(t)) = 0$ and $\int^\infty tQ(t)\Delta t = \infty$, where $\tau(t) : \mathbb{T} \to \mathbb{T}$ is such that $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$ and $\gamma > 1$ is a quotient of odd positive integers.

Let $y(t) = t^\beta$ with $\beta < 1$. Since $\sigma(t) = tq > t$, we have $y^\Delta(t) = t^{\beta-1}\dfrac{q^\beta-1}{q-1}$. After simplifying, we obtain $y^{\Delta\Delta}(t) = \dfrac{q^\beta-1}{(q-1)^2}t^{\beta-2}[q^{\beta-1}-1]$. It follows that

$$Q(t) = \frac{q^\beta - 1}{(q-1)^2} \cdot \frac{t^{\beta-2}[1 - q^{\beta-1}]}{y^\gamma(\tau(t))} =: C_q \frac{t^{\beta-2}}{y^\gamma(\tau(t))}.$$

Now choose $\tau(t) = q^{\lfloor k/2 \rfloor}$, where $\lfloor \cdot \rfloor$ is the greatest integer function. Then $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$. Consequently

$$Q(t) = \begin{cases} C_q t^{(\beta(2-\gamma)-4)/2}, & \text{if } k \text{ is even,} \\ C_q t^{(\beta(2-\gamma)-4)/2}q^{\gamma\beta/2}, & \text{if } k \text{ is odd.} \end{cases}$$

Now

$$\int^\infty tQ(t)dt = C_q \lim_{R\to\infty} \begin{cases} \int_T^R t^{\beta(2-\gamma)/2}\mathrm{dt}, & \text{if } k \text{ is even,} \\ \int_T^R t^{\beta(2-\gamma)/2}q^{\gamma\beta/2}\mathrm{dt}, & \text{if } k \text{ is odd} \end{cases} \quad \text{for } T \text{ sufficiently large}$$

$$= C_q(q-1) \lim_{n\to\infty} \begin{cases} \displaystyle\sum_{k=m}^{n-1} q^{k\cdot\beta(2-\gamma)/2}, & \text{if } k \text{ is even,} \\ q^{\gamma\beta/2}\displaystyle\sum_{k=m}^{n-1} q^{k\cdot\beta(2-\gamma)/2}, & \text{if } k \text{ is odd} \end{cases}$$

$$= \infty$$

if $q^{\beta(2-\gamma)/2} > 1$. That means $\beta(2 - \gamma)$ must be nonnegative. If it were negative, then $y^{\Delta\Delta}(t)$ would be positive, which cannot happen as shown in the proof of Theorem 3.1. Hence, in order for the dynamic equation $y^{\Delta\Delta} + Q(t)y^\gamma(\tau(t)) = 0$ to have a nonoscillatory solution $y(t) = t^\beta$ and for $\int^\infty tQ(t)\Delta t = \infty$, we must choose $0 \leq \beta < 1$ and $1 < \gamma \leq 2$.

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