COMMON FIXED POINT UNDER CONTRACTIVE CONDITION OF ĆIRIĆ'S TYPE IN CONE METRIC SPACES

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A common fixed point theorem is established for a pair of self-mappings of a complete cone metric space. The obtained result is an extension of LJUBOMIR ĆIRIĆ's theorem.[LJ. ĆIRIĆ: *On common fixed points in uniform spaces.* Publ. Inst. Math., **24** (38) (1978), 39–43].

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory in K-metric and K-normed spaces was developed by A. I. PEROV, A. V. KIBENKO and B. V. KVEDARAS [**9, 11, 12**], E. M. MUKHAMADI-JEV and V. J. STETSENKO [**10**], J. S. VANDERGRAFT [**15**] and others. For more details on fixed point theory in K-metric and K-normed spaces, we refer the reader to fine survey paper of P. P. ZABREJKO [**16**]. The main idea is to use an ordered Banach space instead of the set of real numbers, as the codomain for a metric. In 2007, HUANG and ZHANG [**6**] reintroduced such spaces under the name of cone metric spaces and reintroduced definition of convergent and Cauchy sequences in the terms of interior points of the underlying cone. They also proved some fixed point theorems in such spaces in the same work. Recently, in [**1-3, 7, 8, 13, 14**] various common fixed point results were proved for mappings on cone metric spaces.

Consistent with ZABREJKO [**16**], HUANG and ZHANG [**6**], the following definitions and results will be needed in what follows.

Let $(E\|, \| \cdot \|)$ be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, non-empty and $P \neq \{0_E\}$,
(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$,
(c) $P \cap (-P) = \{0_E\}$.
Here, 0_E denotes the zero vector of E.

Given a cone $P \subseteq E$, we define a partial ordering $\leq_E$ with respect to P by $x \leq_E y$ if and only if $y - x \in P$. A cone $P \subseteq E$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0_E \leq_E x \leq_E y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. In [**14**], it is proved that if P is a cone with normal constant K, then $K \geq 1$.

Definition 1.1. *Let X be a non-empty set. Suppose that $d : X \times X \to E$ satisfies*:
(d1) $0_E \leq_E d(x, y)$ *for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$,*
(d2) $d(x, y) = d(y, x)$ *for all $x, y \in X$,*
(d3) $d(x, y) \leq_E d(x, z) + d(z, y)$ *for all $x, y, z \in X$.*
Then d is called a cone metric on X and (X, d) is called a cone metric space.

Let (X, d) be a cone metric space over a cone P with normal constant K. Then, from (d3), for all $x, y, z \in X$, we have that

(1) $$\|d(x, y)\| \leq K\|d(x, z)\| + K\|d(z, y)\|.$$

Lemma 1.2. *Let (X, d) be a cone metric space with a normal cone, and let $\{x_n\}$ be a sequence in X. Then*
(i) $\{x_n\}$ *converges to $x \in X$ if and only if $\|d(x_n, x)\| \to 0$ as $n \to +\infty$,*
(ii)$\{x_n\}$ *is a Cauchy sequence if and only if $\|d(x_n, x_m)\| \to 0$ as $n, m \to +\infty$.*

A cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

In the following, we always suppose that $(E, \|\cdot\|)$ is a Banach space, P is a normal cone in E with $\overset{\circ}{P} \neq \emptyset$, and $\leq_E$ is the partial ordering with respect to P.

In [**4**], Lj. Ćirić proved the following common fixed point result.

Theorem 1.3 (see Lj. Ćirić [**4**]). *Let (X, d) be a complete metric space. Let F and T be a pair of self-mappings of X satisfying the condition:*

$$d(Fx, Ty) \leq q \max \left\{ d(x, y), \frac{1}{2}d(x, Fx), \frac{1}{2}d(y, Ty), d(x, Ty), d(y, Fx) \right\}$$

for some $0 < q < 1$ and all $x, y \in X$. Then, F and T have a unique common fixed point.

In this paper we extend the result given by Theorem 1.3 to complete cone metric spaces.

2. MAIN RESULT

Our main result is the following.

Theorem 2.1. *Let (X, d) be a complete cone metric space, and P a normal cone with normal constant K ($K \geq 1$). Let F and T be a pair of self-mappings of X satisfying the condition:*

$$(2) \qquad \|d(Fx, Ty)\| \leq q \max \left\{ \|d(x, y)\|, \frac{1}{2}\|d(x, Fx)\|, \right.$$
$$\left. \frac{1}{2}\|d(y, Ty)\|, \|d(x, Ty)\|, \|d(y, Fx)\| \right\}$$

for some $0 < q < 1/K$ and all $x, y \in X$. Then, F and T have a unique common fixed point.

Proof. For $x, y \in X$, define $M(x, y)$ by

$$M(x, y) := \max \left\{ \|d(x, y)\|, \frac{1}{2}\|d(x, Fx)\|, \frac{1}{2}\|d(y, Ty)\|, \|d(x, Ty)\|, \|d(y, Fx)\| \right\}.$$

Let x be an arbitrary point in X. We shall show that $\{T^n x\}$ is a Cauchy sequence. For an arbitrary $n \in \mathbb{N}$, let $r = r(n)$ be such that $\|d(Fx, T^r x)\| = \max_{1 \leq i \leq n} \|d(Fx, T^i x)\|$. Then

$$(3) \quad \|d(Fx, T^n x)\| \leq \|d(Fx, T^r x)\| = \|d(Fx, T(T^{r-1} x))\| \leq q M(x, T^{r-1} x).$$

On the other hand, using (1), we obtain that

$$\|d(x, T^{r-1} x)\| \leq K \Big(\|d(x, Fx)\| + \|d(Fx, T^{r-1} x)\| \Big),$$

$$\|d(T^{r-1} x, T^r x)\| \leq K \Big(\|d(T^{r-1} x, Fx)\| + \|d(Fx, T^r x)\| \Big),$$

$$\|d(x, T^r x)\| \leq K \Big(\|d(x, Fx)\| + \|d(Fx, T^r x)\| \Big).$$

Then, we have

$$M(x, T^{r-1} x)$$
$$\leq \max \left\{ K \Big(\|d(x, Fx)\| + \|d(Fx, T^{r-1} x)\| \Big), \frac{1}{2}\|d(x, Fx)\|, \frac{K}{2} \Big(\|d(T^{r-1} x, Fx)\| \right.$$
$$\left. + \|d(Fx, T^r x)\| \Big), K \Big(\|d(x, Fx)\| + \|d(Fx, T^r x)\| \Big), \|d(Fx, T^{r-1} x)\| \right\}$$
$$\leq \max \left\{ K\|d(x, Fx)\| + K\|d(Fx, T^r x)\|, \frac{K}{2} \Big(\|d(T^r x, Fx)\| + \|d(Fx, T^r x)\| \Big), \right.$$
$$\left. \|d(Fx, T^r x)\| \right\} = K \Big(\|d(x, Fx)\| + \|d(Fx, T^r x)\| \Big).$$

Using the above inequality and (3), we get

$$\|d(Fx, T^r x)\| \leq qK \left(\|d(x, Fx)\| + \|d(Fx, T^r x)\| \right),$$

which implies that $\|d(Fx, T^r x)\| \leq \dfrac{qK}{1 - qK} \|d(x, Fx)\|$.

Again, using the above inequality and (3), we have, for all $n \in \mathbb{N}$,

$$(4) \qquad \|d(Fx, T^n x)\| \leq \|d(Fx, T^{r(n)} x)\| \leq \frac{qK}{1 - qK} \|d(x, Fx)\|.$$

Now, let $s = s(n)$ be such that $\|d(F^s x, Tx)\| = \max_{1 \leq i \leq n} \|d(F^i x, Tx)\|$. Analogously to (4) we find that

$$(5) \qquad \|d(F^n x, Tx)\| \leq \|d(F^{s(n)} x, Tx)\| \leq \frac{qK}{1 - qK} \|d(x, Tx)\|$$

for all $n \in \mathbb{N}$. Now, for any fixed $i, j \in \mathbb{N}$, using (1), (4) and (5) we obtain

$$\|d(F^i x, T^j x)\| \leq K \|d(F^i x, Tx)\| + K \|d(Tx, Fx)\| + K \|d(Fx, T^j x)\|.$$

$$\leq K \|d(Tx, Fx)\| + \frac{qK^2}{1 - qK} \left(\|d(x, Tx)\| + \|d(x, Fx)\| \right).$$

Therefore, for all $n \in \mathbb{N}$, $\delta_n := \sup_{i,j \geq n} \|d(F^i x, T^j x)\| < \infty$.
For $i, j \geq n \geq 2$, we have

$$\|d(F^i x, T^j x)\|$$

$$\leq q \max \left\{ \|d(F^{i-1} x, T^{j-1} x)\|, \frac{1}{2} \|d(F^{i-1} x, F^i x)\|, \frac{1}{2} \|d(T^{j-1} x, T^j x)\|, \right.$$

$$\left. \|d(F^{i-1} x, T^j x)\|, \|d(T^{j-1} x, F^i x)\| \right\}$$

$$\leq q \max \left\{ \|d(F^{i-1} x, T^{j-1} x)\|, \frac{K}{2} \left(\|d(F^{i-1} x, T^{j-1} x)\| + \|d(T^{j-1} x, F^i x)\| \right), \frac{K}{2} \right.$$

$$\left. \left(\|d(T^{j-1} x, F^{i-1} x)\| + \|d(F^{i-1} x, T^j x)\| \right), \|d(F^{i-1} x, T^j x)\|, \|d(F^i x, T^{j-1} x)\| \right\}$$

$$\leq qK \max \left\{ \|d(F^{i-1} x, T^{j-1} x)\|, \|d(T^{j-1} x, F^i x)\|, \|d(F^{i-1} x, T^j x)\| \right\} \leq qK \delta_{n-1}.$$

Hence, we get $\delta_n \leq qK \delta_{n-1}$ for all $n \geq 2$. Since $qK < 1$, we obtain that $\delta_n \to 0$ as $n \to +\infty$. Therefore, from (1), for all $p > 0$, we have

$$\|d(T^n x, T^{n+p} x)\| \leq K \left(\|d(F^n x, T^n x)\| + \|d(F^n x, T^{n+p} x)\| \right) \leq 2K \delta_n \to 0$$

as $n \to +\infty$. Hence, $\{T^n x\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $u \in X$ such that $T^n x \to u$ as $n \to +\infty$. Using (1), we have that

$$\|d(F^n x, u)\| \leq K \|d(u, T^n x)\| + K \|d(F^n x, T^n x)\| \leq K \|d(u, T^n x)\| + K \delta_n \to 0,$$

and so, $F^n x \to u$ as $n \to +\infty$, too. Further,

$$\|d(u, Tu)\| \leq K\|d(u, F^{n+1}x)\| + K\|d(FF^n x, Tu)\|$$
$$\leq K\|d(u, F^{n+1}x)\| + KqM(F^n x, u).$$

Letting here $n \to +\infty$ we find that

$$\|d(u, Tu)\| \leq Kq\|d(u, Tu)\|.$$

Since $Kq < 1$, it follows that $d(u, Tu) = 0_E$, i.e.,

$$(6) \qquad\qquad\qquad\qquad Tu = u.$$

In the same way, we can show that

$$(7) \qquad\qquad\qquad\qquad Fu = u.$$

Therefore, from (6) and (7), it follows that u is a common fixed point of F and T.

Now, suppose that v is another common fixed point of F and T, that is, $v = Fv = Tv$. Then,

$$\|d(u, v)\| = \|d(Fu, Tv)\|$$
$$\leq q \max\left\{ \|d(u,v)\|, \frac{1}{2}\|d(u, Fu)\|, \frac{1}{2}\|d(v, Tv)\|, \|d(u, Tv)\|, \|d(v, Fu)\| \right\}$$
$$= q\|d(u, v)\|.$$

Since $q < 1$, we get $d(u, v) = 0_E$, that is, $u = v$. Therefore, the uniqueness of u follows. This completes the proof of Theorem 2.1. $\qquad\qquad\square$

REMARK 2.2. If (F, T) is a pair of self-maps of a complete cone metric space (X, d) over a cone P with normal constant K, satisfying:

$$\|d(Fx, Ty)\| \leq q \max\left\{ \|d(x, y)\|, \|d(x, Fx)\|, \|d(y, Ty)\|, \|d(x, Ty)\|, \|d(y, Fx)\| \right\}$$

for some $q < 1/K$, then the set of common fixed points of F and T can be empty. A counter-example is given by LJ. ĆIRIĆ in [5].

REMARK 2.3. It would be interesting to discuss the validity of Theorem 2.1 with $1/K \leq q < 1$. In the spirit of [2], it would also be interesting to discuss Theorem 2.1 with the condition (2) replaced by

$$d(Fx, Ty) \leq_E q u_{x,y}(F, T) \text{ for all } x, y \in X,$$

where $0 < q < 1$ and

$$u_{x,y}(F, T) \in \left\{ d(x, y), \frac{1}{2}d(x, Fx), \frac{1}{2}d(y, Ty), d(x, Ty), d(y, Fx) \right\}.$$

REFERENCES

1. M. ABBAS, G. JUNGCK: *Common fixed point results for noncommuting mappings without continuity in cone metric spaces.* J. Math. Anal. Appl., **341** (2008), 416–420.

2. M. ABBAS, B. E. RHOADES, T. NAZIR: *Common fixed points for four maps in cone metric spaces.* Appl. Math. Comput., **216** (2010), 80–86.

3. C. DI BARI, P. VETRO: φ-*Pairs and common fixed points in cone metric spaces.* Rend. Circ. Mat. Palermo., **57** (2008), 279–285.

4. LJ. ĆIRIĆ: *On common fixed points in uniform spaces.* Publ. Inst. Math., **24** (38) (1978), 39–43.

5. LJ. ĆIRIĆ: *Contractive type non-self mappings on metric spaces of hyperbolic type.* J. Math. Anal. Appl., **317** (1) (2006), 28–42

6. L. G. HUANG, X. ZHANG: *Cone metric spaces and fixed point theorems of contractive mappings.* J. Math. Anal. Appl., **332** (2) (2007), 1468–1476.

7. D. ILIĆ, V. RAKOČEVIĆ: *Common fixed points for maps on cone metric space.* J. Math. Appl., **341** (2008), 876–882.

8. S. JANKOVIĆ, Z. GOLUBOVIĆ, S. RADENOVIĆ: *Compatible and weakly compatible mappings in cone metric spaces.* Math. Comput. Modelling, **52** (9-10) (2010), 1728–1738.

9. B. V. KVEDARAS, A. V. KIBENKO, A. I. PEROV: *On some boundary value problems.* Litov. matem. sbornik, **5** (1) (1965), 69–84.

10. E. M. MUKHAMADIEV, V. J. STETSENKO: *Fixed point principle in generalized metric space.* Izvestija AN Tadzh. SSR, fiz. mat. i geol. chem. nauki, **10** (1969), 8-19 [Russian].

11. A. I. PEROV: *The Cauchy problem for systems of ordinary differential equations.* In: Approximate methods of solving differential equations, Kiev, Naukova Dumka (1964), 115–134 [Russian].

12. A. I. PEROV, A. V. KIBENKO: *An approach to studying boundary value problems.* Izvestija AN SSSR, ser. math., **30** (2) (1966), 249–264 [Russian].

13. S. RADENOVIĆ: *Common fixed points under contractive conditions in cone metric spaces.* Comput. Math. Appl., **58** (2009), 1273–1278.

14. SH. REZAPOUR, R. HAMLBARANI: *Some notes on the paper: Cone metric spaces and fixed point theorems of contractive mappings.* J. Math. Anal. Appl., **345** (2008), 719–724.

15. J. S. VANDERGRAFT: *Newton's method for convex operators in partially ordered spaces.* SIAM J. Numer. Anal., **4** (1967), 406–432.

16. P. P. ZABREJKO: *K-metric and K-normed linear spaces: survey.* Collect. Math., **48** (4–6) (1997), 825–859.

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