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ON TWO NEW FAMILIES OF ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS WITH OPTIMAL ORDER

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In this paper, two new families of eighth-order iterative methods for solving nonlinear equations are presented. These methods are developed by combining a class of optimal two-point methods and a modified Newton's method in the third step. Per iteration the presented methods require three evaluations of the function and one evaluation of its first derivative and therefore have the efficiency index of 1.682. Kung and Traub conjectured that a multipoint iteration without memory based on n evaluations could achieve optimal convergence order 2^{n-1}. Thus the new families of eighth-order methods agrees with the conjecture of Kung-Traub for the case $n = 4$. Numerical comparisons are made with several other existing methods to show the performance of the presented methods.

1. INTRODUCTION

One of the most important and challenging problems in scientific and engineering applications is to find the solutions of nonlinear equations. The boundary value problems appearing in Kinetic theory of gases, elasticity and other areas are reduced to solving these equations. Many optimization problems also lead to such equations. In this paper, we consider iterative methods to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$.

Newton's method is the well-known iterative method for finding α by using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

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that converges quadratically in some neighborhood of α [**1**].

To improve the local order of convergence, many modified methods have been proposed in open literatures, see [**1, 2, 3, 4, 5, 6**] and references therein. Chun [**2**] developed two one-parameter fourth-order methods, which are given by:

$$\text{(2)} \qquad \textbf{CM1:} \quad \begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)^2}{f(x_n)^2 - 2f(x_n)f(y_n) + 2\beta f(y_n)^2} \dfrac{f(y_n)}{f'(x_n)}, \end{cases}$$

and

$$\text{(3)} \qquad \textbf{CM2:} \quad \begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)^3}{f(x_n)^2 S_2(x_n, y_n) + 2\beta f(y_n)^2 S_\beta(x_n, y_n)} \dfrac{f(y_n)}{f'(x_n)}, \end{cases}$$

where $S_\beta(x_n, y_n) = f(x_n) - \beta f(y_n)$ and $\beta \in \mathbb{R}$ is a constant. We note that the methods defined by (2) and (3) reduce to the Traub-Ostrowski method [**6**] when $\beta = 0$.

Kung and Traub [**7**] constructed two families of iterative methods without memory based on n evaluations of function or its derivatives could achieve optimal convergence order 2^{n-1}. Moreover, they proved that Newton's method (1) and Steffensen's method [**9**] achieve optimal convergence order two based on two evaluations. We present these families, called here KT for brevity, in the form given in [**7, 8, 19**].

KT(1): For any n, we define the iteration function $p_j(f)(j = 0, \dots, n)$ as follows: $p_j(f)(x) = x$ and for $n > 0$,

$$\text{(4)} \qquad \begin{cases} p_1(f)(x) = x + \gamma f(x), \\ \quad \vdots \\ p_{j+1}(f)(x) = R_j(0), \end{cases}$$

for $j = 1, \dots, n-1$, where γ is a nonzero constant, $R_j(y)$ is the inverse interpolatory polynomial of degree at most j such that $R_j(f(p_\eta(f)(x))) = p_\eta(f)(x)(\eta = 0, \dots, j)$. The iterative method is defined by $x_{k+1} = p_n(x_k)$ starting with an initial guess x_0. Let us note that the family KT (1) requires no evaluation of derivatives of f.

KT(2): For any n, we define the iteration function $q_j(f)(j = 0, \dots, n)$ as follows: $q_1(f)(x) = x$ for $n > 1$,

$$\text{(5)} \qquad \begin{cases} q_1(f)(x) = x - \dfrac{f(x)}{f'(x)}, \\ \quad \vdots \\ p_{j+1}(f)(x) = S_j(0), \end{cases}$$

for $j = 2, \dots, n-1$, where $S_j(y)$ is the inverse interpolatory polynomial of degree at most j such that

$$S_j(f(x)) = x, \quad S_j'(f(x)) = \frac{1}{f'(x)}, \quad S_j(f(q_\eta(f)(x))) = q_\eta(x) \quad (\eta = 2, \dots, j).$$

The iterative method is defined by $x_{k+1} = q_n(x_k)$, starting with an initial guess x_0. For a fixed n, the methods KT(1) and KT(2) can be easily constructed using a recursive procedure on a computer (see [**7**]). Here we have taken $n = 4$ to obtain the methods of the eighth order.

In this paper, based on a class of optimal two-point methods, we construct two new families of eighth-order iterative methods free from second derivative. The important characteristic of the new methods is that per iteration they require three evaluations of the function and one of its first derivative. Thus the efficiency, in term of function evaluations, of the new methods is better than that of the classical optimal two-point methods. Finally, some numerical examples are provided to show the performance of the methods presented in this contribution.

2. DERIVATION OF METHODS AND CONVERGENCE ANALYSIS

Consider the following iteration scheme

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(y_n)}. \end{cases} \tag{6}$$

This method is simple and its rate of convergence is four, which is a consequence of the fact that Newton's method is of the second order and the following generalization of Traub's theorem [**6**]:

Theorem 1. *Let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_s(x)$ be iterative functions with the orders $r_1, r_2, \ldots, r_s$, respectively. Then the composition of iterative functions*

$$\varphi(x) = \varphi_1(\varphi_2(\cdots(\varphi_s(x))\cdots))$$

defines the iterative method of the order $r_1 r_2 \ldots r_s$.

Assume that a real function $G(t)$ and its derivatives $G'(t)$ and $G''(t)$ are continuous in the neighborhood of 0. A rather wide class of optimal two-point methods can be obtained starting from the two-step iterative scheme (6) and substituting the derivative $f'(y_n)$ by its approximation $\dfrac{f'(x_n)}{G(\mu_n)}$, where $\mu_n = \dfrac{f(y_n)}{f(x_n)}$. Therefore, the two-step scheme (6) becomes

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - G(\mu_n)\dfrac{f(y_n)}{f'(x_n)}. \end{cases} \tag{7}$$

The function $G(t)$ in (7) has to be determined so that the two-point method (7) attains the optimal order four using only three function evaluations $f(x_n)$, $f'(x_n)$ and $f(y_n)$, which is the subject of the following theorem (see [**19**]).

Theorem 2. *Let $\alpha \in I$ be a simple root of real single-valued function $f : I \to \mathbb{R}$ possessing a certain number of continuous derivatives in the neighborhood of $\alpha \in I$, where I is an open interval. Let $G(t)$ be a function satisfying $G(0) = 1, G'(0) = 2$ and $|G''(0)| < \infty$. If x_0 is sufficiently close to α, then the order of convergence of the family of two-step methods (7) is four and it satisfies the error equation*

$$e_{n+1} = [c_2^3(5 - G''(0)/2) - c_2c_3]e_n^4 + O(e_n^5),$$

where $e_n = x_n - \alpha$ and $c_k = \dfrac{1}{k!}\dfrac{f^{(k)}(\alpha)}{f'(\alpha)}$.

Proof. Presented in [**19**]. □

First, we consider the following iteration scheme

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - G(\mu_n)\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - H(\mu_n)\dfrac{f(z_n)}{f'(x_n)} - M(\delta_n)\dfrac{f(z_n)}{f'(z_n)}, \end{cases} \tag{8}$$

where $\mu_n = \dfrac{f(y_n)}{f(x_n)}, \delta_n = \dfrac{f(z_n)}{f(x_n)}$ and $G(t), H(t)$ and $M(t)$ represent three real-valued functions. It is quite obvious that formula (8) requires five evaluations per iteration. However, we can reduce the number of evaluations to four by using some suitable approximation of the derivative $f'(z_n)$. We obtain this approximation by considering the approximation of $f(x)$ by a rational linear function of the form

$$Y(x) = \frac{(x - x_n) + A}{B(x - x_n) + C} + f'(x_n), \tag{9}$$

where the parameters A, B and C are determined by the condition that f and Y coincide at x_n, y_n and z_n. That means $Y(x)$ satisfies the conditions

$$Y(x_n) = f(x_n), \qquad Y(y_n) = f(y_n), \qquad Y(z_n) = f(z_n). \tag{10}$$

After some simple calculations we obtain

$$A = \frac{P1\,KS\,(P2 - P3)}{P1\,(KP3 - SP2) + P3\,P2\,(S - K)},$$

$$B = \frac{S\,(P1 - P3) + K\,(P2 - P1)}{P1\,(KP3 - SP2) + P3\,P2\,(S - K)},$$

$$C = \frac{KS\,(P3 - P2)}{P1\,(KP3 - SP2) + P3\,P2\,(S - K)},$$

where

$$\begin{cases} P1 = f'(x_n) - f(x_n), \\ P2 = f'(x_n) - f(y_n), \\ P3 = f'(x_n) - f(z_n), \end{cases} \tag{11}$$

and

$$\begin{cases} S = y_n - x_n, \\ K = z_n - x_n. \end{cases} \tag{12}$$

Differentiation of (9) gives

$$Y'(x) = \frac{C - BA}{(B(x - x_n) + C)^2}. \tag{13}$$

We can now approximate the derivative $f'(x)$ with the derivative $Y'(x)$ of rational function (9) and obtain

$$f'(z_n) \approx Y'(z_n). \tag{14}$$

Substituting the values of A, B and C into (9) and then using (13), we get after simplifications

$$f'(z_n) = \frac{f[z_n, x_n]f[z_n, y_n]}{f[y_n, x_n]}, \tag{15}$$

where $f[s,t] = \dfrac{f(s) - f(t)}{s - t}$. Then the iteration scheme (8) in its final form is given by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - G(\mu_n)\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - H(\mu_n)\dfrac{f(z_n)}{f'(x_n)} - M(\delta_n)\dfrac{f(z_n)f[y_n, x_n]}{f[z_n, x_n]f[z_n, y_n]}, \end{cases} \tag{16}$$

where $\mu_n = \dfrac{f(y_n)}{f(x_n)}$, $\delta_n = \dfrac{f(z_n)}{f(x_n)}$ and $G(t), H(t)$ and $M(t)$ represent three real-valued functions. We can state the following convergence theorem for the family of methods (16).

Theorem 3. *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \longrightarrow \mathbb{R}$ for an open interval I, $G(t)$, $H(t)$ and $M(t)$ any real-valued functions with $G(0) = 1, G'(0) = 2, G''(0) = 8, |G'''(0)| < \infty$, $H(0) = H'(0) = H''(0) = H'''(0) = 0, |H^{(4)}(0)| < \infty$, and $M(0) = M'(0) = 1$. If x_0 is sufficiently close to α, then the method defined by* (16) *has eighth-order convergence and it satisfies the error equation*

$$e_{n+1} = K_8 e_n^8 + O(e_n^9), \tag{17}$$

where

$$K_8 = \frac{1}{24}(c_2^2 - c_3)[(4G'''(0) - H^{(4)}(0) - 120)c_2^3 - 96c_2c_3 + 24c_4]c_2^2,$$

and $e_n = x_n - \alpha$ *and* $c_k = \dfrac{1}{k!}\dfrac{f^{(k)}(\alpha)}{f'(\alpha)}$.

Proof. Let α be a simple zero of f. Using Taylor expansion and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)], \tag{18}$$

$$f'(x_n) = f'(\alpha)[1 + 2\,c_2 e_n + 3\,c_3 e_n^2 + 4\,c_4 e_n^3 + O(e_n^4)]. \tag{19}$$

where $e_n = x_n - \alpha$ and $c_k = \dfrac{1}{k!}\dfrac{f^{(k)}(\alpha)}{f'(\alpha)}$. By a simple calculation, we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5), \tag{20}$$

and hence, we have

$$\tilde{e}_n = e_n - \frac{f(x_n)}{f'(x_n)} = c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + O(e_n^5), \tag{21}$$

where $\tilde{e}_n = y_n - \alpha$. By expanding $f(y_n)$ about α, we have

$$f(y_n) = f'(\alpha)[c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + O(e_n^5)]. \tag{22}$$

So, from (18) and (22), we can get

$$\mu_n = \frac{f(y_n)}{f(x_n)} = c_2 e_n + (2c_3 - 3c_2^2)e_n^2 + (3c_4 - 10c_2c_3 + 8c_2^3)e_n^3 + O(e_n^4). \tag{23}$$

Using the Taylor expansion, we have

$$\begin{aligned} G(\mu_n) &= G(0) + G'(0)\mu_n + \frac{G''(0)}{2!}\mu_n^2 + \frac{G'''(0)}{3!}\mu_n^3 + O(\mu_n^4) \\ &= G(0) + G'(0)c_2 e_n + \left[(2c_3 - 3c_2^2)G'(0) + \frac{1}{2}\,G''(0)c_2^2\right] e_n^2 + [(3\,c_4 - 10\,c_2c_3 \\ &\quad +8c_2^3)G'(0) + (-3\,c_2{}^3 + 2\,c_2c_3)G''(0) + \frac{1}{6}\,G'''(0)c_2^3]e_n^3 + O(e_n^4). \end{aligned} \tag{24}$$

From (19), (22) and (24), we obtain

$$\frac{f(y_n)}{f'(x_n)} = c_2 e_n^2 + 2(c_3 - 2c_2^2)e_n^3 + (3c_4 - 14c_2c_3 + 13c_2^3)e_n^4 + O(e_n^5). \tag{25}$$

and

$$\begin{aligned} \hat{e}_n = \tilde{e}_n - G(\mu_n)\frac{f(y_n)}{f'(x_n)} &= \big((1 - G(0))c_2\big)e_n^2 \\ &+ \big[(-2\,c_3 + 4\,c_2^2)G(0) - G'(0)c_2^2 + 2c_3 - 2c_2^2\big]e_n^3 + O(e_n^4), \end{aligned} \tag{26}$$

where $\hat{e}_n = z_n - \alpha$. Similarly to (22), we have

$$(27)\qquad f(z_n) = f'(\alpha)\big[((1-G(0))c_2)e_n^{\,2} + \big[(-2\,c_3 + 4\,c_2^2)G(0) - G'(0)c_2^2 + 2c_3 - 2c_2^2\big]e_n^{\,3} + O(e_n^{\,4})\big],$$

and

$$(28)\qquad f[z_n, y_n] = \frac{f(z_n) - f(y_n)}{\hat{e}_n - \tilde{e}_n} = f'(\alpha)\big[1 - c_2^2(-2+G(0))e_n^{\,2} \big[(4-2G(0))c_3 + (4G(0) - G'(0) - 4)c_2^2\big]e_n^3 + O(e_n^{\,4})\big],$$

$$(29)\qquad f[z_n, x_n] = \frac{f(z_n) - f(x_n)}{\hat{e}_n - e_n} = f'(\alpha)\big[1 + c_2 e_n + (c_3 + (1-G(0))\,c_2^2)e_n^{\,2} + \big[c_4 + 3(1-G(0))c_2c_3 + (4G(0) - G'(0) - 2)c_2^3\big]\,e_n^{\,3} + O(e_n^{\,4})\big],$$

$$(30)\qquad f[y_n, x_n] = \frac{f(y_n) - f(x_n)}{\tilde{e}_n - e_n} = f'(\alpha)\big[1 + c_2 e_n + (c_3 + c_2^2)e_n^{\,2} + (3c_2c_3 - 2c_2^3 + c_4)e_n^{\,3} + O(e_n^{\,4})\big].$$

Using the above results, we obtain

$$(31)\qquad \frac{f(z_n)f[y_n, x_n]}{f[z_n, x_n]f[z_n, y_n]} = -c_2(-1+G(0))e_n^{\,2} + \big[2(1-G(0))c_3 + (4G(0) - G'(0) - 2)c_2^2\big]e_n^{\,3} + O(e_n^{\,4}),$$

also

$$(32)\qquad \delta_n = \frac{f(z_n)}{f(x_n)} = \big(1-G(0)\big)c_2 e_n + \big[2(1-G(0))c_3 + (5G(0) - G'(0) - 3)c_2^2\big]e_n^{\,2} + O(e_n^{\,3}).$$

Using the Taylor expansion, we can get

$$(33)\qquad H(\mu_n) = H(0) + H'(0)\mu_n + \frac{H''(0)}{2!}\mu_n^{\,2} + \frac{H'''(0)}{3!}\mu_n^{\,3} + \frac{H^{(4)}(0)}{4!}\mu_n^{\,4} + O(\mu_n^{\,5})$$

$$= H(0) + H'(0)c_2e_n + \left[2H'(0)c_3 + \Big(-3H'(0) + \frac{1}{2}H''(0)\Big)c_2^2\right]e_n^{\,2} + \Big[3H'(0)c_4 + (2H''(0) - 10H'(0))c_2c_3 + \Big(8H'(0) + \frac{1}{6}H'''(0) - 3\,H''(0)\Big)c_2^3\Big]e_n^{\,3} + O(e_n^{\,4}),$$

and

$$(34)\qquad M(\delta_n) = M(0) + M'(0)\delta_n + O(\delta_n^{\,2}) = M(0) - M'(0)c_2(-1+G(0))e_n + (2M'(0)(1-G(0))c_3 + M'(0)(5G(0) - G'(0) - 3)c_2^2)e_n^{\,2} + O(e_n^{\,3}).$$

Now from (16), (31), (33) and (34), we have

$$e_{n+1} = \hat{e}_n - H(\mu_n)\frac{f(z_n)}{f'(x_n)} - M(\delta_n)\frac{f(z_n)f[y_n, x_n]}{f[z_n, x_n]f[z_n, y_n]} \tag{35}$$
$$= K_2 e_n^2 + K_3 e_n^3 + K_4 e_n^4 + K_5 e_n^5 + K_6 e_n^6 + K_7 e_n^7 + K_8 e_n^8 + O(e_n^9),$$

where

$$K_2 = [1 - G(0) - M(0) + M(0)G(0) - H(0) + H(0)G(0)]\, c_2. \tag{36}$$

Before we list other K_i, $i = 3, 4, \ldots, 8$, we choose $H(0) = 0$ and $G(0) = 1$, so we have $K_2 = 0$ and

$$K_3 = (M(0) - 1)(G'(0) - 2)c_2^2. \tag{37}$$

$$K_4 = \frac{1}{2}c_2((-4\,H'(0) + 2\,H'(0)G'(0) + 14\,G'(0) - G''(0) - 18 + 18\,M(0) \tag{38}$$
$$- 14\,m_0 G'(0) + M(0)G''(0))c_2^2 + (14 - 8\,G'(0) - 14\,M(0) + 8\,M(0)G'(0))c_3).$$

Let $M(0) = 1$ and $G'(0) = 1$, then $K_3 = K_4 = 0$ and

$$K_5 = \frac{1}{2}\,H'(0)c_2^2(2\,c_3 + (G''(0) - 10){c_2}^2). \tag{39}$$

Putting $H'(0) = 0$ into (39), we obtain $K_5 = 0$ and

$$K_6 = \frac{1}{4}\,H''(0)c_2^3(2\,c_3 + (G''(0) - 10)c_2^2). \tag{40}$$

Substituting $H''(0) = 0$ into (40), we obtain $K_6 = 0$ and

$$K_7 = -\frac{1}{12}\big[(72 - 2\,H'''(0) + 12\,M'(0)G''(0) - 120\,M'(0) - 6\,G''(0))c_2^2 c_3 \tag{41}$$
$$+(3\,M'(0)G''(0)^2 + 300\,M'(0) - 60\,M'(0)G''(0) + 6\,G''(0) - H'''(0)G''(0)$$
$$+10\,H'''(0) - 60)c_2^4 + 12\,(M'(0) - 1)c_3^2\big]c_2^2.$$

So that if $H'''(0) = 0, M'(0) = 1, G''(0) = 8, |H^{(4)}(0)| < \infty$, and $|G'''(0)| < \infty$, then from (35)-(41), we obtain the error equation

$$e_{n+1} = K_8 e_n^8 + O(e_n^9), \tag{42}$$

where

$$K_8 = \frac{1}{24}(c_2^2 - c_3)[(4G'''(0) - H^{(4)}(0) - 120)c_2^3 - 96c_2c_3 + 24c_4]c_2^2.$$

This completes the proof. □

Now we consider the following iteration scheme

$$(43)\qquad \begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - G(\mu_n)\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - T(\delta_n)\dfrac{f(z_n)}{f'(z_n)}, \end{cases}$$

where $\mu_n = \dfrac{f(y_n)}{f(x_n)}, \delta_n = \dfrac{f(z_n)}{f(x_n)}$ and $G(t)$ and $T(t)$ represent two real-valued functions. It can be noted that formula (43) requires at least five functional evaluations per iteration, so it is not in the form giving rise to an iteration of optimal order. We now develop a new approximation of the derivative $f'(z_n)$ so that the resulting modification of (43) may require four functional evaluations per step, and so it will be of optimal order.

By Taylor expansion, we have

$$(44)\qquad f(y_n) \approx f(z_n) + f'(z_n)(y_n - z_n) + \frac{1}{2}f''(z_n)(y_n - z_n)^2,$$

and, further,

$$(45)\qquad f(y_n) \approx f(z_n) + f'(z_n)(y_n - z_n) + \frac{1}{2}\frac{f'(z_n) - f'(y_n)}{z_n - y_n}(y_n - z_n)^2.$$

On the other hand, from the linear interpolation between the points $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$, which is given by

$$(46)\qquad f'(x) \approx \frac{x - x_n}{y_n - x_n}f'(y_n) + \frac{x - y_n}{x_n - y_n}f'(x_n),$$

we obtain the following approximation

$$(47)\qquad f'(z_n) \approx \frac{z_n - x_n}{y_n - x_n}f'(y_n) + \frac{z_n - y_n}{x_n - y_n}f'(x_n).$$

This then yields

$$(48)\qquad f'(y_n) \approx \frac{(y_n - x_n)f'(z_n) + (z_n - y_n)f'(x_n)}{z_n - x_n}.$$

By substituting (48) into (45), we get the approximation

$$(49)\qquad f'(z_n) \approx \frac{2(z_n - x_n)f[z_n, y_n] - (z_n - y_n)f'(x_n)}{z_n + y_n - 2x_n},$$

where $f[s,t] = \dfrac{f(s) - f(t)}{s - t}$. Using our approximation (49) in (43), we obtain the following new iteration scheme

$$(50)\qquad \begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - G(\mu_n)\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - T(\delta_n)\dfrac{f(z_n)(z_n + y_n - 2x_n)}{2(z_n - x_n)f[z_n, y_n] - (z_n - y_n)f'(x_n)}, \end{cases}$$

where $\mu_n = \dfrac{f(y_n)}{f(x_n)}, \delta_n = \dfrac{f(z_n)}{f(x_n)}$ and $G(t)$ and $T(t)$ are two real-valued functions. We can state the following convergence theorem for the family of methods (50).

Theorem 4. *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \longrightarrow \mathbb{R}$ for an open interval I, $G(t)$ and $T(t)$ any real-valued functions with $G(0) = 1, G'(0) = 2, G''(0) = 10, |G'''(0)| < \infty$ and $T(0) = 1, T'(0) = \dfrac{3}{2}$. If x_0 is sufficiently close to α, then the method defined by* (50) *has eighth-order convergence and it satisfies the error equation*

$$e_{n+1} = K_8 e_n^8 + O(e_n^9), \tag{51}$$

where

$$K_8 = \frac{1}{4} c_3 c_2^2 (9c_2c_3 + (84 - G'''(0))c_2^3 - 4c_4), \tag{52}$$

and $e_n = x_n - \alpha$ *and* $c_k = \dfrac{1}{k!}\dfrac{f^{(k)}(\alpha)}{f'(\alpha)}$.

Proof. Let α be a simple zero of f. From (32) and using the Taylor expansion, we get

$$\begin{aligned} T(\delta_n) = T(0) + T'(0)\delta_n + O(\delta_n^2) &= T(0) - T'(0)c_2(-1 + G(0))e_n \\ &+ (2\,T'(0)(1 - G(0))c_3 + T'(0)(5\,G(0) - G'(0) - 3)c_2^2)e_n^2 + O(e_n^3). \end{aligned} \tag{53}$$

Furthermore, from (19), (21), (26), (27) and (28) we have

$$\begin{aligned} &\frac{f(z_n)(z_n + y_n - 2x_n)}{2(z_n - x_n)f[z_n, y_n] - (z_n - y_n)f'(x_n)} = \\ &\frac{f(z_n)(\hat{e}_n + \tilde{e}_n - 2e_n)}{2(\hat{e}_n - e_n)f[z_n, y_n] - (\hat{e} - \tilde{e}_n)f'(x_n)} = c_2(1 - G(0))e_n^2 \\ &+(2(1 - G(0))c_3 + (4G(0) - G'(0) - 2)c_2{}^2)e_n^3 + O(e_n^4). \end{aligned} \tag{54}$$

Now, from (50), (53) and (54), we can obtain

$$\begin{aligned} e_{n+1} &= \hat{e}_n - T(\delta_n)\frac{f(z_n)(\hat{e}_n + \tilde{e}_n - 2e_n)}{2(\hat{e}_n - e_n)f[z_n, y_n] - (\hat{e}_n - \tilde{e}_n)f'(x_n)} \\ &= K_2e_n^2 + K_3e_n^3 + K_4e_n^4 + K_5e_n^5 + K_6e_n^6 + K_7e_n^7 + K_8e_n^8 + O(e_n^9), \end{aligned} \tag{55}$$

where

$$K_2 = (G(0) - 1)(T(0) - 1)c_2. \tag{56}$$

Before we list other $K_i, \quad i = 3, 4, \ldots, 8$, we choose $T(0) = 1$ and $G(0) = 1$, so we have $K_2 = K_3 = K_4 = 0$ and

$$K_5 = -T'(0)[4 - 4G'(0) + G'(0)^2]c_2^4. \tag{57}$$

Putting $G'(0) = 2$ into (57), we obtain $K_5 = K_6 = 0$ and

$$K_7 = -\frac{1}{4}c_2^2((4T'(0) - 6)c_3^2 + (30 - 3G''(0) + 4T'(0)G''(0) - 40T'(0))c_2^2c_3 + (100T'(0) - 20T'(0)G''(0) + T'(0)G''(0)^2)c_2^4). \tag{58}$$

So that if $T'(0) = \frac{3}{2}$, $G''(0) = 10$, and $|G'''(0)| < \infty$, then from (55)-(58), we obtain the error equation

$$e_{n+1} = K_8 e_n^8 + O(e_n^9), \tag{59}$$

where

$$K_8 = \frac{1}{4}c_3c_2^2(9\,c_2c_3 + (84 - G'''(0))c_2^3 - 4c_4). \tag{60}$$

This completes the proof. □

Remark 1. Chun's two-point methods CM1 (2) and CM2 (3) are special cases of the more general family of two-point methods (7). Particular cases

$$G(t) = \frac{1}{1 - 2t + 2\beta t^2} \tag{61}$$

and

$$G(t) = \frac{1}{1 - 2t + 2\beta t^2(1 - \beta t)} \tag{62}$$

give CM1 (2) and CM2 (3), respectively.

Remark 2. There are three and two weight functions in schemes (16) and (50), respectively. With weight function (61), theorems 3 and 4 can then be simplified as follows.

Theorem 3′. *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \longrightarrow \mathbb{R}$ for an open interval I, $H(t)$ and $M(t)$ any real-valued functions with $H(0) = H'(0) = H''(0) = H'''(0) = 0, |H^{(4)}(0)| < \infty$, and $M(0) = M'(0) = 1$. If x_0 is sufficiently close to α, then the method defined by* (16) *has eighth-order convergence for $\beta = 0$ and it satisfies the error equation*

$$e_{n+1} = K_8 e_n^8 + O(e_n^9), \tag{63}$$

where

$$K_8 = \frac{1}{24}(c_2^2 - c_3)[(72 - H^{(4)}(0))c_2^3 - 96c_2c_3 + 24c_4]c_2^2,$$

and $e_n = x_n - \alpha$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$.

Theorem 4′. *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \longrightarrow R$ for an open interval I and $T(t)$ any real-valued function with $T(0) = 1, T'(0) = \frac{3}{2}$. If x_0 is sufficiently close to α, then the method defined by (50) has eighth-order convergence for $\beta = -\frac{1}{2}$ and it satisfies the error equation*

$$e_{n+1} = K_8 e_n^8 + O(e_n^9), \tag{64}$$

where

$$K_8 = \frac{1}{4} c_3 c_2^2 (9c_2c_3 + 9c_2^3 - 4c_4), \tag{65}$$

and $e_n = x_n - \alpha$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$.

Similar to weight function $G(t) = \frac{1}{1-2t+2\beta t^2}$, we can get the above results for weight function $G(t) = \frac{1}{1-2t+2\beta t^2(1-\beta t)}$.

Remark 3. Table 1 shows some particular functions $H(t)$ and $M(t)$ that satisfy Theorem 3. Table 2 shows some particular functions $T(t)$ that satisfy Theorem 4. Table 3 shows some particular functions $G(t)$ (with $a = 8$ for Theorem 3 and with $a = 10$ for Theorem 4).

Remark 4. Any method of the families (16) and (50) uses four evaluations per iteration, and has eighth-order convergence with conditions of Theorems 3 and 4, which accord with the conjecture of Kung-Traub that a multipoint iteration without memory based on n evaluations achieves optimal convergence order 2^{n-1} for $n = 4$.

Remark 5. The families (16) and (50) achieve eighth-order convergence. Per iteration the presented methods require three evaluations of the function and one evaluation of its first derivative. We consider the definition of efficiency index [**6**] as $I = p^{\frac{1}{w}}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. If we assume that all the evaluations have the same cost as function one, we have that the families of methods (16) and (50) have the efficiency index $I = \sqrt[4]{8} \simeq 1.682$ which is better than $I = \sqrt[2]{2} \simeq 1.414$ of Newton's method, $I = \sqrt[3]{4} \simeq 1.587$ of Chun's methods (2), $I = \sqrt[4]{6} \simeq 1.565$ of some methods with sixth-order convergence [**3, 4, 5**], and $I = \sqrt[4]{7} \simeq 1.627$ of some variants of Ostrowski's method with seventh-order convergence [**12**].

$H(t)$	$M(t)$
$H_1(t) = \theta t^4 + \lambda t^5$	$M_1(t) = 1 + t + \lambda t^2$
$H_2(t) = \dfrac{t^4}{1 + \lambda t + \theta t^2}$	$M_2(t) = \dfrac{1}{1 - t + \lambda t^2}$
$H_3(t) = \dfrac{\lambda t^5 + t^4}{1 + \theta t^3}$	$M_3(t) = 1 + \dfrac{t}{1 + \lambda t}$
$H_4(t) = \dfrac{\lambda t^4 + \theta t^5}{1 + \theta t^2 + t^4}$	$M_4(t) = (1 + \lambda t)^{\frac{1}{\lambda}}, \quad \lambda \neq 0$

Table 1. Typical forms of $H(t)$ and $M(t)$ with $\theta, \lambda \in \mathbb{R}$

$T(t)$
$T_1(t) = 1 + \dfrac{(3/2)t}{1+\lambda t}$
$T_2(t) = 1 + \dfrac{3}{2}t + \lambda t^2 + \gamma t^3$
$T_3(t) = \dfrac{1}{1-(3/2)t + \lambda t^2 + \gamma t^3}$
$T_4(t) = (1+\lambda t)^{\frac{3}{2\lambda}}$

Table 2. Typical forms of $T(t)$ with $\lambda, \gamma \in R$

$G(t)$
$G_1(t) = \dfrac{(a-8)t-4}{-4+at}$
$G_2(t) = 1 + 2t + \dfrac{a}{2}t^2$
$G_3(t) = \dfrac{2}{(8-a)\,t^2 - 4\,t + 2}$
$G_4(t) = \dfrac{8}{-(a-8)^2t^3 + (32-4a)t^2 - 16t + 8}$
$G_5(t) = \left(1 + \left(2 - \dfrac{1}{2}\,a\right)t\right)^{\frac{4}{4-a}}$

Table 3. Typical forms of $G(t)$ ($a = 8$ for Theorem 3 and $a = 10$ for Theorem 4)

3. NUMERICAL EXAMPLES

Now, we give some numerical examples that confirm the theoretical results. Compared were Newton's method (1) (NM), the methods of Chun and Ham [**5**] (CM) defined by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n)}{f(x_n) - 2f(y_n)}\,\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - H(\mu_n)\dfrac{f(z_n)}{f'(x_n)}, \end{cases} \tag{66}$$

with $\mu_n = \dfrac{f(y_n)}{f(x_n)}$ and $H(t) = \dfrac{1}{1-2t}$, methods of Kou et al. [**12**] (KM) defined by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ H_2(x_n, y_n) = [f(x_n) - 2f(y_n)]^{-1} f(y_n), \\ z_n = y_n - H_2(x_n, y_n)(x_n - y_n), \\ H_\beta(y_n, z_n) = [f(y_n) - \beta f(z_n)]^{-1} f(z_n), \\ x_{n+1} = z_n - [(1 + H_2(x_n, y_n))^2 + H_\beta(y_n, z_n)]\dfrac{f(z_n)}{f'(x_n)}, \end{cases} \tag{67}$$

with $\beta = 0$, methods of Bi et al. [**13**] (BM) defined by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)}\,\dfrac{f(y_n)}{f'(x_n)}, \\ x_{n+1} = z_n - H(\mu_n)\dfrac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}, \end{cases} \tag{68}$$

where

$$(69)\qquad \begin{cases} \mu_n = \dfrac{f(z_n)}{f(x_n)}, \\ f[z_n, y_n] = \dfrac{f(z_n) - f(y_n)}{z_n - y_n}, \\ f[z_n, x_n, x_n] = \dfrac{f[z_n, x_n] - f'(x_n)}{z_n - x_n}, \end{cases}$$

with $\beta = -\frac{1}{2}$ and $H(t) = \frac{1}{(1-t)^2}$, Kung and Traub methods (4) with $\gamma = 0.01$ (KT(1)), Kung and Traub method (5) (KT(2)) and

PM1: $H_1(t), M_3(t), G_3(t)$ with $\lambda = 30, \quad \theta = 6, \quad a = 8$ in (16),

PM2: $T_1(t), G_4(t)$ and $\lambda = 1, \quad a = 10$ in (50),

PM3: $T_2(t), G_4(t)$ and $\lambda = 1, \quad \gamma = 1, \quad a = 10$ in (50),

PM4: $T_3(t), G_4(t)$ and $\lambda = 1, \quad \gamma = 1, \quad a = 10$ in (50),

PM5: $T_1(t), G_3(t)$ and $\lambda = 1, \quad a = 10$ in (50),

were introduced in the present contribution.

All computations were done using MAPLE 12 using 750 digit floating point arithmetics (Digits := 750). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. For numerical illustrations in this section we used the fixed stopping criterion $\epsilon = 10^{-30}$. We used the test functions as in [**5, 8, 17, 18, 19**].

$$f_1(x) = x^3 + 4x^2 - 10, \qquad x_* = 1.3652300134140968457608068290,$$

$$f_2(x) = \sin^2(x) - x^2 + 1, \qquad x_* = 1.4044916482153412260350868178,$$

$$f_3(x) = 10xe^{-x^2} - 1, \qquad x_* = 1.6796306104284499406749203388,$$

$$f_4(x) = (x+2)e^x - 1, \qquad x_* = -0.4428544010023885831413280000,$$

$$f_5(x) = (x-1)^3 - 2, \qquad x_* = 2.2599210498948731647672106073,$$

$$f_6(x) = e^{x^2+7x-30} - 1, \qquad x_* = 3.0,$$

$$f_7(x) = \mathrm{e}^{-x^2+x+2} - \cos(x+1) + x^3 + 1, \qquad x_* = -1.0,$$

$$f_8(x) = (x-2)\left(x^{10} + x + 1\right)\mathrm{e}^{-x-1}, \qquad x_* = 2.0.$$

As a convergence criterion, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-30}. Also displayed is the number of iterations to approximate the zero (IT), the number of functional evaluations (NFEs) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative, the approximate zero x_*, and the value $|f(x_*)|$. Note that the approximate zeroes were displayed only up to the 28th decimal places, so making all look the same though they may in fact differ. The test results in Table 4 show that for most of the functions we tested, the methods introduced in this paper for numerical tests have equal or better performance compared to the other methods.

$f(x)$	IT	NFE	$\|f(x_*)\|$	δ	$f(x)$	IT	NFE	$\|f(x_*)\|$	δ
$f_1, x_0 = 1.8$					$f_5, x_0 = 2.5$				
NM	7	14	1.56e−93	1.39e−47	NM	7	14	5.18e−99	3.70e−50
CM	4	16	0	2.25e−154	CM	4	16	2.00e−749	6.32e−165
KM	3	12	7.78e−234	5.50e−34	KM	3	12	1.94e−251	1.37e−36
K-T(1)	3	12	7.33e−309	2.93e−39	K-T(1)	3	12	3.09e−344	9.14e−44
K-T(2)	3	12	1.23e−327	1.42e−41	K-T(2)	3	12	3.56e−350	1.69e−44
BM	3	12	4.70e−426	9.11e−54	BM	3	12	7.11e−415	1.67e−52
PM1	3	12	4.50e−502	2.46e−63	PM1	3	12	6.40e−415	1.48e−52
PM2	3	12	1.93e−434	8.38e−55	PM2	3	12	1.39e−421	2.49e−53
PM3	3	12	1.94e−434	8.38e−55	PM3	3	12	1.43e−421	2.50e−53
PM4	3	12	1.94e−434	8.38e−55	PM4	3	12	1.43e−421	2.50e−53
PM5	3	12	6.72e−491	7.13e−62	PM5	3	12	9.03e−429	3.06e−55
$f_2, x_0 = 1.6$					$f_6, x_0 = 3.2$				
NM	7	14	7.82e−112	2.00e−56	NM	10	20	9.57e−107	1.06e−54
CM	4	16	0	3.04e−178	CM	5	20	0	3.04e−164
KM	3	12	6.26e−271	3.33e−39	KM	4	16	2.11e−292	3.35e−43
K-T(1)	3	12	4.73e−392	1.06e−49	K-T(1)	5	20	0	93.11e−208
K-T(2)	3	12	6.49e−389	2.57e−49	K-T(2)	4	16	1.99e−354	7.59e−46
BM	3	12	5.22e−484	4.68e−61	BM	4	16	4.04e−257	1.10e−33
PM1	3	12	1.86e−492	3.25e−62	PM1	4	16	2.36e−489	1.09e−62
PM2	3	12	8.78e−491	6.84e−62	PM2	4	16	9.05e−301	3.93e−39
PM3	3	12	8.89e−491	6.85e−62	PM3	4	16	1.20e−299	5.43e−39
PM4	3	12	8.90e−491	6.86e−62	PM4	4	16	1.52e−299	5.59e−39
PM5	3	12	2.88e−477	3.25e−60	PM5	4	16	2.07e−570	7.56e−73
$f_3, x_0 = 1.5$					$f_7, x_0 = -0.7$				
NM	7	14	5.61e−108	1.46e−54	NM	6	12	4.52e−73	6.72e−37
CM	4	16	0	2.83e−165	CM	3	12	1.16e−201	4.50e−34
KM	3	12	7.10e−250	1.93e−36	KM	3	12	1.39e−284	3.90e−41
K-T(1)	3	12	3.56e−372	2.67e−47	K-T(1)	3	12	1.68e−439	2.18e−55
K-T(2)	3	12	9.11e−369	7.03e−47	K-T(2)	3	12	3.04e−444	5.73e−56
BM	3	12	3.26e−434	6.96e−55	BM	3	12	1.21e−412	5.34e−52
PM1	3	12	1.77e−431	1.12e−54	PM1	3	12	6.77e−417	1.50e−52
PM2	3	12	1.45e−441	8.61e−56	PM2	3	12	1.35e−412	5.38e−52
PM3	3	12	1.83e−441	8.86e−56	PM3	3	12	2.72e−412	5.87e−52
PM4	3	12	1.87e−441	8.89e−56	PM4	3	12	2.92e−412	5.92e−52
PM5	3	12	9.13e−435	5.93e−55	PM5	3	12	8.09e−412	6.72e−52
$f_4, x_0 = 0$					$f_8, x_0 = 2.1$				
NM	7	14	9.53e−73	9.13e−37	NM	7	14	5.30e−60	5.10e−33
CM	4	16	0	1.89e−131	CM	4	16	3.60e−645	6.52e−109
KM	4	16	0	6.42e−204	KM	4	16	0	3.45e−165
K-T(1)	3	12	1.31e−260	3.64e−33	K-T(1)	4	16	0	9.44e−173
K-T(2)	3	12	6.12e−263	1.88e−33	K-T(2)	4	16	1.65e−529	7.27e−221
BM	3	12	8.35e−264	1.51e−33	BM	3	12	1.42e−262	3.20e−34
PM1	3	12	5.98e−321	1.11e−40	PM1	3	12	1.03e−277	3.84e−36
PM2	3	12	1.78e−269	3.01e−34	PM2	3	12	1.28e−269	4.31e−35
PM3	3	12	2.54e−269	3.14e−34	PM3	3	12	1.38e−269	4.35e−35
PM4	3	12	2.63e−269	3.16e−34	PM4	3	12	1.39e−269	4.35e−35
PM5	3	12	9.90e−273	8.38e−34	PM5	3	12	7.71e−284	7.00e−37

Table 4. Numerical results for different functions.

4. CONCLUSIONS

In this paper, we suggest and analyze two new families of eighth-order iterative methods for solving nonlinear equations. Per iteration each class member requires three function and one first derivative evaluations. We observed from numerical examples that the proposed methods have at least equal performance as compared with the other methods in Table 4. Our approach can be continuously applied to develop higher order iterative methods.

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