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EDGE BIPARTITENESS AND SIGNLESS LAPLACIAN SPREAD OF GRAPHS

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Let G be a connected graph, and let $\epsilon_b(G)$ and $\mathsf{S}_Q(G)$ be the edge bipartiteness and the signless Laplacian spread of G, respectively. We establish some important relationships between $\epsilon_b(G)$ and $\mathsf{S}_Q(G)$, and prove $\mathsf{S}_Q(G) \geq 2\Big(1+\cos\dfrac{\pi}{n}\Big)$, with equality if and only if $G = P_n$ or $G = C_n$ in case of odd n. In addition, we show that if $G \neq P_n$ or $G \neq C_{2k+1}$, then $\mathsf{S}_Q(G) \geq 4$, with equality if and only if G is one of the following graphs: $K_{1,3}$, K_4, two triangles connected by an edge, and C_n for even n. As a consequence, we prove a conjecture of Cvetković, Rowlinson and Simić on minimal signless Laplacian spread [Eigenvalue bounds for the signless Laplacian, Publ. Inst. Math. (Beograd), **81** (95) (2007), 11–27].

1. INTRODUCTION

Let G be a simple graph of order n with the vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = E(G)$. The *adjacency matrix* of the graph G is defined to be a matrix $A(G) = [a_{ij}]$ of order n, where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. Let $D(G) = \text{diag}\{d_{v_1}, \ldots, d_{v_n}\}$, be a diagonal matrix, where d_v denotes the degree of the vertex v in the graph G. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, and the matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* (or *Q-matrix*) of G. The eigenvalues of $A(G), L(G)$ and $Q(G)$ are denoted by $\lambda_i^A(G), \lambda_i^L(G)$ and $\lambda_i^Q(G)$ $(i = 1, 2, \ldots, n)$ respectively, all written in non-increasing order with respect to the natural ordering on the reals.

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The *spread* of a graph G of order n (now called *adjacency spread* of G to avoid confusion in the following discussion) is defined in [**16**] as

$$\mathsf{S}_A(G) = \lambda_1^A(G) - \lambda_n^A(G).$$

The *Laplacian spread* of G is defined in [**14**] as

$$\mathsf{S}_L(G) = \lambda_1^L(G) - \lambda_{n-1}^L(G).$$

The *signless Laplacian spread* of G is defined in [**22**] (also called Q-spread in [**24**]) as

$$\mathsf{S}_Q(G) = \lambda_1^Q(G) - \lambda_n^Q(G).$$

Note that in the definition of Laplacian spread, as $\lambda_n^L(G)$ always equals zero, the second smallest eigenvalue $\lambda_{n-1}^L(G)$ is used instead; and $\lambda_{n-1}^L(G) > 0$ if and only if G is connected. By Lemma 2.3 in next section, if G is connected, then $\lambda_n^Q(G) = 0$ if and only if G is bipartite, and hence $\mathsf{S}_Q(G) > \mathsf{S}_L(G)$ if G is connected and bipartite. However, there also exists a non-bipartite connected graph G such that $\mathsf{S}_Q(G) < \mathsf{S}_L(G)$; see [**22**]. In addition, if G is regular, then $\mathsf{S}_A(G) = \mathsf{S}_Q(G)$.

The spread of a graph has received much attention recently. Petrović [**25**] determines all minimal graphs whose adjacency spread does not exceed four. Gregory et al. [**16**] present some lower and upper bounds for the spread of a graph. They show that the path is the unique graph with minimal adjacency spread among all connected graphs of given order. However the graph(s) with maximal adjacency spread is still unknown, and some conjectures are presented in their paper. Lower and upper bounds for the adjacency spread of graphs can also be found in [**20**]. The maximum adjacency spread of unicyclic or bicyclic graphs can be found in [**18**], [**13**] and [**27**]. Further discussion on the maximum adjacency spread of graphs can be found in [**26**].

For the Laplacian spread of graphs, Fan et al. [**14**] prove that among all trees of fixed order, the star is the unique one with maximum Laplacian spread, and the path is the unique one with the minimum Laplacian spread. The maximum Laplacian spread of unicyclic graphs, bicyclic graphs, tricyclic graphs, quasi-trees and cacti are discussed in [**1**], [**4**], [**11**], [**23**], and [**30**], respectively. The ordering of Laplacian spread of trees or unicyclic graphs is discussed in [**21**]. The graph with minimum Laplacian spread among all unicyclic graphs with fixed order is determined in [**31**], and is a cycle.

For the signless Laplacian spread, the unique graph with maximum signless Laplacian spread is determined in [**24**] and [**22**], which is a union of a complete graph and an isolated vertex. Among connected graphs, the one with maximum or minimum signless Laplacian spread is still unknown. Recent work has just appeared on bounding the signless Laplacian spread and investigating graphs with minimum signless Laplacian spread in [**8**]. Liu and Liu [**22**] determined the unicyclic graph with maximum signless Laplacian spread. Cvetković, Rowlinson and Simić have posed the following conjecture on the minimum signless Laplacian spread:

Conjecture 1.1. [**5**] *Among all connected graphs of order n, $\mathsf{S}_Q(G)$ is minimized by the path P_n, and in the case that n is odd, by the cycle C_n.*

To prove this conjecture, we use a very important notion called the edge bipartiteness of a graph, which was introduced in [**9**], and used to measure how close a graph is to being bipartite. Furthermore a number of relationships were established between the edge bipartiteness of a graph and the least signless Laplacian eigenvalue (see [**9**]).

Definition 1.2. [**9**] *The edge bipartiteness of a graph G, denoted by $\epsilon_b(G)$, is the minimum number of edges of G whose deletion yields a bipartite graph.*

In this work we show that if $\epsilon_b(G) \leq 1$, then $\mathsf{S}_Q(G) \geq 2\Big(1 + \cos\dfrac{\pi}{n}\Big)$, with equality if and only if $G = P_n$ or $G = C_n$ in case of odd n; and if $\epsilon_b(G) \geq 2$, then $\mathsf{S}_Q(G) \geq 4$, which implies the validity of Conjecture 1.1. Furthermore, if $G \neq P_n$ and $G \neq C_n$ in case of odd n, then $\mathsf{S}_Q(G) \geq 4$, with equality if and only if G is one of the following graphs: $K_{1,3}$, K_4, two triangles connected by an edge, and an even cycle.

2. EDGE BIPARTITENESS AND SIGNLESS LAPLACIAN SPREAD OF GRAPHS

We first introduce some terminology on mixed graphs. A *mixed graph G* is a graph containing oriented edges and unoriented edges. The Laplacian matrix associated with a mixed graph G is defined as $D(G) + \bar{A}(G)$, where $\bar{A}(G)_{uv} = 1$ if uv is an unoriented edge, $\bar{A}(G)_{uv} = -1$ if uv is an oriented edge, and $\bar{A}(G)_{uv} = 0$ otherwise; see [**2**] for more details. The notion of a Laplacian matrix of a mixed graph generalizes both the classical Laplacian matrix and signless Laplacian matrix if all edges are oriented or unoriented, respectively.

In [**28**] a parameter called the *edge singularity* of a mixed graph G was defined as the minimum number of edges of G such that deletion of these edges produces components whose Laplacian matrices are all singular. Using Lemma 2.3, within the setting of the signless Laplacian of graphs, the edge singularity is exactly the edge bipartiteness. Thus we may recast some results from [**28**] as in Lemma 2.4, Lemma 2.6(5) and Lemma 2.12 all stated below.

Lemma 2.3. [**6**] *Let G be a connected graph of order n. Then $\lambda_n^Q(G) = 0$ if and only if G is bipartite, and in this case, $Q(G)$ is similar to $L(G)$ by a signature matrix (a diagonal matrix with ± 1 on its diagonal).*

Lemma 2.4. [**28**] *Let G be a connected graph of order n with m edges. Then $\epsilon_b(G) \leq m - n + 1$, with equality if and only if all cycles of G are odd, and any two cycles share no common edges.*

By Lemma 2.4, any connected graph G of order n contains at least $n-1+\epsilon_b(G)$ edges; and if it contains exactly $n - 1 + \epsilon_b(G)$ edges, then it is obtained from a tree of order n by adding $\epsilon_b(G)$ edges to produce $\epsilon_b(G)$ odd cycles, and any two cycles share no common edges. We also have the following fact.

Corollary 2.5. *Let G be a connected graph of order n with m edges. Then $m \geq n + \epsilon_b(G)$ if and only if G contains at least one even cycle.*

We now list some bounds for the extreme signless Laplacian eigenvalues of graphs. The first result can be obtained by the fact $\lambda_1^Q(G) \geq \frac{1}{n}\mathbf{1}_n^\top Q(G)\mathbf{1}_n$ with equality if and only if $\mathbf{1}_n$ is a Perron vector of $Q(G)$, where $\mathbf{1}_n \in \mathbb{R}^n$ consisting of all ones.

Lemma 2.6. *Let G be a graph of order n with m edges, and let $\Delta(G), \delta(G)$ be the maximum degree and the minimum degree of the vertices of G. Then*
(1) $\lambda_1^Q(G) \geq \frac{4m}{n}$, *with equality if and only if G is regular.*
(2) [**12**] $\lambda_1^Q(G) \geq \Delta(G) + 1$, *with equality in the connected case if and only if G is a star.*
(3) [**7**] $\lambda_n^Q(G) < \delta(G)$.
(4) [**19**] $\lambda_n^Q(G) \leq \min_{uv \in E(G)} \frac{d_u + d_v - 2}{2}$.
(5) [**28**] $\lambda_n^Q(G) \leq \frac{4\epsilon_b(G)}{n}$.

Let G be a graph, and let $U \subset V(G)$. Denote by $G[U]$ the subgraph of G induced by the vertices of U, and by $\mathrm{diam}(G)$ the diameter of G. Denote by $P_n, C_n, K_{1,n-1}, K_n$ the path, the cycle, the star and the complete graph all of order n, respectively. We let $\mathbf{I}_n$ be the identity matrix of order n and $|S|$ be the cardinality of a finite set S.

The eigen-equation $Q(G)X = \lambda X$ can be interpreted as

$$(\lambda - d_u)X_u = \sum_{v \in N(u)} X_v, \text{ for each vertex } u \in V(G), \tag{2.1}$$

where X_v denotes the entry of X corresponding the vertex v, and $N(u)$ denotes the neighborhood of u. Similarly, $L(G)X = \lambda X$ can be interpreted as

$$(d_u - \lambda)X_u = \sum_{v \in N(u)} X_v, \text{ for each vertex } u \in V(G). \tag{2.2}$$

Corollary 2.7. *If G is a connected graph of order n with $m \geq n + \epsilon_b(G)$ edges, or equivalently G contains an even cycle, then*

$$\mathsf{S}_Q(G) \geq 4,$$

with equality if and only if $G = K_4$ or $G = C_n$ in case of even n.

Proof. By Lemma 2.6 (1) and (5),

$$\mathsf{S}_Q(G) \geq \frac{4(n + \epsilon_b(G))}{n} - \frac{4\epsilon_b(G)}{n} = 4. \tag{2.3}$$

To prove the equality case, we restate the proof of the inequality $\lambda_n^Q(G) \leq \frac{4\epsilon_b(G)}{n}$ in [**28**]. Let $F \subset E(G)$ of size $\epsilon_b(G)$ such that $G-F$ is bipartite. Then $G-F$ is connected; otherwise we can delete fewer edges from G to yield a bipartite graph. Let (U,W) be the bipartition of $V(G-F)$ such that each edge of $G-F$ links one vertex in U and one in W. Observe that the edges F lie within U or W. Define a vector X such that $X_v = 1$ if $v \in U$ and $X_v = -1$ otherwise. Then

$$\lambda_n^Q(G) \leq \|X\|^{-2} X^\top Q(G) X = \frac{1}{n} \sum_{uv \in E(G)} (X_u + X_v)^2 = \frac{4\epsilon_b(G)}{n},$$

with equality if and only if X is an eigenvector corresponding to $\lambda_n^Q(G)$.

Now if the equality (2.3) holds, then $\lambda_1^Q(G) = \frac{4m}{n} = \frac{4(n+\epsilon_b(G))}{n}$, which implies G contains exactly $n+\epsilon_b(G)$ edges and is regular of some degree d by Lemma 2.6(1). Also $\lambda_n^Q(G) = \frac{4\epsilon_b(G)}{n}$, and X is a corresponding eigenvector defined as above. By the eigen-equation as in (2.1), $Q(G)X = \lambda_n^Q(G)X$, letting d'_u be the degree of a vertex u in the graph $G[U]$,

$$(\lambda_n^Q(G) - d) \cdot 1 = d'_u \cdot 1 + (d - d'_u) \cdot (-1).$$

Hence $\lambda_n^Q(G) = 2d'_u$, which implies $G[U]$ is regular of some degree d'. Similarly $G[W]$ is regular of degree d'. So $G-F$ is also regular as G is regular. Since $G-F$ is connected, bipartite, and contains exactly n edges, $G-F$ is an even cycle. So $d' = d-2$, and $\lambda_n^Q(G) = 2d' = 2(d-2)$. However by Lemma 2.6(3), $\lambda_n^Q(G) < d$. So $d \leq 3$ or $d' \leq 1$.

If $d=2$, then G is exactly an even cycle. Assume $d=3$. Then $\lambda_n^Q(G) = 2$. Noting that $Q(G) = 3\mathbf{I}_n + A(G)$, so $\lambda_n^A(G) = -1$, which implies G is a complete graph K_n or a disjoint union of two or more complete graphs (see [**15**]). As G is connected, $G = K_n$, and $n=4$. The result then follows.

Lemma 2.8. *If G is a connected graph of order n with $m = n-1+\epsilon_b(G)$ edges* (*or equivalently G contains only odd cycles*), *and G contains a vertex of degree at least* 4 *or a cycle whose vertices all have degree* 3, *then* $\mathsf{S}_Q(G) > 4$.

Proof. By Lemma 2.6 (2), if the maximum degree $\Delta(G) \geq 4$, then $\lambda_1^Q(G) \geq 5$. If G contains a cycle C whose vertices all have degree 3, considering the proper principal submatrix Q' of $Q(G)$ indexed by the vertices of C, the spectral radius of Q' is exactly 5 as it has constant row sum 5. So $\lambda_1^Q(G) > 5$ by the Perron-Frobenius theory on nonnegative matrices (see [**17**]).

In addition, as G contains a pendent vertex (of degree 1) or a pendant cycle (with exactly one vertex of degree 3), by Lemma 2.6 (3) or (4), $\lambda_n^Q(G) \leq 1$. So $\mathsf{S}_Q(G) \geq 5-1 = 4$. If $\mathsf{S}_Q(G) = 4$, then $\lambda_1^Q(G) = 5$, $\Delta(G) = 4$, and G is a star of order 5 by Lemma 2.6 (2). However in this case $\lambda_n^Q(G) = 0$ by Lemma 2.3, and hence $\mathsf{S}_Q(G) = 5 > 4$, a contradiction.

Lemma 2.9. *Let G be a connected graph with a pendant vertex v. Then $\mathsf{S}_Q(G) > \mathsf{S}_Q(G-v)$.*

Proof. Let e be the edge between v and $G-v$ in the graph G. Then the signless Laplacian eigenvalues of $G-e$ interlace those of G (see e.g. [**10**]), and hence

$$\lambda_n^Q(G-e) \le \lambda_n^Q(G) \le \lambda_{n-1}^Q(G-e). \tag{2.4}$$

Observing that $G-e$ is union of v and $G-v$, so $\lambda_n^Q(G-e) = 0$ and $\lambda_{n-1}^Q(G-e) = \lambda_{n-1}^Q(G-v)$ (the least eigenvalue of $Q(G-v)$), which implies $\lambda_n^Q(G) \le \lambda_{n-1}^Q(G-v)$. By the Perron-Frobenius theory we have $\lambda_1^Q(G) > \lambda_1^Q(G-v)$, which completes the proof.

Lemma 2.10. *Let G be a connected graph of order n with a cut edge e, and let G_1, G_2 be the components of $G-e$. Then $\mathsf{S}_Q(G) > \min\{\mathsf{S}_Q(G_1), \mathsf{S}_Q(G_2)\}$.*

Proof. Assume G_1 contains m vertices, and without loss of generality, $\lambda_m^Q(G_1) \le \lambda_{n-m}^Q(G_2)$ (both are least eigenvalues of $Q(G_1)$ and $Q(G_2)$ respectively). Thus $\lambda_n^Q(G-e) = \lambda_m^Q(G_1)$ and $\lambda_{n-1}^Q(G-e) \le \lambda_{n-m}^Q(G_2)$. So, by (2.4) and a similar discussion as in the proof of Lemma 2.9, we have $\mathsf{S}_Q(G) > \mathsf{S}_Q(G_2)$. □

At the end of this section, we will establish a relationship between the signless Laplacian spectrum of a graph G and the Laplacian spectrum of a graph ${}^dG$, and prove $\mathsf{S}_Q(G) = \mathsf{S}_L({}^dG)$ when $\epsilon_b(G) = 1$, where ${}^dG$ is the double graph of G defined in [**9**].

Suppose the vertices of G are partitioned into two sets S and T, where we allow the possibility that $S = V(G)$ and $T = \emptyset$. Let G' be a copy of G, whose vertices are labeled as u', corresponding to the vertex $u \in V(G)$. For each edge uw within S or T (if it exists), replace it by two new edges uw' and $u'w$, and preserve the remaining edges that exist between S and T, or S' and T'. The resulting graph is called the *double graph of* G, denoted by ${}^dG$; see Figure 1. When $S = V(G)$ and $T = \emptyset$, the graph ${}^dG$ is exactly the Kronecker product $G \otimes K_2$ (see [**29**]), also called *bipartite double cover* of G.

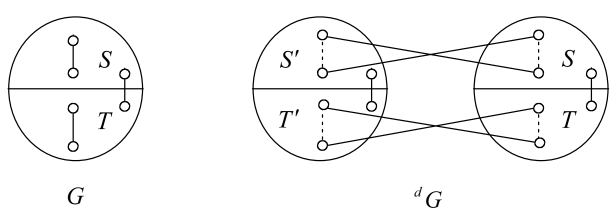

Figure 1. The graph G and its double graph ${}^dG$.

Obviously ${}^dG$ is bipartite, so $Q({}^dG)$ is similar to $L({}^dG)$, and hence $\lambda_{2n}^Q({}^dG) = \lambda_{2n}^L({}^dG) = 0$. It was proved in [**9**] that ${}^dG$ is independent of the choice of the bipartition (S,T) of $V(G)$ so that ${}^dG = G \otimes K_2$; and ${}^dG$ has exactly two components (each being a copy of G) if G is bipartite, and is connected if G is non-bipartite. It was also proved in [**9**] that the spectrum of $Q({}^dG)$ is the union of those of $Q(G)$ and $L(G)$. We restate the proof here for completeness.

Theorem 2.11. *Let G be a connected non-bipartite graph of order n. Then the spectrum of $Q({}^dG)$ is the union of the spectrum of $Q(G)$ and $L(G)$, and*

$$\lambda_1^Q({}^dG) = \lambda_1^Q(G),\ \lambda_{2n-1}^Q({}^dG) = \min\{\lambda_n^Q(G), \lambda_{n-1}^L(G)\}.$$

Proof. We label the vertices of G as $v_1, v_2, \ldots, v_n$, and arrange the vertices of ${}^dG$ in the order $v_1, v_2, \ldots, v_n, v_1', v_2', \ldots, v_n'$. Let λ be an eigenvalue of $Q(G)$ corresponding to an eigenvector X. Then by the eigen-equation (2.1), the vector $(X^\top, X^\top)^\top$ is an eigenvector of $Q({}^dG)$ corresponding to the same eigenvalue λ.

Let μ be an eigenvalue of $L(G)$ corresponding to an eigenvector Y. By the eigen-equation (2.2), the vector $(Y^\top, Y^\top)^\top$ is an eigenvector of $L({}^dG)$ corresponding to the eigenvalue μ. As ${}^dG$ is bipartite, $Q({}^dG) = \Gamma L({}^dG)\Gamma$, where $\Gamma = \mathbf{I}_{|S|} \oplus (-\mathbf{I}_{|T|}) \oplus (-\mathbf{I}_{|S'|}) \oplus \mathbf{I}_{|T'|}$. So the vector $\Gamma(Y^\top, Y^\top)^\top$ is an eigenvector of $Q({}^dG)$ corresponding to the eigenvalue μ.

Noting that $\Gamma(Y^\top, Y^\top)^\top = (Y_S^\top, -Y_T^\top, -Y_{S'}^\top, Y_{T'}^\top)^\top =: (Z^\top, -Z^\top)$ since $Y_S = Y_{S'}$ and $Y_T = Y_{T'}$, so this vector is orthogonal to $(X^\top, X^\top)^\top$. By the above discussion, we can find $2n$ eigenvectors of $Q({}^dG)$, orthogonal to each other, which correspond to the eigenvalues of $Q(G)$ and eigenvalues of $L(G)$, respectively. So the spectrum of $Q({}^dG)$ is the union of the spectrum of $Q(G)$ and $L(G)$.

As $Q(G)$ is nonnegative and irreducible, the eigenvector X of $Q(G)$ corresponding to $\lambda_1^Q(G)$ can be chosen positive by the Perron-Frobenius theory. By the above discussion, $(X^\top, X^\top)^\top$ is a positive eigenvector of $Q({}^dG)$ corresponding to the eigenvalue $\lambda_1^Q(G)$. So $\lambda_1^Q({}^dG) = \lambda_1^Q(G)$. The equality $\lambda_{2n-1}^Q({}^dG) = \min\{\lambda_n^Q(G), \lambda_{n-1}^L(G)\}$ follows obviously. □

A problem arises in which case $\lambda_{2n-1}^Q({}^dG) = \lambda_n^Q(G)$. The authors [**28**] give a sufficient condition for the above equality.

Lemma 2.12. [**28**] *Let G be a connected graph with $\epsilon_b(G) = 1$. Then $\lambda_{2n-1}^Q({}^dG) = \lambda_n^Q(G)$.*

Theorem 2.13. *Let G be a connected graph with $\epsilon_b(G) = 1$. Then $\mathsf{S}_Q(G) = \mathsf{S}_L({}^dG)$.*

Proof. Suppose G has order n. By Theorem 2.11 and Lemma 2.12,

$$\mathsf{S}_Q(G) = \lambda_1^Q(G) - \lambda_n^Q(G) = \lambda_1^Q({}^dG) - \lambda_{2n-1}^Q({}^dG) = \lambda_1^L({}^dG) - \lambda_{2n-1}^L({}^dG) = \mathsf{S}_L({}^dG),$$

where the third equality follows from Lemma 2.3 as ${}^dG$ is bipartite.

Corollary 2.14. *Let G be a non-bipartite unicyclic graph. Then $\mathsf{S}_Q(G) = \mathsf{S}_L({}^dG)$.*

Proof. Obviously $\epsilon_b(G) = 1$, so the result follows from Theorem 2.13.

3. MINIMUM SIGNLESS LAPLACIAN SPREAD OF GRAPHS

In this section, we will establish some new lower bounds for the signless Laplacian spread. With these inequalities, we establish a proof of Conjecture 1.1. We divide the analysis according to the value $\epsilon_b(G)$ of a graph G.

3.1. Graphs G with $\epsilon_b(G) \leq 1$

If a connected graph G is bipartite (i.e., $\epsilon_b(G) = 0$), then $\mathsf{S}_Q(G) = \lambda_1^Q(G)$ by Lemma 2.3. The following results are known.

Lemma 3.15. [**3**] [**6**] *Let G be a connected graph of order n. Then $\lambda_1^Q(G) \geq \lambda_1^Q(P_n) = 2\left(1 + \cos\frac{\pi}{n}\right)$, with equality if and only if $G = P_n$. If $G \neq P_n$, then $\lambda_1^Q(G) \geq 4$, with equality if and only if $G = C_n$ or $G = K_{1,3}$.*

Corollary 3.16. [**24**] *Let G be a connected bipartite graph of order n. Then*

$$\mathsf{S}_Q(G) \geq \mathsf{S}_Q(P_n) = 2\left(1 + \cos\frac{\pi}{n}\right),$$

with equality if and only if $G = P_n$. If $G \neq P_n$, then

$$\mathsf{S}_Q(G) \geq 4,$$

with equality if and only if $G = C_n$ for even n or $G = K_{1,3}$.

Now we consider graphs G with $\epsilon_b(G) = 1$ and begin with unicyclic graphs. We need a result from [**31**] on the Laplacian spread of unicyclic graphs.

Theorem 3.17. [**31**] *Let G be a unicyclic graph of order n. Then $\mathsf{S}_L(G) \geq \mathsf{S}_L(C_n)$, with equality if and only if $G = C_n$.*

Theorem 3.18. *Let G be a non-bipartite unicyclic graph of order n. Then*

$$\mathsf{S}_Q(G) \geq 2\left(1 + \cos\frac{\pi}{n}\right),$$

with equality if and only if $G = C_n$ and n is odd. If G is not an odd cycle, then $\mathsf{S}_Q(G) > 4$.

Proof. By Corollary 2.14 and Theorem 3.17, noting that ${}^dG$ is a bipartite unicyclic graph of order $2n$, we have

$$\mathsf{S}_Q(G) = \mathsf{S}_L({}^dG) \geq \mathsf{S}_L(C_{2n}) = 2\left(1 + \cos\frac{\pi}{n}\right).$$

The equality case is easily verified.

Now suppose that G contains an odd cycle C_k ($k < n$). If the maximum degree $\Delta(G) \geq 4$, then by Lemma 2.8, $\mathsf{S}_Q(G) > 4$. So assume $\Delta(G) = 3$. By Theorem 2.8, Lemmas 2.11 and 2.12 in [**31**], if $k \geq 9$, then $\mathsf{S}_L(G) > 4$. So, if $k \geq 5$, then ${}^dG$ contains a cycle C_{2k} with length greater than or equal to 10, and hence $\mathsf{S}_Q(G) = \mathsf{S}_L({}^dG) > 4$ by Corollary 2.14 and the above discussion. So we can assume $k = 3$. In this case, G contains a triangle with a pendant edge, denote such a graph by H. By Lemma 2.9, we have $\mathsf{S}_Q(G) \geq \mathsf{S}_Q(H) = \sqrt{17} > 4$. □

By Theorem 3.18 and Corollary 2.7, we have the following consequence.

Corollary 3.19. *Let G be a unicyclic graph of order n. Then*

$$\mathsf{S}_Q(G) \geq 2\left(1 + \cos\frac{\pi}{n}\right),$$

with equality if and only if $G = C_n$ and n is odd. Furthermore, excluding odd cycles, $\mathsf{S}_Q(G) \geq 4$ with equality if and only $G = C_n$ and n is even.

Theorem 3.20. *Let G be a connected graph of order n with $\epsilon_b(G) = 1$. Then*

$$\mathsf{S}_Q(G) \geq 2\left(1 + \cos\frac{\pi}{n}\right),$$

with equality if and only if $G = C_n$ and n is odd. Furthermore, excluding odd cycles, $\mathsf{S}_Q(G) > 4$.

Proof. As $\epsilon_b(G) = 1$, the graph G contains at least one odd cycle. If G is unicyclic, the result follows from Theorem 3.18. Otherwise, G contains an even cycle, and hence $\mathsf{S}_Q(G) > 4$ by Corollary 2.7 since $\epsilon_b(G) = 1$.

3.2. Graphs G with $\epsilon_b(G) \geq 2$

We break the discussion into some special cases. By Corollary 2.7, if a connected graph G contains $m \geq n + \epsilon_b(G)$ edges, then $\mathsf{S}_Q(G) \geq 4$. Also if G satisfies the property as in Lemma 2.8, then $\mathsf{S}_Q(G) > 4$; and if G contains a pendant vertex, then $\mathsf{S}_Q(G) > \mathsf{S}_Q(G - v)$ by Lemma 2.9. Thus we first consider the graphs G of order n with following property: *contain exactly $n - 1 + \epsilon_b(G)$ edges, contain only vertices of degree 2 or 3, and contain no cycles with each vertex having degree 3.* Let $\mathcal{G}_n^0$ be the set of graphs G on n vertices with above property, and with $\epsilon_b(G) \geq 2$.

Lemma 3.21. *If $G \in \mathcal{G}_n^0$, then*

$$\lambda_n^Q(G) \leq \frac{2\epsilon_b(G)}{n - \epsilon_b(G)}.$$

Proof. Pick one vertex of degree two from each odd cycle of G, for a total of $\epsilon_b(G)$ vertices which will form the set U. Let $W = V(G) - U$. As $G[W]$ is bipartite, there exists a vector X with entries 1 and -1 defined on W, such that $X^\top Q(G[W])X = 0$. Now define a vector Y on $V(G)$ such that $Y_v = 0$ if $v \in U$ and $Y_v = X_v$ if $v \in W$. Then

$$\lambda_n^Q(G) \leq \left(\sum_{v\in V(G)} Y_v^2\right)^{-1} \sum_{uw\in E(G)} (Y_u + Y_w)^2 = \frac{2\epsilon_b(G)}{n - \epsilon_b(G)}.$$

□

For a graph $G \in \mathcal{G}_n^0$, contract each cycle of G into a single vertex, and each path not on a cycle with the following property into an edge: if it has length greater than one and connects two vertices of degree 3 and contains no other vertices of degree 3. We then arrive at a *contracted graph of* G, denoted by ${}^cG$. Evidently, ${}^cG$ is a tree of order at least $\epsilon_b(G)$. See Figure 2 for an example of a graph and its contraction as described above

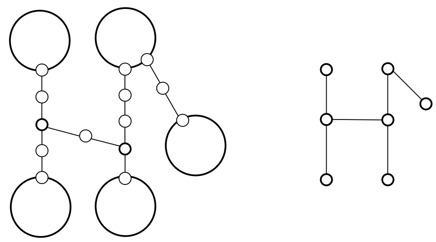

Figure 2. Example of a graph and its contraction.

Lemma 3.22. *If* $G \in \mathcal{G}_n^0$ *satisfies* $\operatorname{diam}({}^cG) \geq 3$, *then there exists a subgraph* $H \in \mathcal{G}_m^0$ *of* G *for some positive integer* m, *such that* $\operatorname{diam}({}^cH) < \operatorname{diam}({}^cG)$ *and* $\mathsf{S}_Q(G) > \mathsf{S}_Q(H)$.

Proof. In the tree ${}^cG$, find a path of length $\operatorname{diam}({}^cG)$, and choose a non-pendant edge e of this path. Then the two components of ${}^cG - e$ both have order greater than or equal to 2. The edge e of ${}^cG$ corresponds to a path P of G connecting two vertices of degree 3. Deleting an (arbitrary) edge on the path P will yield two components of G, denoted by G_1 and G_2, respectively. Noting that each vertex of ${}^cG$ corresponds to a cycle or a vertex of degree 3. So, in any case, $\epsilon_b(G_1) \geq 2$ and $\epsilon_b(G_2) \geq 2$.

By Lemma 2.10, without loss of generality, $\mathsf{S}_Q(G) > \mathsf{S}_Q(G_2)$. If G_2 contains a pendant vertices, by Lemma 2.9, we have a subgraph H of G_2 such that $H \in \mathcal{G}_m^0$ for some positive integer m and $\mathsf{S}_Q(G_2) > \mathsf{S}_Q(H)$. If $\operatorname{diam}({}^cH) = \operatorname{diam}({}^cG)$, repeating the discussion for H, we finally arrive at a graph $H' \in G_{m'}^0$ for some positive integer m', with $\operatorname{diam}({}^cH') < \operatorname{diam}({}^cG)$, and $\mathsf{S}_Q(G) > \mathsf{S}_Q(H')$. The result now follows. □

By Lemma 3.22, we may further reduce the discussion to graphs $G \in \mathcal{G}_n^0$ with $\operatorname{diam}({}^cG) \leq 2$. There are exactly two such types of graphs: type (i)-*graphs* G *with* $\epsilon_b(G) = 3$ *obtained from* 3 *cycles each connected to a fixed vertex by a path*; type (ii)-*graphs* G *with* $\epsilon_b(G) \geq 2$ *obtained from* $\epsilon_b(G) - 1$ *cycles each connected to a fixed cycle at different* (*but not all*) *vertices by a path.* Denote by $\mathcal{G}_n^1$ the set of graphs on n vertices of the two types described above. See Figure 3 for an illustration of the two types of graphs described above.

Lemma 3.23. *Let* $G \in \mathcal{G}_n^1$, *and let* α *be the number of edges of the subgraph of* G *induced by the vertices of degree* 3. *Then*

$$\lambda_1^Q(G) \geq 4 + \frac{15(\epsilon_b(G) - 1) + \alpha}{2n + 5(\epsilon_b(G) - 1)}.$$

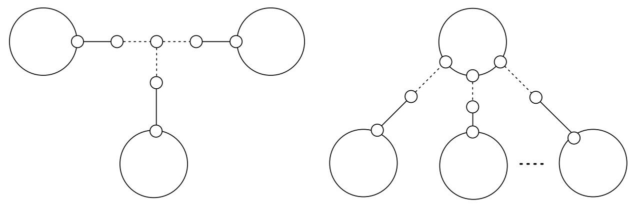

Figure 3. Type (i) and Type (ii) graphs.

Proof. First suppose G is of type (ii) in $\mathcal{G}_n^1$. Let U, W be the sets of vertices of G with degree 2 and 3, respectively. Then U, W form a bipartition of the vertex set $V(G)$. Note that the contracted graph ${}^cG$ contains exactly $\epsilon_b(G)$ vertices and $\bar{\epsilon}_b(G) := \epsilon_b(G) - 1$ edges. So G has exactly $2\bar{\epsilon}_b(G)$ vertices of degree 3, which implies $|W| = 2\bar{\epsilon}_b(G), |U| = n - 2\bar{\epsilon}_b(G)$. As there are α edges in $G[W]$, there are exactly $6\bar{\epsilon}_b(G) - 2\alpha$ edges in $E(W, U)$ (the set of edges between W and U), and $n - 5\bar{\epsilon}_b(G) + \alpha$ edges in $G[U]$. Define a vector X such that $X_v = 2$ if $v \in U$ and $X_v = 3$ if $v \in W$. Then

$$\begin{aligned}
\lambda_1^Q(G) &\geq \left(\sum_{v \in V(G)} X_v^2\right)^{-1} \sum_{uw \in E(G)} (X_u + X_w)^2 \\
&= \frac{\sum\limits_{uw \in E(G[W]) \cup E[W,U] \cup E(G[U])} (X_u + X_w)^2}{\sum\limits_{v \in W \cup U} X_v^2} \\
&= \frac{6^2\alpha + 5^2 \cdot [6\bar{\epsilon}_b(G) - 2\alpha] + 4^2 \cdot [n - 5\bar{\epsilon}_b(G) + \alpha]}{3^2 \cdot 2\bar{\epsilon}_b(G) + 2^2 \cdot [n - 2\bar{\epsilon}_b(G)]} \\
&= 4 + \frac{15\bar{\epsilon}_b(G) + \alpha}{2n + 5\bar{\epsilon}_b(G)}.
\end{aligned}$$

If G is of type (i) in $\mathcal{G}_n^1$, the result also holds by retracing the above proof. □

Lemma 3.24. *Let $G \in \mathcal{G}_n^1$, and let α be the number of edges of the subgraph of G induced by the vertices of degree* 3*. If $\alpha = 0$, then $n \geq 6\epsilon_b(G) - 5$; otherwise, $n \geq 4\epsilon_b(G) - 3$.*

Proof. First suppose G is of type (ii) in $\mathcal{G}_n^1$. Let C be the cycle of G such that other cycles are connected to C by paths, respectively. We count the vertices of G. If $\alpha = 0$, then every two vertices of degree 3 are not adjacent. So, $|V(C)| \geq 2(\epsilon_b(G) - 1) + 1$, and $|V(G - C)| \geq 3(\epsilon_b(G) - 1) + (\epsilon_b(G) - 1) = 4(\epsilon_b(G) - 1)$, which implies $n \geq 6(\epsilon_b(G) - 1) + 1$. Now suppose $\alpha > 0$. Then $|V(C)| \geq (\epsilon_b(G) - 1) + 1$; and there are at least $3(\epsilon_b(G) - 1)$ vertices in total for the pendant cycles of G attached to C. So, $n \geq 4(\epsilon_b(G) - 1) + 1$.

If G is of type (i) in $\mathcal{G}_n^1$, the above result also holds by a simple counting argument. □

Note that a graph $G \in \mathcal{G}_n^0$ with $\epsilon_b(G) = 2$ or 3 must satisfy $\mathrm{diam}({}^cG) \leq 2$, and hence $G \in \mathcal{G}_n^1$.

Lemma 3.25. *Let $G \in \mathcal{G}_n^0$ (or $\mathcal{G}_n^1$) with $\epsilon_b(G) = 2$ or 3. Then $\mathsf{S}_Q(G) \geq 4$, with equality if and only if G is obtained from two triangles connected by an edge.*

Proof. First we consider the case $\epsilon_b(G) = 2$. By Lemma 3.21 and Lemma 3.23,

$$\mathsf{S}_Q(G) \geq 4 + \frac{15+\alpha}{2n+5} - \frac{4}{n-2} = 4 + \frac{7n - 50 + \alpha(n-2)}{(2n+5)(n-2)},$$

where α is as defined in Lemma 3.23. If $\alpha = 0$, then $n \geq 7$, with equality if and only if G is obtained from two triangles connected by a path of length 2; denote this graph by H_0. Using basic mathematical software it can be checked that $\mathsf{S}_Q(H_0) \approx 4.15653 > 4$. Using the formula above, it follows that if $\alpha = 0$ and $n \geq 8$, then $\mathsf{S}_Q(G) > 4$.

If $\alpha = 1$, then $\mathsf{S}_Q(G) \geq 4 + \frac{8n-52}{(2n+5)(n-2)}$. In this case, $n \geq 6$ with equality if and only if G is obtained from 2 triangles connected by an edge. Denoting this graph by H_1, we find that $\mathsf{S}_Q(H_1) = 4$. Furthermore, if $n \geq 7$, then $\mathsf{S}_Q(G) > 4$.

Next suppose $\epsilon_b(G) = 3$. Similarly, by Lemma 3.21 and Lemma 3.23, we have

$$\mathsf{S}_Q(G) \geq 4 + \frac{30+\alpha}{2n+10} - \frac{6}{n-3} = 4 + \frac{18n - 150 + \alpha(n-3)}{(2n+10)(n-3)}.$$

If $\alpha = 0$, then $n \geq 13$ by Lemma 3.24, and hence $\mathsf{S}_Q(G) > 4$. If $\alpha \geq 1$, then $18n - 150 + \alpha(n-3) \geq 19n - 153 > 0$ as $n \geq 9$, which implies $\mathsf{S}_Q(G) > 4$.

Lemma 3.26. *Let $G \in \mathcal{G}_n^1$, where $\epsilon_b(G) \geq 4$. Then $\mathsf{S}_Q(G) > 4$.*

Proof. To avoid a complicated computation, we use Lemma 2.6(1) for the lower bound of $\lambda_1^Q(G)$. By Lemma 2.6(1) and Lemma 3.21 we observe that,

$$\mathsf{S}_Q(G) \geq 4 + \frac{4(\epsilon_b(G)-1)}{n} - \frac{2\epsilon_b(G)}{n - \epsilon_b(G)} = 4 + \frac{n(2\epsilon_b(G) - 4) - 4\epsilon_b(G)(\epsilon_b(G)-1)}{n(n-\epsilon_b(G))}.$$

By Lemma 3.24, $n \geq 4\epsilon_b(G) - 3$. Combining this with the fact $\epsilon_b(G) \geq 4$, we have

$$n(2\epsilon_b(G) - 4) - 4\epsilon_b(G)(\epsilon_b(G)-1) \geq 4\epsilon_b(G)^2 - 18\epsilon_b(G) + 12 > 0.$$

So $\mathsf{S}_Q(G) > 4$. □

We are now in a position to confirm one of our key observations regarding the signless Laplacian spread of a graph, which is needed for our main result.

Theorem 3.27. *Let G be a graph of order n with $n - \epsilon_b(G) + 1$ edges, where $\epsilon_b(G) \geq 2$. Then $\mathsf{S}_Q(G) \geq 4$, with equality if and only if G is obtained from two triangles connected by an edge.*

Proof. First assume G contains no pendant vertices. If G has the property as in Lemma 2.8, then $\mathsf{S}_Q(G) > 4$. So it is enough to consider $G \in \mathcal{G}_n^0$. If $\operatorname{diam}({}^cG) \leq 2$, then $G \in \mathcal{G}_n^1$, and $\mathsf{S}_Q(G) \geq 4$ by Lemma 3.25 and Lemma 3.26, with equality if and only if G is obtained from two triangles connected by an edge. Otherwise, by repeated use of Lemma 3.22, there exists a subgraph $H \in \mathcal{G}_m^1$ for some positive integer m such that $\operatorname{diam}({}^cH) \leq 2$ and $\mathsf{S}_Q(G) > \mathsf{S}_Q(H) \geq 4$ by the above discussion.

If G contains a pendant vertex v, repeatedly using Lemma 2.9, we arrive at a subgraph H of G without pendant vertices, and $\mathsf{S}_Q(G) > \mathsf{S}_Q(H) \geq 4$ by the above discussion. □

Now we are in a position to confirm the validity of Conjecture 1.1.

Theorem 3.28. *Let G be a graph of order n. Then*

$$\mathsf{S}_Q(G) \geq 2\left(1 + \cos\frac{\pi}{n}\right),$$

with equality if and only if $G = P_n$ or $G = C_n$ in case of odd n. Furthermore, except for P_n or C_{2k+1},

$$\mathsf{S}_Q(G) \geq 4,$$

with equality if and only if G is one of the following graphs: $K_{1,3}$, K_4, two triangles connected by an edge, and C_n for even n.

Proof. If G is bipartite, then by Corollary 3.16, $\mathsf{S}_Q(G) \geq \mathsf{S}_Q(P_n) = 2\Big(1 + \cos\frac{\pi}{n}\Big)$, with equality if and only if $G = P_n$. If $G \neq P_n$, then $\mathsf{S}_Q(G) \geq 4$, with equality if and only if $G = K_{1,3}$ or $G = C_n$.

Suppose G is non-bipartite. If G contains $m \geq n + \epsilon_b(G)$ edges, then $\mathsf{S}_Q(G) \geq 4$ by Corollary 2.7, with equality if and only if $G = K_4$. So we can assume G contains exactly $n - 1 + \epsilon_b(G)$ edges. If $\epsilon_b(G) = 1$, by Theorem 3.18, $\mathsf{S}_Q(G) \geq 2\Big(1 + \cos\frac{\pi}{n}\Big)$, with equality if and only if $G = C_n$ and n is odd; and if $G \neq C_n$ for odd n then $\mathsf{S}_Q(G) > 4$. Otherwise, $\epsilon_b(G) \geq 2$, and by Theorem 3.27, $\mathsf{S}_Q(G) \geq 4$, with equality if and only if G is obtained from two triangles connected by an edge.

Remark 3.29 (Added During Review). The paper [**8**] was brought to our attention by the referee during review of our paper and we acknowledge that the first part of Theorem 3.28 was also proved in [**8**], but here we provide a different proof technique using edge bipartiteness.

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