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WEIGHTED SPACE METHOD FOR THE STABILITY OF SOME NONLINEAR EQUATIONS

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Dedicated to Professor Th. M. Rassias, on the occasion of his 60th birthday

We prove the stability of some equations of a single variable, including a nonlinear functional equation, a linear functional equation as well as a Volterra integral equation, by using the weighted space method. Our results generalize and extend some recent theorems given in this field, with simplified proofs. Several direct applications of these results are also obtained.

1. INTRODUCTION

The study of the functional equations stability originated from a question of S. M. Ulam ([**39**], 1940) in a talk at the University of Wisconsin, concerning the stability of group homomorphisms.

In 1941 D. H. Hyers [**24**] gave an affirmative answer to the question of Ulam for Cauchy functional equation in Banach spaces. The result of D. H. Hyers was generalized in 1950 by T. Aoki [**3**] for approximately additive mappings and in 1978 by Th. M. Rassias [**36**] for approximately linear mappings. G. L. Forti [**15**] extended in 1980 a part of Th. M. Rassias's result for a general class of functional equations. Independently, P. Găvruţa [**20**] obtained a generalization of Th. M. Rassias's theorem, by replacing the Cauchy differences by a control mapping φ satisfying a very simple condition of convergence. These papers were provided a lot of influence in the development of what is now known as *generalized Hyers-Ulam-Rassias stability* of the functional equations.

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Some results and informations which complete the history of the subject are described in the papers [**33**], [**5**], [**19**], [**35**], [**12**], [**21**], [**26**], [**32**], [**14**], [**17**], [**18**] and [**31**]. Also, we refer the reader to the expository papers [**16**] and [**37**] or to the books [**13**], [**25**] and [**28**].

It should be noted that almost all proofs in this topic used the direct method (of Hyers): the exact solution of the functional equation is explicitly constructed as a limit of a sequence, starting from the given approximate solution. On the other hand, in 1991 J. A. Baker [**4**] used the Banach fixed point theorem to give Hyers-Ulam stability results for a nonlinear functional equation. In 2003, V. Radu [**34**] proposed a new method, successively developed in [**7, 8, 9**], to obtain the existence of the exact solutions and the error estimations, based on the fixed point alternative. Subsequently, these results were generalized by D. Miheţ [**30**], L. Găvruţa [**22**] and L. Cădariu and V. Radu [**11**].

Our main purpose is to study the generalized stability for some nonlinear functional equations by using *the weighted space method*, considered in [**23**] for the first time. Actually, we prove in Theorem 1 the generalized Hyers-Ulam stability of the single variable functional equation

$$y(x) = F\left(x, y(x), y\left(\eta(x)\right)\right).$$

Thereafter, we show that the theorem extends some results in [**10**], [**23**] and [**4**]. As a consequence of Theorem 1, the generalized Hyers-Ulam stability for the linear equation

$$y(x) = g(x) \cdot y\left(\eta(x)\right) + h(x)$$

is highlighted in the next section. Moreover, as direct applications of our theorem for this linear equation, we obtain several results given in [**4**] and [**9**]. Notice that in all these equations y is the unknown function and the other ones are given mappings. In the last section of the paper we discuss the generalized Hyers-Ulam stability for a general class of the nonlinear Volterra integral equations in Banach spaces.

We recall that *the weighted space method* consists of the use of a classical mathematical result in a space endowed with a weighted distance. Specifically, in this paper, the *Banach's fixed point theorem* is the classical result used to obtain the stability of the equations above-mentioned.

2. THE GENERALIZED HYERS-ULAM STABILITY OF A NONLINEAR EQUATION

In this section we obtain the generalized Hyers-Ulam stability results for the nonlinear equation

$$y(x) = F\left(x, y(x), y\left(\eta(x)\right)\right) \tag{1}$$

by using the weighted space method. A function $y : S \to X$, is the unknown, S is a nonempty set, (X, d) is a complete metric space, $F : S \times X \times X \to X$ and $\eta : S \to S$

are given functions. Moreover, we prove that some results of the stability given in [**4**], [**10**] and [**23**] can be obtained directly from the following theorem:

Theorem 1. *Suppose that there exists $L \in [0,1)$ such that the mappings $\lambda, \mu : S \to [0,\infty)$ satisfy*

$$\lambda(x)\varphi(x) + \mu(x)\varphi(\eta(x)) \leq L\varphi(x),\ \ \forall x \in S, \tag{2}$$

for some given function $\varphi : S \to (0,\infty)$. Suppose also that the given mapping $F : S \times X \times X \to X$ satisfies

$$\begin{aligned} &d\left(F(x,u(x),u(\eta(x))),F(x,v(x),v(\eta(x)))\right)\\ &\leq \lambda(x)d(u(x),v(x)) + \mu(x)d(u(\eta(x)),v(\eta(x))),\end{aligned} \tag{3}$$

for all $x \in S$ and for all $u,v \in X$.

If $y : S \to X$ is a fixed mapping with the property

$$d\left(y(x),F(x,y(x),y(\eta(x)))\right) \leq \varphi(x), \forall x \in S, \tag{4}$$

then there exists a unique $y_0 : S \to X$ such that

$$y_0(x) = F(x,y_0(x),y_0(\eta(x))), \forall x \in S$$

and the inequality

$$d(y(x),y_0(x)) \leq \frac{\varphi(x)}{1-L} \tag{5}$$

holds for all $x \in S$.

Proof. Let us consider the set

$$\mathcal{Y} := \left\{ u : S \to X : \sup_{x\in S} \frac{d(u(x),y(x))}{\varphi(x)} < \infty \right\}.$$

Then $\mathcal{Y}$ is a complete metric space with the *weighted metric*

$$\rho(u,v) = \sup_{x\in S} \frac{d(u(x),v(x))}{\varphi(x)}.$$

Now, define the (nonlinear) mapping

$$(Tu)(x) := F(x,u(x),u(\eta(x))).$$

For $u,v \in \mathcal{Y}$, by using (3) and (2), we have

$$\begin{aligned}\frac{d\left((Tu)(x),(Tv)(x)\right)}{\varphi(x)} &\leq \frac{\lambda(x)d(u(x),v(x)) + \mu(x)d(u(\eta(x)),v(\eta(x)))}{\varphi(x)}\\ &= \lambda(x)\cdot\frac{d(u(x),v(x))}{\varphi(x)} + \mu(x)\cdot\frac{\varphi(\eta(x))}{\varphi(x)}\cdot\frac{d(u(\eta(x)),v(\eta(x)))}{\varphi(\eta(x))}\\ &\leq \rho(u,v)\cdot\left(\lambda(x)+\mu(x)\cdot\frac{\varphi(\eta(x))}{\varphi(x)}\right) \leq L\rho(u,v).\end{aligned}$$

On the other hand, by using (4), we obtain

$$\frac{d\left((Tu)(x),y(x)\right)}{\varphi(x)} \le \frac{d\left((Tu)(x),(Ty)(x)\right)}{\varphi(x)} + \frac{d\left((Ty)(x),y(x)\right)}{\varphi(x)} \le L\rho(u,y)+1<\infty.$$

It results that if $u \in \mathcal{Y}$, then $Tu \in \mathcal{Y}$, so $T : \mathcal{Y} \to \mathcal{Y}$ is well-defined. Also, we obtain that $\rho(Tu,Tv) \le L\rho(u,v)$, hence T is a *strictly contractive* self-mapping of $\mathcal{Y}$, with the constant $L<1$.

We can apply *the Banach's fixed point theorem* on the *weighted space* $\mathcal{Y}$ and we obtain the existence of a mapping $y_0 : S \to X$ such that:

(i) y_0 is the unique fixed point of T, that is

$$y_0(x) = F(x, y_0(x), y_0(\eta(x))), x \in S;$$

(ii) $\rho(T^n y, y_0) \xrightarrow[n\to\infty]{} 0$, which implies

$$y_0(x) = \lim_{n\to\infty} (T^n y)(x), \forall x \in S;$$

(iii) $\rho(y,y_0) \le \dfrac{1}{1-L}\rho\left(y,Ty\right)$, which implies the inequality

$$\rho(y,y_0) \le \frac{1}{1-L},$$

that is, the estimation relation (5) holds. □

It is easy to see that the above theorem extends the recent result in [**10**]. Actually, if we take in Theorem 1 a particular form of the given mapping F (here denoted by G) and if the mappings λ and μ are constants, we obtain the generalized Hyers-Ulam stability for the nonlinear equation

$$y(x) = G\big(y(x), y(\eta(x))\big).$$

Corollary 2 (Cădariu, Moslehian, Radu [**10**], Theorem 2.2)**.** *Suppose that S is a nonempty set, (X,d) is a complete metric space, $\eta : S \to S$ and $G : X\times X \to X$ are given mappings, λ, μ are nonnegative real numbers and the following condition is verified:*

$$\text{(6)} \qquad d\Big(G\big(s,u\big), G\big(t,v\big)\Big) \le \lambda d(s,t) + \mu d(u,v), \quad \forall s,t,u,v \in X.$$

Let $y : S \to X$ be a $\varphi-$solution, for some given function $\varphi : S \to (0,\infty)$, that is

$$\text{(7)} \qquad d\Big(y(x), G\big(y(x), y(\eta(x))\big)\Big) \le \varphi(x), \quad \forall x \in S.$$

Suppose also that, for some $L \in [0,1)$,

$$\text{(8)} \qquad \mu\cdot\varphi(\eta(x)) + \lambda\cdot\varphi(x) \le L\cdot\varphi(x), \quad \forall x \in S.$$

Then there exists a unique function $y_0 : S \to X$ *such that*

$$y_0(x) = G\big(y_0(x), y_0(\eta(x))\big),\ \forall x \in S$$

and

$$d\Big(y(x), y_0(x)\Big) \le \frac{\varphi(x)}{1-L}, \quad \forall x \in S. \tag{9}$$

As a direct consequence of Theorem 1, the generalized Hyers-Ulam stability result proved in [**23**] for the nonlinear equation

$$y(x) = F\big(x, y(\eta(x))\big) \tag{10}$$

can be obtained. In fact, if we consider a nonempty set S, a complete metric space (X, d), the given functions $\eta : S \to S$, $F : S \times X \to X$, $\varphi : S \to (0, \infty)$, $\lambda(x) \equiv 0$ and $\mu(x) = \dfrac{L \cdot \varphi(x)}{\varphi(\eta(x))}$ in Theorem 1, we have:

Corollary 3 (P. Găvruţa, L. Găvruţa [**23**], Theorem 2.1)**.** *Suppose that there exists* $L \in [0, 1)$ *such that*

$$\varphi(\eta(x)) \cdot d\left(F(x, u(\eta(x))), F(x, v(\eta(x)))\right) \le L \cdot \varphi(x) \cdot d(u(\eta(x)), v(\eta(x))),$$

holds for all $x \in S$ *and for all* $u, v \in X$.

If the mapping $y : S \to X$ *has the property*

$$d\left(y(x), F(x, y(\eta(x)))\right) \le \varphi(x),\ \forall x \in S,$$

then there exists a unique $y_0 : S \to X$ *such that*

$$y_0(x) = F(x, y_0(\eta(x))),\ \forall x \in S,$$

and the inequality

$$d(y(x), y_0(x)) \le \frac{\varphi(x)}{1-L}$$

holds for all $x \in S$.

It should be noted that the following Hyers-Ulam stability result [cf. (Baker [**4**], Theorem 2) or (Agarwal et al. [**1**], Theorem 13)] for the nonlinear equation (10) it is obtained directly, from the above Corollary, by taking $\varphi(x) = \delta > 0$:

Corollary 4. *Let* S *be a nonempty set and* (X, d) *be a complete metric space. Let be* $\eta : S \to S$ *and* $F : S \times X \to Y$ *some given mappings and* $0 \le L < 1$. *Suppose that*

$$d(F(x, u), F(x, v)) \le L \cdot d(u, v),\ \forall x \in S, \forall u, v \in X.$$

If $y : S \to X$ *has the property*

$$d(y(x), F\left(x, y\left(\eta(x)\right)\right)) \le \delta,\ \forall x \in S,$$

with a fixed constant $\delta > 0$, then there exists a unique mapping $y_0 : S \to X$ which satisfies both the equation

$$y(x) = F\left(x, y\left(\eta(x)\right)\right), \forall x \in S$$

and the inequality

$$d(y(x), y_0(x)) \leq \frac{\delta}{1-L}, \forall x \in S.$$

3. THE GENERALIZED HYERS-ULAM STABILITY OF A LINEAR FUNCTIONAL EQUATION

In this section we emphasize the significance of Theorem 1. In fact, if

$$F\left(x, y(x), y\left(\eta(x)\right)\right) = g(x) \cdot y\left(\eta(x)\right) + h(x),$$

the equation (1) becomes

$$y(x) = g(x) \cdot y\left(\eta(x)\right) + h(x), \tag{11}$$

where g, η, h are given mappings and y is the unknown function. The above equation is called *linear functional equation* and was intensively investigated by Kuczma, Choczewski and Ger [**29**]. They obtained some results concerning monotonic, regular and convex solutions of (11).

In what follows we prove a generalized Hyers-Ulam stability result for the equation (11), as a particular case of Theorem 1.

Let us consider a nonempty set S, a real (or complex) Banach space X, endowed with norm the $||\cdot||$ and the given functions $\eta : S \to S$, $g : S \to \mathbb{R}$ (or $\mathbb{C}$) and $h : S \to X$.

Theorem 5. *Suppose that there exists $L \in [0, 1)$ such that the functions $\lambda, \mu : S \to [0,\infty)$ satisfy*

$$\lambda(x)\varphi(x) + \mu(x)\varphi(\eta(x)) \leq L\varphi(x),\ \forall x \in S, \tag{12}$$

for some fixed mapping $\varphi : S \to (0,\infty)$. Suppose also that the given function $F : S \times X \times X \to X$ verifies

$$(|g(x)| - \mu(x)) \cdot ||u(\eta(x))) - v(\eta(x))|| \leq \lambda(x)||u(x) - v(x)||, \tag{13}$$

for all $x \in S$ and for all u, v from S into X.

If $y : S \to X$ has the property

$$||y(x) - g(x)y(\eta(x)) - h(x)|| \leq \varphi(x),\ \forall x \in S, \tag{14}$$

then there exists a unique mapping $y_0 : S \to X$ defined by

$$y_0(x) = h(x) + \lim_{n\to\infty}\left(y(\eta^n(x)) \cdot \prod_{i=0}^{n-1} g\left(\eta^i(x)\right) + \sum_{j=0}^{n-2}\left(h\left(\eta^{j+1}(x)\right) \cdot \prod_{i=0}^{j} g\left(\eta^i(x)\right)\right)\right),$$

for all $x \in S$, which satisfies the equation

$$y(x) = g(x) \cdot y(\eta(x)) + h(x), \forall x \in S$$

and the inequality

$$||y(x) - y_0(x)|| \leq \frac{\varphi(x)}{1-L}, \forall x \in S. \tag{15}$$

Proof. We consider in Theorem 1 the metric d on X, given by $d(u,v) = ||u - v||$ and the function $F(x, y(x), y(\eta(x))) := g(x) \cdot y(\eta(x)) + h(x), \forall x \in S$, with g, η, h as in the hypotheses of Theorem 5. Applying Theorem 1, there exists a unique mapping y_0 which satisfies the equation (11) and the estimation (15). Moreover,

$$y_0(x) = \lim_{n\to\infty} (T^n y)(x), \forall x \in S,$$

where

$$\begin{aligned}(T^n y)(x) &= g(x) \cdot \left(T^{n-1} y\right)(\eta(x)) + h(x) \\ &= g(x) \cdot g(\eta(x)) \cdot (T^{n-2} y)(\eta^2(x)) + g(x) \cdot h(\eta(x)) + h(x), \forall x \in S,\end{aligned}$$

whence, for all $x \in S$,

$$(T^n y)(x) := h(x) + y(\eta^n(x)) \cdot \prod_{i=0}^{n-1} g\left(\eta^i(x)\right) + \sum_{j=0}^{n-2} \left(h\left(\eta^{j+1}(x)\right) \cdot \prod_{i=0}^{j} g\left(\eta^i(x)\right) \right). \quad \square$$

It is easy to see that we can obtain the generalized stability result given in [**9**] (see also [**38**]) for the linear functional equation (11), by taking in Theorem 5 $\lambda(x) \equiv 0$ and $\mu(x) = \dfrac{L \cdot \varphi(x)}{\varphi(\eta(x))}$:

Corollary 6 (Cădariu, Radu [**9**], Theorem 5.1)**.** *Let us consider a nonempty set S and a real (or complex) Banach space X. Suppose that $\eta : S \to S, h : S \to X$ and $g : S \to \mathbb{R}$ (or $\mathbb{C}$) are given mappings. If $y : S \to X$ satisfies*

$$||y(x) - g(x) y(\eta(x)) - h(x)|| \leq \varphi(x), \ \forall x \in S,$$

with some fixed mapping $\varphi : S \to (0, \infty)$ and there exists $L \in [0,1)$ such that

$$|g(x)| \cdot \varphi(\eta(x)) \leq L\varphi(x), \forall x \in S,$$

then there exists a unique mapping $y_0 : S \to X$

$$y_0(x) = h(x) + \lim_{n\to\infty} \left(y(\eta^n(x)) \cdot \prod_{i=0}^{n-1} g\left(\eta^i(x)\right) + \sum_{j=0}^{n-2} \left(h\left(\eta^{j+1}(x)\right) \cdot \prod_{i=0}^{j} g\left(\eta^i(x)\right) \right) \right),$$

for all $x \in S$, which satisfies both
the equation

$$y_0(x) = g(x) \cdot y_0(\eta(x)) + h(x), \forall x \in S$$

and the inequality

$$||y(x) - y_0(x)|| \leq \frac{\varphi(x)}{1-L}, \forall x \in S.$$

If $\varphi(x) = \delta > 0$ in the above Corollary, then we obtain the Hyers-Ulam stability result of Baker ([**4**], Theorem 3) (see also (Agarwal et al. [**1**], Theorem 7)) for the linear equation (11):

Corollary 7. *Let us consider a nonempty set S and a real (or complex) Banach space X. Suppose that $\eta : S \to X$, $g : S \to \mathbb{R}$ (or $\mathbb{C}$) and $h : S \to X$ are given mappings. If $y : S \to S$ satisfies*

$$||y(x) - g(x)y(\eta(x)) - h(x)|| \leq \delta,\ \forall x \in S,$$

with a fixed constant $\delta > 0$ and there exists $L \in [0, 1)$ such that

$$|g(x)| \leq L, \forall x \in S,$$

then there exists a unique mapping $y_0 : S \to X$ defined by

$$y_0(x) = h(x) + \lim_{n\to\infty}\left(f(\eta^n(x)) \cdot \prod_{i=0}^{n-1} g\left(\eta^i(x)\right) + \sum_{j=0}^{n-2}\left(h\left(\eta^{j+1}(x)\right) \cdot \prod_{i=0}^{j} g\left(\eta^i(x)\right)\right)\right),$$

for all $x \in S$, which satisfies

$$y_0(x) = g(x) \cdot y_0(\eta(x)) + h(x), \forall x \in S$$

and

$$||f(x) - y_0(x)|| \leq \frac{\delta}{1-L}, \forall x \in S.$$

4. THE STABILITY OF THE NONLINEAR VOLTERRA INTEGRAL EQUATIONS

In this section we obtain the generalized Hyers-Ulam stability for a general class of the nonlinear Volterra integral equations in Banach spaces, by using the weighted space method. Also, we prove that some stability theorems given in [**2**] and [**23**] for these integral equations are consequences of our result.

Let us consider a Banach space X over the (real or complex) field $\mathbb{K}$, an interval $I = [a, b]$ $(a < b)$ and the continuous given functions $L : I \times I \to [0, \infty)$ and $\varphi : I \to (0, \infty)$. We denote by $\mathcal{C}(I, X) := \{f : I \to X, f \text{ is continuous}\}$ and by $||\cdot||$ the norm on the Banach space X.

The result of stability for the nonlinear Volterra integral equation

$$y(x) = h(x) + \lambda \int_a^x G(x, t, y(t))\mathrm{d}t, \forall x \in I, \tag{16}$$

($y : I \to X$ is the unknown function, $h : I \to X$ and $G : I \times I \times X \to X$ are continuous given mappings and λ is a fixed nonzero scalar in $\mathbb{K}$), can be read as follows:

Theorem 8. *Suppose that there exists a positive nonzero constant* α *such that*

$$\int_a^x L(x,t)\varphi(t)\mathrm{d}t \le \alpha\varphi(x),\ \ \forall x \in I. \tag{17}$$

Suppose also that $G : I \times I \times X \to X$ *is a continuous function, which satisfies*

$$||G(x,t,u(t))-G(x,t,v(t))|| \le L(x,t)\cdot||u(t)-v(t)||, \forall x,t \in I, \forall u,v \in \mathcal{C}(I,X). \tag{18}$$

If the continuous mapping $y : I \to X$ *has the property*

$$\left\| y(x) - h(x) - \lambda \int_a^x G(x,t,y(t))\mathrm{d}t \right\| \le \varphi(x), \forall x \in I \tag{19}$$

and if

$$|\lambda| < \frac{1}{\alpha}$$

then there exists a unique $y_0 \in \mathcal{C}(I,X)$ *such that*

$$y_0(x) = h(x) + \lambda \int_a^x G(x,t,y_0(t))\mathrm{d}t,\ \ \forall x \in I$$

and the inequality

$$||y(x) - y_0(x)|| \le \frac{\varphi(x)}{1 - |\lambda| \cdot \alpha},\ \ \forall x \in I \tag{20}$$

holds.

Proof. We apply the *Fixed point theorem of Banach* on the space $Y := \mathcal{C}(I,X)$. In fact, if we consider the *weighted metric*

$$d(u,v) = \sup_{x\in I} \frac{||u(x) - v(x)||}{\varphi(x)},$$

then (Y,d) is a complete metric space.

Now, define the operator $T : Y \to Y$

$$(Tu)(x) := h(x) + \lambda \int_a^x G(x,t,u(t))\mathrm{d}t.$$

From the relations (17) and (18) it results that T *is strictly contractive* operator on Y. Indeed, for any $u, v \in Y$ we have:

$$d(Tu,Tv) = \sup_{x\in I} \frac{|\lambda| \cdot \left\| \int\limits_a^x \left(G(x,t,u(t)) - G(x,t,v(t))\right)\mathrm{d}t \right\|}{\varphi(x)}$$

$$\leq |\lambda| \sup_{x\in I} \frac{\int_a^x L(x,t)\cdot ||u(t)-v(t)||\mathrm{d}t}{\varphi(x)}$$
$$= |\lambda| \sup_{x\in I} \frac{\int_a^x L(x,t)\cdot \varphi(t)\cdot \frac{||u(t)-v(t)||}{\varphi(t)}\mathrm{d}t}{\varphi(x)}$$
$$\leq |\lambda|\cdot \sup_{t\in I} \frac{||u(t)-v(t)||}{\varphi(t)}\cdot \sup_{x\in I} \frac{\int_a^x L(x,t)\cdot \varphi(t)\mathrm{d}t}{\varphi(x)} \leq \alpha\cdot|\lambda|\cdot d(u,v).$$

Therefore T is a *strictly contractive* self-mapping of Y, with Lipschitz constant $\alpha|\lambda| < 1$.

On the other hand, by using (19), we have that $d(y, Ty) < 1$.

We can apply the *Banach's fixed point theorem* on the complete metric space Y and we obtain the existence of a mapping $y_0 \in Y$ such that:

(i) y_0 is the unique fixed point of T, that is

$$y_0(x) = h(x) + \lambda \int_a^x G(x,t,y_0(t))\mathrm{d}t, \ \forall x \in I;$$

(ii) $d(T^n y, y_0) \xrightarrow[n\to\infty]{} 0$, which implies

$$y_0(x) = \lim_{n\to\infty} (T^n y)(x), \forall x \in I;$$

(iii) $d(y, y_0) \leq \frac{1}{1-L} d\left(y, Ty\right)$, which implies the inequality

$$d(y, y_0) \leq \frac{1}{1-|\lambda|\cdot\alpha},$$

hence the estimation relation (20) is true. □

As a direct consequence of Theorem 8, the Hyers-Ulam stability result recent proved in [**2**] by M. Akkouchi for the integral equation (16) it is obtained. Actually, we consider a Banach space X over the (real or complex) field $\mathbb{K}$, $I = [a, b]$ an interval $(a < b)$, the continuous given functions $h : I \to X$ and $\varphi : I \to (0, \infty)$ and a fixed scalar λ in $\mathbb{K}$. We also denote by $\mathcal{C}(I, X) := \{f : I \to X, f \text{ is continuous}\}$. If we take in Theorem 8 $L(x,t) \equiv L$ and $\alpha = K \cdot L$ (K, L are fixed nonzero positive constants), we obtain:

Corollary 9 (Akkouchi [**2**], Theorem 4.1)**.** *Suppose that there exist positive constants* K, L, *with* $0 < |\lambda|KL < 1$ *such that*

$$\int_a^x \varphi(t)\mathrm{d}t \leq K\varphi(x), \ \forall x \in I.$$

Suppose also that $G : I \times I \times X \to X$ *is a continuous function, which satisfies*

$$||G(x,t,u(t)) - G(x,t,v(t))|| \leq L \cdot ||u(t) - v(t)||, \forall x, t \in I, \forall u, v \in \mathcal{C}(I, X).$$

If the continuous mapping $y : I \to X$ *has the property*

$$\left\| y(x) - h(x) - \lambda \int_a^x G(x,t,y(t))\mathrm{d}t \right\| \leq \varphi(x), \forall x \in I,$$

then there exists a unique $y_0 \in \mathcal{C}(I, X)$ *such that*

$$y_0(x) = h(x) + \lambda \int_a^x G(x,t,y_0(t))\mathrm{d}t,\ \forall x \in I$$

and the inequality

$$||y(x) - y_0(x)|| \leq \frac{\varphi(x)}{1 - |\lambda| \cdot K \cdot L},\ \forall x \in I$$

holds.

Also, it is necessary to mention that our Theorem 8 extends another recent result in [**23**], where Hyers-Ulam stability for the following nonlinear Volterra integral equation

$$y(x) = h(x) + \int_a^x f(t,y(t))\mathrm{d}t,\ \forall x \in I \tag{21}$$

is proved ($y : I \to \mathbb{C}$ is the unknown function, $h : I \to \mathbb{C}$ and $f : I \times \mathbb{C} \to \mathbb{C}$ are given mappings). Indeed, if we denote by $\mathcal{C}(I)$ the space of all complex-valued continuous functions on I and if we take in Theorem 8

$$G(x,t,y(t)) := \frac{f(t,y(t))}{\lambda}\ , \quad L(x,t) := |\lambda| \cdot L(t) \ \text{ and } \ \alpha = \frac{\beta}{|\lambda|},$$

we obtain the generalized Hyers-Ulam stability for the nonlinear integral equation (21):

Corollary 10 (P. Găvruţa, L. Găvruţa [**23**], Theorem 3.1)**.** *Assume that there exist a positive constant* $\beta \in (0,1)$ *such that the continuous mapping* $\varphi : I \to (0,\infty)$ *satisfies*

$$\int_a^x L(t)\varphi(t)\mathrm{d}t \leq \beta\varphi(x),\ \forall x \in I.$$

Suppose also that $L : I \to [0,\infty)$ *and* $f : I \times X \to X$ *are continuous functions, which satisfy*

$$|f(t,u(t)) - f(t,v(t))| \leq L(t) \cdot |u(t) - v(t)|, \forall t \in I, \forall u, v \in \mathcal{C}(I).$$

If the continuous mapping $y : I \to \mathbb{C}$ *has the property*

$$\left| y(x) - h(x) - \int_a^x f(t, y(t)) \mathrm{d}t \right| \le \varphi(x), \forall x \in I,$$

then there exists a unique $y_0 \in \mathcal{C}(I)$ *such that*

$$y_0(x) = h(x) + \int_a^x f(t, y_0(t)) \mathrm{d}t, \ \forall x \in I$$

and

$$|y(x) - y_0(x)| \le \frac{\varphi(x)}{1-\beta}, \ \forall x \in I.$$

REMARK 11. Some particular cases of the above Corollary was obtained in [**27**]. See also [**6**].

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