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NONLOCAL SYSTEMS OF BVPS WITH ASYMPTOTICALLY SUBLINEAR BOUNDARY CONDITIONS

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In this paper we consider a coupled system of second-order boundary value problems with nonlocal, nonlinear boundary conditions. By imposing only a condition of asymptotic sublinear growth on the nonlinear boundary functions, we are able to achieve generalizations over existing works and, in particular, we allow for the nonlocal terms to be able to be represented as Lebesgue-Stieltjes integrals possessing signed Borel measures. Because we only suppose the sublinearity of the the nonlinear boundary functions at positive infinity, we also remove many of the restrictive growth assumptions found in other recent works on closely related problems. We conclude with a numerical example to explicate the consequences of our main result.

1. INTRODUCTION

In this paper we consider a system of nonlocal boundary value problems with nonlinear boundary conditions. In particular, we consider the existence of at least one positive solution to the system

$$\begin{aligned} x''(t) &= -\lambda_1 a_1(t) g_1(x(t), y(t)),\ t \in (0,1)\\ y''(t) &= -\lambda_2 a_2(t) g_2(x(t), y(t)),\ t \in (0,1)\\ x(0) &= H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right)\\ y(0) &= H_2\left(\phi_2(y) + \varepsilon_0^1 x\left(\xi_0^2\right) + \varepsilon_0^2 y\left(\xi_0^2\right)\right)\\ x(1) &= 0 = y(1), \end{aligned} \tag{1.1}$$

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where $\lambda_1,\,\lambda_2 > 0$ are eigenvalues, $\varepsilon_0^1,\,\varepsilon_0^2 > 0$ are constants, which shall be specified later, $\xi_0^1, \xi_0^2 \in (0,1)$ are fixed, $\phi_1,\,\phi_2\ :\ \mathcal{C}([0,1]) \to \mathbb{R}$ are functionals, which are realizations of the nonlocal nature of the boundary conditions, $H_1,\,H_2\ :\ \mathbb{R} \to \mathbb{R}$ are continuous functions, which are realizations of the nonlinear nature of the boundary conditions, $g_1,\,g_2\ :\ [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ are continuous, and $a_1,\,a_2\ :\ [0,1] \to [0,+\infty)$ are continuous functions that are not identically zero on any subinterval of $[0,1]$. The nonlocal terms here are very general, being as they are realized as Lebesgue-Stieltjes integrals – that is,

$$\phi_1(x) := \int_{[0,1]} x(t)\,\mathrm{d}\alpha_1(t) \quad \text{and} \quad \phi_2(y) := \int_{[0,1]} y(t)\,\mathrm{d}\alpha_2(t), \tag{1.2}$$

with $\alpha_1,\,\alpha_2 \in BV([0,1])$. It may be assumed without loss that, in fact, $\alpha_1,\,\alpha_2 \in NBV([0,1])$. Consequently, we observe that to each of $\alpha_1,\,\alpha_2$, there exists a unique Borel measure, say μ_{α_1} and μ_{α_2}, respectively. In our context, these measures may be *signed*.

Our novel approach to problem (1.1) is twofold. We first introduce the perturbation terms $\varepsilon_0^1 x\left(\xi_0^1\right),\, \varepsilon_0^2 y\left(\xi_0^1\right),\, \varepsilon_0^1 x\left(\xi_0^2\right)$, and $\varepsilon_0^2 y\left(\xi_0^2\right)$ appearing in (1.1). These perturbation terms allows us in turn to introduce a second novelty – namely, to utilize much less restrictive growth conditions on each of H_1 and H_2 appearing in (1.1). Indeed, we require that, for each $i = 1, 2$,

$$\lim_{z\to\infty} \frac{\left|H_i(z) - \kappa_0^i z\right|}{|z|} = 0, \tag{1.3}$$

for some $\kappa_0^i \in [0,+\infty)$. Note that condition (1.3) implies that H_i may grow either sub- or superlinearly at $z = 0$. These two relatively simple modifications allow for considerably weaker conditions on problem (1.1), for we may now assume that each of the measures μ_{α_1} and μ_{α_2} is signed and that neither H_1 nor H_2 is sublinear at $z = 0$, assumptions that seem to be made in most problems related to (1.1) as we indicate in the sequel. Furthermore, it turns out that we do not even require the perturbation terms provided that we assume that each of $H_1(z)$ and $H_2(z)$ is monotone increasing for $z \geq 0$.

Closely related to these observations, we should point out at this juncture that Yang [**17, 18**] actually introduced asymptotic conditions in those works not entirely dissimilar to (1.3) above. In particular, in [**17**] a system of equations, which are very similar to (1.1), was studied. Among a variety of other conditions, Yang was able to employ an asymptotic condition of the general form

$$\limsup_{z\to\infty} \frac{H(z)}{z} < \frac{1}{\varphi}, \tag{1.4}$$

for some positive, finite constant φ. Certainly, (1.4) is more general than our condition (1.3). However, a careful examination of the proof in [**17**] reveals that the positivity of the measures μ_{α_1} and μ_{α_2} is essential. Consequently, it does not seem possible at present to give a simple modification of Yang's techniques in the case where the measures may be signed (i.e., our situation here).

Thus, we employ two different strategies to overcome these difficulties. Our first strategy is via condition (1.3) and the perturbation terms in (1.1), whereas our second strategy is via a monotonicity assumption on each of H_1 and H_2. In any case, we should also point out that although Yang [**17**] achieves a more general condition in (1.4), in [**17**] much more complicated structural conditions are instead assumed on the nonlinearities g_1, g_2 than we assume here, and the eigenvalue problem is not studied in [**17**] either.

Prior to enumerating specifically the contributions of this paper, let us briefly review the relevant existing literature on problems similar to (1.1). Recently, Infante and Webb [**13**] provided an elegant theory for nonlocal BVPs in the case where the boundary conditions are *linear*; furthermore, one may consult the introduction of [**13**] for a thorough review of the recent literature on multipoint BVPs prior to the contribution of Infante and Webb. Related extensions may be found in recent papers by Webb [**14, 15, 16**] as well as by Graef and Webb [**6**].

On the other hand, recently there has been some attempts by Infante [**7**], Infante and Pietramala [**8, 9, 10**], Kang and Wei [**11**], and Yang [**17, 18**] to consider in fairly general contexts BVPs with nonlinear BCs. However, insofar as these papers are concerned, while they do make a connection to the linear boundary condition theory, they do so under some limiting assumptions, namely that H, which is the function capturing the nonlinearity of the BCs, is strictly positive, that the Borel measure associated to the Lebesgue-Stieltjes integral $\phi(y) = \int\limits_E y(t)\,\mathrm{d}\alpha(t)$ is *positive*, and, in nearly all cases ([**17, 18**] being partial exceptions), that H satisfies a uniform growth condition of the form $\zeta_1 z \leq H(z) \leq \zeta_2 z$, for $0 \leq \zeta_1 \leq \zeta_2 < +\infty$, *for all* $z \geq 0$.

In particular, our work here directly generalizes and improves [**8, 17**] since those works are very closely related to our work here. Indeed, Infante and Pietramala [**8**] and Yang [**17**] each considered a system almost identical to (1.1) but with the nonlocal condition at $t = 1$ rather than at $t = 0$, which is a trivial difference. Here we achieve in the particular case of problem (1.1) the following generalizations over various aspects of the results presented in [**8, 17**] and, more tangentially, in [**7, 9, 10, 11, 18**]. Our results here also complement those which we have given recently for related problems – see [**4, 5**]. We enumerate these generalizations and improvements as follows.

1. For the first of our two existence results, we do not assume that either H_1 or H_2 is monotone, unlike some works in the literature involving nonlinear boundary conditions. Where we do assume monotonicity, this assumption, as noted above, allows us to dispense with the perturbation terms appearing in (1.1) above.

2. We allow for each of μ_{α_1} and μ_{α_2} to be *signed* measures rather than merely positive. This is a notable generalization over preceding works on related problems – specifically, [**7, 8, 9, 10, 17, 18**].

3. We do *not* assume a uniform linear growth condition on either H_1 or H_2.

While condition (1.3) does imply linear growth of the H_i's at $+\infty$, this is only an asymptotic condition, which is much weaker than the uniform condition proposed in other works on related problems – specifically, [**7, 8, 9, 10**].

4. We believe that our techniques here allow for H to be only eventually positive, though we do not prove such a theorem here – see [**3**] for an exemplar of this extension.

5. While we present our results in the somewhat simpler setting of Dirichlet-type boundary conditions, we believe that our techniques can be extended to include some of the other types of boundary conditions considered by other authors.

6. Finally, we exhibit an explicit and direct connection to the linear BC theory developed originally in [**13**]. Indeed, condition (1.3) essentially shows that if the boundary conditions merely possess asymptotically sublinear growth at $+\infty$ (i.e., are asymptotically similar to the sorts of conditions considered in [**13**]), then this is sufficient, together with some other relatively standard assumptions, to deduce that problem (1.1) has at least one positive solution. Heuristically, then, if $\phi(y)$ is a linear functional to which the theory of [**13**] applies and if $H_i(\phi(y)) \approx \phi(y)$ for $\phi(y) \gg 1$, for each i, then we recover the existence of at least one positive solution to problem (1.1). We feel that this is both a novel and interesting observation.

2. PRELIMINARIES

We consider in this work the space $\mathcal{X} := \mathcal{B}\times\mathcal{B}$, where $\mathcal{B}$ represents the Banach space $\mathcal{C}([0,1])$ when equipped with the usual supremum norm, $\|\cdot\| := \|\cdot\|_\infty$. Note – see Dunninger and Wang [**2**] – that $\mathcal{X}$ becomes a Banach space when equipped with the norm $\|(x,y)\| := \|x\| + \|y\|$. It is then known that a fixed point in $\mathcal{X}$ of

$$\begin{aligned} S(x,y)(t) &:= \left(T_1(x,y), T_2(x,y)\right)(t) \\ &= \Bigg((1-t)H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\ &\qquad + \lambda_1 \int_0^1 G(t,s)a_1(s)g_1(x(s),y(s))\,\mathrm{d}s, \\ &\qquad (1-t)H_2\left(\phi_2(y) + \varepsilon_0^1 x\left(\xi_0^2\right) + \varepsilon_0^2 y\left(\xi_0^2\right)\right) \\ &\qquad + \lambda_2 \int_0^1 G(t,s)a_2(s)g_2(x(s),y(s))\,\mathrm{d}s\Bigg) \end{aligned} \tag{2.1}$$

is a solution of problem (1.1), where $S : \mathcal{X} \to \mathcal{X}$ and $T_i : \mathcal{X} \to \mathcal{B}$, for each $i = 1, 2$. Here $G : [0,1]\times[0,1] \to \mathbb{R}$ appearing in (2.1) is the Green's function associated to the two-point conjugate problem – that is,

$$G(t,s) := \begin{cases} t(1-s), & 0 \le t \le s \le 1 \\ s(1-t), & 0 \le s \le t \le 1, \end{cases} \tag{2.2}$$

which can be found, for example, in [**12**].

Now, in the sequel the set E will be a fixed but arbitrary set such that $E \Subset (0,1)$. For simplicity, we may just as well assume that $E := [a,b]$ such that $0 < a < b < 1$. In any case, with this declaration it is then well-known that there is a constant $\gamma := \gamma(E)$ such that

$$\min_{t\in E} G(t,s) \geq \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(s,s), \tag{2.3}$$

for each $s \in [0,1]$. Note that $\gamma \in (0,1)$. Finally, let us also recall as a preliminary lemma Krasnosel'skiĭ's fixed point theorem (see [**1**]).

Lemma 2.1. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are bounded open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T : \mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathcal{K}$ is a completely continuous operator. If either*

1. *$\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$; or*
2. *$\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$;*

then T has at least one fixed point in $\mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

3. MAIN RESULTS AND NUMERICAL EXAMPLE

Before stating and proving our two main results, which are Theorem 3.6 and Theorem 3.8, we introduce some structural conditions on the various functions and functionals in (1.1). They are as follows.

H1: For each i, let $H_i : \mathbb{R} \to \mathbb{R}$ be a real-valued, continuous function. Moreover, $H_i : [0,+\infty) \to [0,+\infty)$ – i.e., H_i is nonnegative when restricted to $[0,+\infty)$.

H2: For each i, the functional $\phi_i(y)$ appearing in (1.1) is linear and, in particular, has the form

$$\phi_i(y) := \int_{[0,1]} y(t)\,\mathrm{d}\alpha_i(t), \tag{3.1}$$

where $\alpha_i : [0,1] \to \mathbb{R}$ satisfies $\alpha_i \in BV([0,1])$.

H3: For each i, there is a constant ε_1^i such that the functional ϕ_i in (1.1) satisfies the inequality

$$|\phi_i(y)| \leq \varepsilon_1^i \|y\| \tag{3.2}$$

for all $y \in \mathcal{C}([0,1])$.

H4: For each i, there is $\kappa_0^i \geq 0$ such that

$$\lim_{z\to+\infty} \frac{\left|H_i(z) - \kappa_0^i z\right|}{|z|} = 0 \tag{3.3}$$

holds.

H5: We find that

$$\lim_{(x,y)\to(+\infty,+\infty)} g_1(x,y) = +\infty \text{ and } \lim_{(x,y)\to(+\infty,+\infty)} g_2(x,y) = +\infty. \tag{3.4}$$

H6: We find that

$$\lim_{(x,y)\to(+\infty,+\infty)} \frac{g_1(x,y)}{x+y} = 0 \text{ and } \lim_{(x,y)\to(+\infty,+\infty)} \frac{g_2(x,y)}{x+y} = 0. \tag{3.5}$$

H7: Each of the following holds.

$$\begin{aligned} 0 &\leq \varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_1^2 < \frac{1}{2} \\ 0 &\leq \kappa_0^1\left(\varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^1\right) < \frac{1}{2} \\ 0 &\leq \kappa_0^2\left(\varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^2\right) < \frac{1}{2} \end{aligned} \tag{3.6}$$

H8: For each i, each of

$$\int_{[0,1]} (1-t)\,\mathrm{d}\alpha_i(t) \geq 0 \tag{3.7}$$

and

$$\int_{[0,1]} G(t,s)\,\mathrm{d}\alpha_i(t) \geq 0 \tag{3.8}$$

holds, where the latter holds for each $s \in [0,1]$.

H9: The nonlinearities g_1 and g_2 satisfy either the relationship $g_1(x,y) \leq g_2(x,y)$ or the relationship $g_2(x,y) \leq g_1(x,y)$, for all x, $y \geq 0$.

Let us make some brief remarks regarding certain of the preceding conditions.

Remark 3.1. Regarding conditions (H2)–(H3), we observe that there are many nontrivial functions satisfying these conditions. Indeed, consider the following collection of functionals.

$$\begin{aligned} \phi_1^i(y) &:= \int_F y(t)\,\mathrm{d}t \\ \phi_2^i(y) &:= \sum_{m=1}^{n} a_m y\left(\xi_m\right) \\ \phi_3^i(y) &:= \int_{[0,1]} y(t)\,\mathrm{d}\alpha(t) \end{aligned} \tag{3.9}$$

Since each of $(3.9)_1$–$(3.9)_3$ is linear, each satisfies (H2). On the other hand, so long as $m(F) \leq \varepsilon_1^i$, where m is the Lebesgue measure, then $(3.9)_1$ satisfies (H3). As long as $\sum_{m=1}^{n} |a_m| \leq \varepsilon_1^i$, then $(3.9)_2$ satisfies (H3). Finally, as long as $V_{[0,1]}(\alpha)$, which is the total variation of α over $[0,1]$, satisfies $V_{[0,1]}(\alpha) \leq \varepsilon_1^i$, then functional $(3.9)_3$ satisfies condition (H3). The example at the end of this paper presents two other functionals, each of which satisfies conditions (H2) and (H3).

Remark 3.2. Regarding condition (H4), this is the asymptotic condition, which is key to our arguments in the sequel. Note that if the condition

$$\lim_{z\to+\infty} |H(z) - z| = 0, \tag{3.10}$$

which implies that $H(z)$ converges to z at $+\infty$, holds, then it follows that condition (H4) holds, too. It should also be noted that there are many nontrivial functions which do not satisfy condition (3.10) but do satisfy condition (H4) for some κ_0^i. For instance, consider the function $H_1 : [0,+\infty) \to [0,+\infty)$ defined by

$$H_1(z) := 2\sqrt{z}\cos\left(\frac{1}{1+z}\right) \tag{3.11}$$

Then it is clear that H_1 satisfies (3.3) in case $\kappa_0^1 = 0$ but fails to satisfy the condition (3.10).

Remark 3.3. Note that in (3.6) above, depending upon the values of the various constants, it may be that each of conditions $(3.6)_2$ and $(3.6)_3$ is superfluous.

Remark 3.4. Observe that we do *not* require any growth conditions on H_i except asymptotically as given in (3.3) above. This is in contrast to nearly all other recent papers on BVPs with nonlinear, nonlocal boundary conditions – see, for instance, [**7, 8, 9, 10, 11**]. Indeed, as mentioned in Section 1, it seems to be assumed frequently that the function capturing the nonlinear aspect of the boundary conditions satisfy a condition of the sort $\alpha z \leq H_i(z) \leq \beta z$, for $0 \leq \alpha \leq \beta$ and all $z \geq 0$. Here we remove such restrictions entirely. Indeed, we only really need sublinearity at $+\infty$, and we consider this observation to be an interesting contribution of this work.

Remark 3.5. Observe that no growth conditions are required of either H_1 or H_2 at 0. In particular, $H_1(z)$ could be sublinear at $z = 0$, whilst $H_2(z)$ is superlinear at $z = 0$. In particular, the nonlinearities H_1, H_2 may exhibit mixed behavior at $z = 0$. The same comment may be given for the nonlinearities g_1 and g_2.

Now, let γ_0 be the constant defined by

$$\gamma_0 := \min\left\{\gamma, \min_{t\in E}(1-t)\right\}, \tag{3.12}$$

where $\gamma_0 \in (0,1)$ and γ is the constant from (2.3). Then the cone, $\mathcal{K}$, we shall use in the sequel is defined by

$$\begin{aligned} \mathcal{K} := \Big\{(x,y) \in \mathcal{X} \ : x, y \geq 0,\ \min_{t\in E}[x(t)+y(t)] \geq \gamma_0\|(x,y)\|, \\ \phi_1(x), \phi_2(y) \geq 0\Big\}, \end{aligned} \tag{3.13}$$

which is a simple modification of a cone first introduced by Infante and Webb [**13**]. Let us point out at this juncture that $\mathcal{K}$ is not just the trivial subspace of $\mathfrak{X}$. Indeed, it is easy to verify that if we put $\boldsymbol{\beta}(t) := (1-t, 1-t)$, then $\boldsymbol{\beta} \in \mathcal{K}$. In fact, it is also true, of course, that if we put $\boldsymbol{\beta}_1(t) := (1-t, 0)$ and $\boldsymbol{\beta}_2(t) := (0, 1-t)$, then $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2 \in \mathcal{K}$. With this in hand, we now state and prove our main result.

Theorem 3.6. *Let conditions* (H1)–(H9) *hold. Assume that* ξ_0^1, $\xi_0^2 \in E$, *where the set* E *is fixed as in Section* 2. *Then for all* λ_1, $\lambda_2 > 0$ *sufficiently large problem* (1.1) *has at least one positive solution.*

Proof. We consider the problem

$$\begin{aligned} x''(t) &= -\lambda_1 a_1(t) g_1\left(x(t), y(t)\right), \ t \in (0,1) \\ y''(t) &= -\lambda_2 a_2(t) g_2\left(x(t), y(t)\right), \ t \in (0,1) \\ x(0) &= H_1\left(\phi_1\left(x_j\right) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\ y(0) &= H_2\left(\phi_2\left(y_j\right) + \varepsilon_0^1 x\left(\xi_0^2\right) + \varepsilon_0^2 y\left(\xi_0^2\right)\right) \\ x(1) &= 0 = y(1). \end{aligned} \tag{3.14}$$

We first show that $S(\mathcal{K}) \subseteq \mathcal{K}$. To this end, let $(x,y) \in \mathcal{K}$. Then it is obvious that $T_i\left(x,y\right)(t) \geq 0$, for each $t \in [0,1]$ and for each $i = 1, 2$. On the other hand, note that

$$\begin{aligned} \min_{t \in E} T_1\left(x,y\right)(t) &\geq \gamma_0 H_1\left(\phi_1\left(x\right) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\ &\quad + \lambda_1 \gamma \max_{t \in [0,1]} \int_0^1 G(t,s) a_1(s) g_1\big(x(s), y(s)\big)\, \mathrm{d}s \geq \gamma_0 \| T_1\left(x,y\right) \|. \end{aligned} \tag{3.15}$$

It similarly holds that $\min_{t \in E} T_2\left(x,y\right)(t) \geq \gamma_0 \| T_2\left(x,y\right) \|$. We thus conclude that

$$\min_{t \in E} \left[\left(T_1\left(x,y\right)\right)(t) + \left(T_2\left(x,y\right)\right)(t) \right] \geq \gamma_0 \| S\left(x,y\right) \|. \tag{3.16}$$

Finally, note that

$$\begin{aligned} \phi_1\left(T_1\left(x,y\right)\right) &= H_1\left(\phi_1\left(x\right) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \int_{[0,1]} (1-t)\, \mathrm{d}\alpha_1(t) \\ &\quad + \lambda_1 \int_{[0,1]} \int_0^1 G(t,s) a_1(s) g_1\left(x(s), y(s)\right)\, \mathrm{d}s\ \mathrm{d}\alpha_1(t) \\ &= H_1\left(\phi_1\left(x\right) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \int_{[0,1]} (1-t)\, \mathrm{d}\alpha_1(t) \\ &\quad + \lambda_1 \int_0^1 \left[\int_{[0,1]} G(t,s)\, \mathrm{d}\alpha_1(t) \right] a_1(s) g_1\left(x(s), y(s)\right)\, \mathrm{d}s \geq 0, \end{aligned} \tag{3.17}$$

where the final inequality from assumption (H8). Similarly, $\phi_2\left(T_2\left(x,y\right)\right) \geq 0$. Thus, $S : \mathcal{K} \to \mathcal{K}$, as claimed. Furthermore, since it is standard to show that S is a completely continuous operator, we omit the proof of this claim.

We next make a simple observation. For each $(x, y) \in \mathcal{K}$, we have that

$$\min_{t\in E}[x(t) + y(t)] \geq \gamma_0 \|(x, y)\|, \tag{3.18}$$

and, thus, since $\phi_1(x) \geq 0$ it follows that

$$\begin{aligned} \phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right) &\geq \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right) \\ &\geq \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\} \min_{t\in E}\left[x(t) + y(t)\right] \\ &\geq \min\left\{\varepsilon_0^1, \varepsilon_0^2\right\} \gamma_0 \|(x, y)\|. \end{aligned} \tag{3.19}$$

Of course, the same inequality holds if we replace $\phi_1(x)$ with $\phi_2(y)$, $x\left(\xi_0^1\right)$ with $x\left(\xi_0^2\right)$, and $y\left(\xi_0^1\right)$ with $y\left(\xi_0^2\right)$. In any case, observation (3.19) will be very important in the sequel.

Now, we note that by condition (H5), there is $r_1 > 0$ sufficiently large such that whenever $x + y \geq r_1$, we find that

$$g_1(x, y) \geq \frac{1}{\int\limits_E G\left(t_0, s\right) a_1(s)\, \mathrm{d}s}, \tag{3.20}$$

where $t_0 \in \overset{\circ}{E}$ is any fixed but otherwise arbitrary point; note that since $E \Subset (0, 1)$, it follows that $0 < t_0 < 1$. Similarly, there is $r_1^* > 0$ such that for $x + y \geq r_1^*$, it follows that

$$g_2(x, y) \geq \frac{1}{\int\limits_E G\left(t_0, s\right) a_2(s)\, \mathrm{d}s}. \tag{3.21}$$

Define the number $r_1^{**} > 0$ by

$$r_1^{**} := \max\left\{\frac{r_1}{\gamma_0}, \frac{r_1^*}{\gamma_0}\right\} \tag{3.22}$$

and the set $\Omega_{r_1^{**}} \subset \mathfrak{X}$ by

$$\Omega_{r_1^{**}} := \left\{(x, y) \in \mathfrak{X}\ :\ \|(x, y)\| < r_1^{**}\right\}. \tag{3.23}$$

Observe that for $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$ it follows that

$$\min_{t\in E}\left[x(t) + y(t)\right] \geq \gamma_0 \|\left(x, y\right)\| = \gamma_0 r_1^{**} = \max\left\{r_1, r_1^*\right\}. \tag{3.24}$$

In particular, both (3.20) and (3.21) hold. Therefore, it follows that for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$ we have

$$\begin{aligned} T_1\left(x, y\right)\left(t_0\right) &\geq \lambda_1 \int_0^1 G\left(t_0, s\right) a_1(s) g_1\left(x(s), y(s)\right)\, \mathrm{d}s \\ &\geq \lambda_1 \int_E G\left(t_0, s\right) a_1(s) g_1\left(x(s), y(s)\right)\, \mathrm{d}s \geq \lambda_1, \end{aligned} \tag{3.25}$$

where we have used the fact that $H(z) \geq 0$, for each $z \geq 0$. By now making λ_1 sufficiently large, we get

$$\|T_1(x,y)\| \geq \frac{1}{2}\|(x,y)\|. \tag{3.26}$$

Similarly, by making λ_2 sufficiently large we deduce that

$$\|T_2(x,y)\| \geq \frac{1}{2}\|(x,y)\|. \tag{3.27}$$

So, from (3.26)–(3.27) we conclude that

$$\|S(x,y)\| \geq \|(x,y)\|, \tag{3.28}$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$.

On the other hand, select numbers ε_2^1, $\varepsilon_2^2 > 0$ sufficiently small such that each of the following inequalities holds.

$$\begin{aligned} &\kappa_0^1\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^1\right)+\varepsilon_2^1<\frac{1}{2} \\ &\kappa_0^1\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^2\right)+\varepsilon_2^2<\frac{1}{2} \end{aligned} \tag{3.29}$$

Evidently, these inequalities may be satisfied because condition (H7) holds. Then condition (H6) implies that there is $r_2 > 0$ sufficiently large such that for each $i = 1, 2$ it holds that

$$g_i(x,y) \leq \eta_1(x+y), \tag{3.30}$$

whenever $x + y \geq r_2$, where η_1 satisfies both

$$\eta_1 \int_0^1 G(s,s)a_1(s)\,\mathrm{d}s \leq \frac{\varepsilon_2^1}{2\lambda_1} \quad \text{and} \quad \eta_1 \int_0^1 G(s,s)a_2(s)\,\mathrm{d}s \leq \frac{\varepsilon_2^2}{2\lambda_2}. \tag{3.31}$$

Additionally, for a given number $\varepsilon_3^1 > 0$ condition (H4) implies the existence of a number $r_2^* := r_2^*\left(\varepsilon_3^1\right) > 0$ such that

$$\begin{aligned} &\left|H_1\left(\phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right)+\epsilon_0^2 y\left(\xi_0^1\right)\right)-\kappa_0^1\left(\phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2 y\left(\xi_0^1\right)\right)\right| \\ &<\varepsilon_3^1\|(x,y)\| \end{aligned} \tag{3.32}$$

whenever

$$\phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2 y\left(\xi_0^1\right) \geq r_2^*. \tag{3.33}$$

Note that to get (3.32) we have used the fact that

$$\begin{aligned} 0 &\leq \phi_1(x)+\varepsilon_0^1 x\left(\xi_0^1\right)+\varepsilon_0^2 y\left(\xi_0^1\right) \\ &\leq \varepsilon_1^1\|x\|+\varepsilon_0^1\|x\|+\varepsilon_0^2\|y\| \leq \max\left\{\varepsilon_0^1,\varepsilon_0^2,\varepsilon_1^1\right\}\|(x,y)\|<\|(x,y)\|. \end{aligned} \tag{3.34}$$

Furthermore, in the same manner as in the preceding paragraph, we may select ε_3^1 in such a way so that

$$\kappa_0^1\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^1\right)+\varepsilon_2^1+\varepsilon_3^1<\frac{1}{2} \tag{3.35}$$

holds. In any case, recalling (3.19) and the fact that $\phi_1(x)\geq 0$ since $(x,y)\in\mathcal{K}$, we have that (3.32) is satisfied provided that

$$\left\|(x,y)\right\|\geq\frac{r_2^*}{\gamma_0\min\left\{\varepsilon_0^1,\varepsilon_0^2\right\}} \tag{3.36}$$

holds. A dual argument reveals that (3.32) also holds for the function H_2 whenever (3.36) holds replacing r_2^* with some (possibly larger) constant r_2^{**}, by making the obvious changes in the various subscripts appearing in (3.32)–(3.34), and changing ε_3^1 to some ε_3^2 – i.e., provided that

$$\left\|(x,y)\right\|\geq\frac{r_2^{**}}{\gamma_0\min\left\{\varepsilon_0^1,\varepsilon_0^2\right\}} \tag{3.37}$$

holds. Here, of course, analogous to (3.35) we choose ε_3^2 so that

$$\kappa_0^1\left(\varepsilon_0^1+\varepsilon_0^2+\varepsilon_1^2\right)+\varepsilon_2^2+\varepsilon_3^2<\frac{1}{2} \tag{3.38}$$

is satisfied. So, both conditions hold provided that

$$\left\|(x,y)\right\|\geq\max\left\{\frac{r_2^*}{\gamma_0\min\left\{\varepsilon_0^1,\varepsilon_0^2\right\}},\frac{r_2^{**}}{\gamma_0\min\left\{\varepsilon_0^1,\varepsilon_0^2\right\}}\right\}. \tag{3.39}$$

Now, assume by condition (H9) and without loss of generality that $g_2(x,y)\leq g_1(x,y)$, for all $x,\ y\geq 0$. Then because g_1 is unbounded at infinity in the sense of condition (H5), we may select a number $R_1>0$, where R_1 satisfies

$$R_1>\max\left\{2r_1^{**},r_2,\frac{r_2^*}{\gamma_0\min\left\{\varepsilon_0^1,\varepsilon_0^2\right\}},\frac{r_2^{**}}{\gamma_0\min\left\{\varepsilon_0^1,\varepsilon_0^2\right\}}\right\}=:\vartheta, \tag{3.40}$$

such that

$$g_1(x,y)\leq g_1\left(\rho_1,\rho_2\right), \tag{3.41}$$

for all $(x,y)\in[0,R_1]\times[0,R_1]$, where either $\rho_1=R_1$ and $0\leq\rho_2\leq R_1$ or $0\leq\rho_1\leq R_1$ and $\rho_2=R_1$. To prove this claim, pick a number $\theta^*>0$ such that

$$\theta^*>\vartheta. \tag{3.42}$$

By the extreme value theorem, the function g_1 attains its maximum on the square $[0,\theta^*]\times[0,\theta^*]$, say

$$\max_{(x,y)\in[0,\theta^*]\times[0,\theta^*]}g_1(x,y)=g_1\left(x_0,y_0\right). \tag{3.43}$$

Now, if

$$(x_0, y_0) \in [0, \theta^*]^2 \setminus [0, \vartheta]^2 \tag{3.44}$$

holds, then we may put $R_1 := \max\{x_0, y_0\}$; for instance, if $x_0 > y_0$, then $\rho_1 = x_0 = R_1$ and $\rho_2 = y_0 \leq R_1$. On the other hand, if (3.43) is not true, then because of condition (H5), there must be a number $h > 0$ sufficiently large and a point (x_1, y_1) satisfying

$$(x_1, y_1) \in [0, \theta^* + h]^2 \setminus [0, \vartheta]^2 \tag{3.45}$$

such that $g_1(x_1, y_1) \geq g_1(x_0, y_0)$. In this case, put $R_1 := \max\{x_1, y_1\}$, with $\vartheta < R_1 \leq \theta^* + h$. We then have that, say, $\rho_1 := \max\{x_1, y_1\}$ and $\rho_2 := \min\{x_1, y_1\}$. We conclude, therefore, that we can always construct a square $[0, R_1] \times [0, R_1]$ with R_1 chosen sufficiently large such that either

$$g_1(x, y) \leq g_1(R_1, \rho_2) \tag{3.46}$$

holds for some $0 \leq \rho_2 \leq R_1$ or

$$g_1(x, y) \leq g_1(\rho_1, R_1) \tag{3.47}$$

holds for some $0 \leq \rho_1 \leq R_1$, and such that R_1 satisfies the inequality

$$R_1 > \vartheta. \tag{3.48}$$

Notice, then, for x, $y \leq R_1$, it follows that if in (3.41) we have that $\rho_1 = R_1$ and $0 \leq \rho_2 \leq R_1$, then for all $(x, y) \in [0, R_1] \times [0, R_1]$ it holds that

$$g_1(x, y) \leq g_1(\rho_1, \rho_2) = g_1(R_1, \rho_2) \leq \eta_1(R_1 + \rho_2) \leq 2\eta_1 R_1, \tag{3.49}$$

where the second-to-last inequality follows from invoking (3.30), which is valid since $R_1 > \vartheta$, whence $R_1 + \rho_2 \geq r_2$. On the other hand, if $0 \leq \rho_1 \leq R_1$ and $\rho_2 = R_1$ in (3.30), then inequality (3.49) still holds. Inequality (3.49) is the key observation, for we observe that if $\|(x, y)\| = R_1$, then

$$g_1(x(t), y(t)) \leq 2\eta_1 R_1 \tag{3.50}$$

holds for $t \in [0, 1]$. Since, by assumption, $g_2(x, y) \leq g_1(x, y)$ for each x, $y \geq 0$, it also follows that for $\|(x, y)\| = R_1$ the inequality

$$g_2(x(t), y(t)) \leq 2\eta_1 R_1 \tag{3.51}$$

holds.

So, let R_1 be the number constructed in the previous paragraph. Define the set Ω_{R_1} by

$$\Omega_{R_1} := \{(x, y) \in \mathcal{X} \ : \ \|(x, y)\| < R_1\}. \tag{3.52}$$

Then for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{R_1}$ we find that

$$
\begin{aligned}
(3.53)\quad & \big\|T_1(x,y)\big\| \\
& \le H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) + \lambda_1 \int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\,\mathrm{d}s \\
& \le \Big| H_1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right)\,\mathrm{d}s + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\
& - \kappa_0^1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right)\Big| + \kappa_0^1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\
& + \lambda_1 \int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\,\mathrm{d}s \\
& \le \varepsilon_3^1 \|(x,y)\| + \kappa_0^1\left(\phi_1(x) + \varepsilon_0^1 x\left(\xi_0^1\right) + \varepsilon_0^2 y\left(\xi_0^1\right)\right) \\
& + \lambda_1 \int_0^1 G(s,s)a_1(s)2\eta_1 R_1\,\mathrm{d}s \\
& \le \varepsilon_3^1\big\|(x,y)\big\| + \kappa_0^1\left(\varepsilon_1^1\|x\| + \varepsilon_0^1\|x\| + \varepsilon_0^2\|y\|\right) + 2\eta_1 R_1 \lambda_1 \int_0^1 G(s,s)a_1(s)\,\mathrm{d}s \\
& \le \varepsilon_3^1\big\|(x,y)\big\| + \kappa_0^1\left(\varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^1\right)\big\|(x,y)\big\| + \varepsilon_2^1 R_1 \\
& = \left(\varepsilon_3^1 + \kappa_0^1\left(\varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^1\right) + \varepsilon_2^1\right)\big\|(x,y)\big\| < \frac{1}{2}\big\|(x,y)\big\|,
\end{aligned}
$$

where we have used the fact that

$$
(3.54)\qquad 0 \le \varepsilon_3^1 + \kappa_0^1\left(\varepsilon_0^1 + \varepsilon_0^2 + \varepsilon_1^1\right) + \varepsilon_2^1 < \frac{1}{2}
$$

by construction. Similarly, we estimate

$$
(3.55)\qquad \big\|T_2(x,y)\big\| \le \frac{1}{2}\big\|(x,y)\big\|,
$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{R_1}$. Consequently, from (3.53) and (3.55) we conclude that

$$
(3.56)\qquad \big\|S(x,y)\big\| \le \big\|(x,y)\big\|,
$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{R_1}$.

Putting the preceding part of the proof together, we see that may thus invoke Lemma 2.1 to deduce the existence of a function

$$
(3.57)\qquad (x_0,y_0) \in \mathcal{K} \cap \left(\overline{\Omega}_{R_1} \setminus \Omega_{r_1^{**}}\right)
$$

such that $S(x_0,y_0) = (x_0,y_0)$. The functions $x_0(t)$ and $y_0(t)$ from (3.57) represent a positive solution to problem (1.1); in fact, it satisfies the *a priori* bounds

$$
(3.58)\qquad 0 < r_1^{**} \le \|(x_0,y_0)\| \le R_1 < +\infty.
$$

Thus, in particular, we have shown that problem (1.1) has at least one positive solution. And this completes the proof.

Remark 3.7. Although not explicitly stated in either the statement or the proof of Theorem 3.6, it is possible to write an explicit formula for the admissible range of the eigenvalues, λ_1 and λ_2. In particular, put

$$\begin{aligned}\alpha_1 := \frac{1}{\gamma_0}\inf\Bigg\{y\in[0,+\infty)\ :\ g_1\left(z_1,z_2\right)\geq\left[\int_E G\left(t_0,s\right)a_1(s)\,\mathrm{d}s\right]^{-1},\\ \text{for all } z_1+z_2\in[y,+\infty)\Bigg\}\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}\alpha_2 := \frac{1}{\gamma_0}\inf\Bigg\{y\in[0,+\infty)\ :\ g_2\left(z_1,z_2\right)\geq\left[\int_E G\left(t_0,s\right)a_2(s)\,\mathrm{d}s\right]^{-1},\\ \text{for all } z_1+z_2\in[y,+\infty)\Bigg\}.\end{aligned} \tag{3.60}$$

Now, define α_0 by

$$\alpha_0 := \frac{1}{2\gamma_0}\max\left\{\alpha_1,\alpha_2\right\}. \tag{3.61}$$

Then it follows that whenever

$$\lambda_1,\ \lambda_2\in[\alpha_0,+\infty) \tag{3.62}$$

we have that the pair λ_1, λ_2 is a pair of admissible eigenvalues for problem (1.1). In particular, (3.59)–(3.60) demonstrate that the range of admissible eigenvalues for problem (1.1) is explicitly computable.

We next state our second existence theorem, which provides an alternative approach to problem (1.1). Indeed, as intimated in Section 1, here we give up the assumption that H need not be monotone increasing. In return, however, we are able to recover an existence theorem for the *unperturbed* problem (1.1) – i.e., the case in which $\varepsilon_0^1=\varepsilon_0^2=0$. Moreover, we may still retain the other upshots of Theorem 3.6 such as the fact that the measures μ_{α_1} and μ_{α_2} are possibly signed and that H need only be asymptotically sublinear.

Theorem 3.8. *Suppose that conditions* (H1)–(H9) *hold but with $\kappa_0^i=0$ for each i in condition* (H4). *In addition, assume that each of $H_1(z)$ and $H_2(z)$ is a monotone increasing function for all $z\geq0$. Let $\varepsilon_0^1=\varepsilon_0^2=0$. Then problem* (1.1) *has at least one positive solution.*

Proof. As in the proof of Theorem 3.6, the operator S is completely continuous and satisfies $S(\mathcal{K})\subseteq\mathcal{K}$. So, since these facts still hold, we need only show that S has at least one nontrivial fixed point in $\mathcal{K}$.

To this end, observe that the first part of the proof of Theorem 3.6 may be repeated *verbatim* in spite of the fact that $\varepsilon_0^1=\varepsilon_0^2=0$ here. Indeed, this is because estimate (3.19) was not used in the first part of the proof of Theorem 3.6

but rather only in the second part. In any case, in the same exact way as in the proof of Theorem 3.6, we arrive at a number r_1^{**} such that inequality (3.28) holds for each $(x, y) \in \mathcal{K} \cap \partial\Omega_{r_1^{**}}$, provided that the numbers λ_1, λ_2 are chosen sufficiently large, say according to (3.62).

We next diverge somewhat with respect to the proof of Theorem 3.6. Indeed, because each of H_1 and H_2 is monotone increasing by assumption, by means of assumption (H3) we may estimate both

$$H_1\left(\phi_1(x)\right) \le H_1\left(\varepsilon_1^1\|x\|\right) \le H_1\left(\varepsilon_1^1\|(x,y)\|\right) \tag{3.63}$$

and

$$H_2\left(\phi_2(y)\right) \le H_2\left(\varepsilon_1^2\|y\|\right) \le H_2\left(\varepsilon_1^2\|(x,y)\|\right), \tag{3.64}$$

for each each $(x, y) \in \mathcal{K}$. Next, as in the proof of Theorem 3.6, we may assume that inequalities (3.30)–(3.31) hold whenever $x+y \ge r_2$. Moreover, by assumption (H4) with $\kappa_0^i = 0$, there is a number $r_2^* > 0$ sufficiently large such that, for each i,

$$H_i(z) \le z, \tag{3.65}$$

provided that $z \ge r_2^*$. Now, define the number r_2^{**} by

$$r_2^{**} := \max\left\{2r_1^{**}, r_2, \frac{r_2^*}{\min\left\{\varepsilon_1^1, \varepsilon_1^2\right\}}\right\}. \tag{3.66}$$

Note that for $\|(x, y)\| = r_2^{**}$, by means of (3.63) and (3.65) we may thus estimate

$$H_1\left(\phi_1(x)\right) \le H_1\left(\varepsilon_1^1\|(x,y)\|\right) = H_1\left(\varepsilon_1^1 r_2^{**}\right) \le \varepsilon_1^1 r_2^{**} \tag{3.67}$$

since

$$\varepsilon_1^1 r_2^{**} \ge \frac{\varepsilon_1^1 r_2^*}{\min\left\{\varepsilon_1^1, \varepsilon_1^2\right\}} \ge r_2^*. \tag{3.68}$$

Reasoning similarly, we also deduce the estimate

$$H_2\left(\phi_2(y)\right) \le \varepsilon_1^2 r_2^{**} \tag{3.69}$$

whenever $\|(x, y)\| = r_2^{**}$. Finally, we may assume that r_2^{**} is chosen sufficiently large such that inequality (3.50) and hence inequality (3.51) hold for the number r_2^{**}.

Now, define the set $\Omega_{r_2^{**}} \subseteq \mathfrak{X}$ by

$$\Omega_{r_2^{**}} := \left\{(x, y) \in \mathfrak{X} \ : \ \|(x, y)\| < r_2^{**}\right\}. \tag{3.70}$$

Then, for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$, similar to inequality (3.53) we estimate

$$\begin{aligned} \|T_1(x,y)\| &\le H_1\left(\phi_1(x)\right) + \lambda_1 \int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\,\mathrm{d}s \\ &\le \varepsilon_1^1 r_2^{**} + \lambda_1 \int_0^1 G(s,s)a_1(s)g_1(x(s),y(s))\,\mathrm{d}s \\ &\le \varepsilon_1^1 r_2^{**} + 2\eta_1 r_2^{**}\lambda_1 \int_0^1 G(s,s)a_1(s)\,\mathrm{d}s \\ &\le \varepsilon_1^1 r_2^{**} + \varepsilon_2^1 r_2^{**} = \left(\varepsilon_1^1 + \varepsilon_2^1\right)\|(x,y)\| \le \frac{1}{2}\|(x,y)\|. \end{aligned} \tag{3.71}$$

In a completely similar manner, we deduce that

$$\|T_2(x,y)\| \le \frac{1}{2}\|(x,y)\|, \tag{3.72}$$

whence

$$\|S(x,y)\| \le \|(x,y)\|, \tag{3.73}$$

for each $(x,y) \in \mathcal{K} \cap \partial\Omega_{r_2^{**}}$.

Consequently, we may invoke Lemma 2.1 to deduce the existence of a fixed point $(x_0,y_0) \in \mathcal{K} \cap \left(\overline{\Omega}_{r_2^{**}} \setminus \Omega_{r_1^{**}}\right)$ of the operator S. And this completes the proof.

Remark 3.9. We note that inequality (3.71) reveals that the slightly weaker condition

$$0 \le \max\left\{\varepsilon_1^1, \varepsilon_1^2\right\} < \frac{1}{2} \tag{3.74}$$

may replace the slightly stronger hypothesis (H7) in the statement of Theorem 3.8. In this way, it is unnecessary to assume that $\varepsilon_1^1 + \varepsilon_1^2 \in \left[0, \frac{1}{2}\right)$ since as long as inequality (3.74) holds, we may always choose $\varepsilon_2^i > 0$ sufficiently small such that $\varepsilon_1^i + \varepsilon_2^i \in \left(0, \frac{1}{2}\right]$. However, we omit the statement of this slightly more general result.

We conclude with an explicit numerical example, which explicates the use of Theorem 3.6, together with some final remarks.

Example 3.10. Consider the boundary value problem

$$\begin{aligned} x''(t) &= -\lambda_1\left(e^t - 1\right)\left(\sqrt{x+y} + 2\right) \\ y''(t) &= -\lambda_2\left(t^2 + 1\right)\sqrt{x+y} \\ x(0) &= H_1\left(\phi_1(x) + \frac{1}{200}x\left(\frac{2}{5}\right) + \frac{1}{30}y\left(\frac{2}{5}\right)\right) \\ y(0) &= H_2\left(\phi_2(y) + \frac{1}{200}x\left(\frac{2}{5}\right) + \frac{1}{30}y\left(\frac{2}{5}\right)\right) \\ x(1) &= 0 = y(1), \end{aligned} \tag{3.75}$$

where we make the following declarations:

$$\begin{aligned}
\phi_1(x) &:= \frac{1}{60}x\left(\frac{1}{3}\right) - \frac{1}{200}x\left(\frac{2}{5}\right) - \frac{1}{120}x\left(\frac{3}{5}\right) + \frac{1}{20}\int_{\left[\frac{13}{20},\frac{3}{4}\right]} x(s)\,\mathrm{d}s\\
\phi_2(y) &:= \frac{1}{6}y\left(\frac{3}{10}\right) - \frac{1}{30}y\left(\frac{2}{5}\right) - \frac{1}{15}y\left(\frac{11}{20}\right) + \frac{2}{3}\int_{\left[\frac{3}{5},\frac{7}{10}\right]} y(s)\,\mathrm{d}s\\
H_1(z) &:= z\cos\left(\frac{1}{z+1}\right)\\
H_2(z) &:= z^{\frac{1}{3}} + z.
\end{aligned} \tag{3.76}$$

Obviously, each of H_1 and H_2 satisfies condition (H4) with $\kappa_0^1 = \kappa_0^2 = 1$. Moreover, it is clear that each of $g_1(x,y) := \sqrt{x+y}+2$ and $g_2(x,y) := \sqrt{x+y}$ satisfies conditions (H5), (H6), and (H9). Incidentally, we remark that if we define $\alpha_1,\ \alpha_2\ :\ \mathbb{R}\to\mathbb{R}$ by

$$\alpha_1(t) := \begin{cases} 0, & t < \frac{1}{3}\\ \frac{1}{60}, & \frac{1}{3} \le t < \frac{2}{5}\\ \frac{7}{600}, & \frac{2}{5} \le t < \frac{3}{5}\\ \frac{1}{300}, & \frac{3}{5} \le t < \frac{13}{20}\\ t - \frac{97}{150}, & \frac{13}{20} \le t < \frac{3}{4}\\ \frac{31}{300}, & t \ge \frac{3}{4}\end{cases} \tag{3.77}$$

and

$$\alpha_2(t) := \begin{cases} 0, & t < \frac{3}{10}\\ \frac{1}{6}, & \frac{3}{10} \le t < \frac{2}{5}\\ \frac{2}{15}, & \frac{2}{5} \le t < \frac{11}{20}\\ \frac{1}{15}, & \frac{11}{20} \le t < \frac{3}{5}\\ t - \frac{8}{15}, & \frac{3}{5} \le t < \frac{7}{10}\\ \frac{1}{6}, & t \ge \frac{7}{10}\end{cases}, \tag{3.78}$$

then we may write

$$\phi_1(x) := \int_{[0,1]} x(s)\,\mathrm{d}\alpha_1(s) \quad \text{and} \quad \phi_2(y) := \int_{[0,1]} y(s)\,\mathrm{d}\alpha_2(s), \tag{3.79}$$

where the unique Borel measures associated to the Lebesgue-Stieltjes integrals in (3.79) are

$$\begin{aligned}\mu_{\alpha_1}((-\infty,t]) :=& \frac{1}{60}\delta_{\frac{1}{3}}((-\infty,t]) - \frac{1}{200}\delta_{\frac{2}{5}}((-\infty,t]) \\ &- \frac{1}{120}\delta_{\frac{3}{5}}((-\infty,t]) + \frac{1}{20}m\left((-\infty,t]\cap\left(\frac{13}{20},\frac{3}{4}\right)\right)\end{aligned} \tag{3.80}$$

and

$$\begin{aligned}\mu_{\alpha_2}((-\infty,t]) :=& \frac{1}{6}\delta_{\frac{3}{10}}((-\infty,t]) - \frac{1}{30}\delta_{\frac{2}{5}}((-\infty,t]) \\ &- \frac{1}{15}\delta_{\frac{11}{20}}((-\infty,t]) + \frac{2}{3}m\left((-\infty,t]\cap\left(\frac{3}{5},\frac{7}{10}\right)\right),\end{aligned} \tag{3.81}$$

respectively. Importantly, we observe that each of the measures μ_{α_1} and μ_{α_2} is *signed*.

Now, it is easy to check numerically that condition (H8) holds. Furthermore, we may select $\varepsilon_1^1 := \frac{7}{200}$ here and $\varepsilon_1^2 := \frac{1}{3}$ here; for instance, we observe both that

$$\phi_1(x) \leq \frac{1}{60}\|x\| + \frac{1}{200}\|x\| + \frac{1}{120}\|x\| + \frac{1}{20}\left[\frac{3}{4} - \frac{13}{20}\right]\|x\| \leq \frac{7}{200}\|x\| \tag{3.82}$$

and that

$$\phi_2(y) \leq \frac{1}{6}\|y\| + \frac{1}{30}\|y\| + \frac{1}{15}\|y\| + \frac{2}{3}\left[\frac{7}{10} - \frac{3}{5}\right]\|y\| \leq \frac{1}{3}\|y\|. \tag{3.83}$$

Since, in addition, $\varepsilon_0^1 = \frac{1}{200}$ and $\varepsilon_0^2 = \frac{1}{30}$, it follows that condition (H7) holds, too. Moreover, we find that

$$\int_{[0,1]} 1 - t\,\mathrm{d}\alpha_1(t) = \frac{87}{900} \quad \text{and} \quad \int_{[0,1]} 1 - t\,\mathrm{d}\alpha_2(t) = \frac{9}{100}. \tag{3.84}$$

Since the remaining conditions clearly hold, it follows that problem (3.75) has at least one positive solution. Finally, we remark that problem (3.75) may be recast in the form

$$\begin{aligned}
&x''(t) = -\lambda_1\left(e^t - 1\right)\left(\sqrt{x+y} + 2\right)\\
&y''(t) = -\lambda_2\left(t^2 + 1\right)\sqrt{x+y}\\
&x(0) = \left[\psi_1(x) + \frac{1}{30}y\left(\frac{2}{5}\right)\right]\cos\left(\frac{1}{1 + \psi_1(x) + \frac{1}{30}y\left(\frac{2}{5}\right)}\right)\\
&y(0) = \left[\psi_2(y) + \frac{1}{200}x\left(\frac{2}{5}\right)\right]^{\frac{1}{3}} + \left[\psi_2(y) + \frac{1}{200}x\left(\frac{2}{5}\right)\right]\\
&x(1) = 0 = y(1),
\end{aligned} \tag{3.85}$$

if we put

$$\psi_1(x) := \frac{1}{60}x\left(\frac{1}{3}\right) - \frac{1}{120}x\left(\frac{3}{5}\right) + \frac{1}{20}\int_{\left[\frac{13}{20},\frac{3}{4}\right]} x(s)\,\mathrm{d}s \tag{3.86}$$

and

$$\psi_2(y) := \frac{1}{6}y\left(\frac{3}{10}\right) - \frac{1}{15}y\left(\frac{11}{20}\right) + \frac{2}{3}\int_{\left[\frac{3}{5},\frac{7}{10}\right]} y(s)\,\mathrm{d}s. \tag{3.87}$$

Remark 3.11. We note that problem (3.75) could not be addressed by any existing results in the literature. This is due to several reasons, among which are the following. Firstly, as (3.80)–(3.81) demonstrate, each of the measures μ_{α_1} and μ_{α_2} is signed; contenting ourselves with the papers on systems with nonlocal, nonlinear boundary conditions, this removes from straightforward modification the results of [**8, 9, 17**]. Secondly, since

$$H_2'(z) = \frac{1}{3}z^{-\frac{2}{3}} + 1, \tag{3.88}$$

it is clear that there is no $\beta \in \mathbb{R}$ satisfying $+\infty > \beta > 0$ such that $H_2(z) \leq \beta z$, for all $z \geq 0$. Thus, in particular, the results of [**8**] (and related works) cannot straightforwardly modified. In summary, the fact that the measures are signed rather than positive and that H_2 does not satisfy uniform linear growth seems to remove from consideration any simple modification of the existing results in the literature.

Remark 3.12. We have elected not to give an explicit example of Theorem 3.8. However, we emphasize that this theorem recovers at least one positive solution to the *unperturbed* problem, namely

$$\begin{aligned} x''(t) &= -\lambda_1 a_1(t) g_1(x(t), y(t)),\ t \in (0,1)\\ y''(t) &= -\lambda_2 a_2(t) g_2(x(t), y(t)),\ t \in (0,1)\\ x(0) &= H_1\left(\phi_1(x)\right)\\ y(0) &= H_2\left(\phi_2(y)\right)\\ x(1) &= 0 = y(1). \end{aligned} \tag{3.89}$$

It ought to be noted that problem (3.89) is very nearly the problem studied by Infante and Pietramala [**8**] as well as Yang [**17**]. Consequently, we have here obtained a direct generalization and improvement of their results.

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