

**CHARACTERIZATIONS OF PARTIAL METRIC  
COMPLETENESS IN TERMS OF WEAKLY  
CONTRACTIVE MAPPINGS HAVING FIXED POINT**

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We characterize both complete and 0-complete partial metric spaces in terms of weakly contractive mappings having a fixed point. Our results extend a well-known characterization of metric completeness due to SUZUKI and TAKAHASHI to the partial metric framework.

**1. INTRODUCTION AND PRELIMINARIES**

Throughout this paper the letters  $\mathbb{N}$  and  $\omega$  will denote the set of positive integer numbers and the set of non-negative integer numbers, respectively.

The problem of characterizing complete metric spaces by means of fixed point theorems has been discussed by many authors (see [16, 20, 21, 28, 29, 30, 34], etc). In particular, KIRK [16] solved this problem in terms of the celebrated Caristi fixed point theorem [7]. However, the Banach contraction principle does not characterize metric completeness: indeed, CONELL gave in [10] an example of a non-complete metric space for which every contraction has a fixed point. Despite this fact, SUZUKI and TAKAHASHI obtained in [30] a generalization of Banach's contraction principle that characterizes metric completeness by replacing in the contraction condition the (complete) metric by a certain w-distance. Recall that the notion of w-distance on a metric space was introduced in [14], where the authors improved the Caristi-Kirk fixed point theorem [8], the Ekeland variational principle [11], and the nonconvex minimization theorem [31], for w-distances.

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Several authors have recently contributed to a vigorous development of the theory of fixed point for some classes of generalized metric spaces, as cone metric spaces, quasi-metric spaces and partial metric spaces (see [1, 2, 3, 4, 5, 6, 9, 13, 15, 17, 18, 23, 26], etc.). In particular, ROMAGUERA [23], and ACAR, ALTUN and ROMAGUERA [2], have obtained characterizations of 0-complete and complete partial metric spaces, respectively, in the style of the aforementioned Kirk characterization of metric completeness.

The purpose of this note is to show that both 0-complete and complete partial metric spaces can be also characterized by means of generalizations of the Banach contraction principle by using appropriate notions of w-distance in this context. Thus, our results extend Suzuki and Takahashi's characterization of metric completeness to the partial metric framework.

Let us recall that partial metric spaces were introduced by MATTHEWS [19] in the context of his studies of denotational semantics of dataflow networks. Since then partial metric spaces have turned into a very efficient tool in constructing computational models for metric spaces and other related structures via domain theory (see [12, 22, 24, 25, 27, 32, 33], etc.).

In the rest of this section we give some well-known properties and facts in partial metric spaces, which will be useful later on. A greater part of them may be found in [19].

A partial metric on a set  $X$  is a function  $p : X \times X \rightarrow [0, \infty)$  such that for all  $x, y, z \in X$  :

(i)  $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$ ; (ii)  $p(x, x) \leq p(x, y)$ ; (iii)  $p(x, y) = p(y, x)$ ; (iv)  $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ .

A partial metric space is a pair  $(X, p)$  such that  $X$  is a set and  $p$  is a partial metric on  $X$ .

Each partial metric  $p$  on a set  $X$  induces a  $T_0$  topology  $\tau_p$  on  $X$  which has as a base the family open  $p$ -balls  $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ , where  $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$  for all  $x \in X$  and  $\varepsilon > 0$ . If  $p$  is a partial metric on  $X$ , then the function  $p^s : X \times X \rightarrow [0, \infty)$  given by  $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$  is a metric on  $X$ . Moreover, a sequence  $(x_n)_{n \in \mathbb{N}}$  in a partial metric space  $(X, p)$  converges, with respect to  $\tau_{p^s}$ , to a point  $x \in X$  if and only if  $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = \lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$ .

A sequence  $(x_n)_{n \in \mathbb{N}}$  in a partial metric space  $(X, p)$  is called a Cauchy sequence if there exists (and is finite)  $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ . If  $\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0$ , then  $(x_n)_{n \in \mathbb{N}}$  is said to be a 0-Cauchy sequence in  $(X, p)$  [23].  $(X, p)$  is called complete if every Cauchy sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  converges, with respect to  $\tau_p$ , to a point  $x \in X$  such that  $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$ .  $(X, p)$  is called 0-complete [23] if every 0-Cauchy sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  converges, with respect to  $\tau_p$ , to a point  $x \in X$  such that  $p(x, x) = 0$ . It is clear that every 0-Cauchy sequence in  $(X, p)$  is a Cauchy sequence in  $(X, p)$ , and thus, if  $(X, p)$  is complete then it is 0-complete. In p. 3 of [23] it is given an easy example of a 0-complete partial metric space that

is not complete.

Observe that a partial metric space  $(X, p)$  is 0-complete if and only if every 0-Cauchy sequence converges with respect to the topology  $\tau_{p^s}$ , induced by the metric  $p^s$ .

Finally, we recall the following crucial properties which are given in p. 194 of [19].

Let  $(X, p)$  be a partial metric space. Then:

(a) A sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  is a Cauchy sequence in  $(X, p)$  if and only if it is a Cauchy sequence in the metric space  $(X, p^s)$ .

(b)  $(X, p)$  is complete if and only if  $(X, p^s)$  is complete.

## 2. THE RESULTS

According to [14], a w-distance on a metric space  $(X, d)$  is a function  $q : X \times X \rightarrow [0, \infty)$  satisfying the following conditions: (i)  $q(x, z) \leq q(x, y) + q(y, z)$  for all  $x, y, z \in X$ ; (ii)  $q(x, \cdot) : X \rightarrow [0, \infty)$  is lower semicontinuous for  $(X, \tau_d)$  for all  $x \in X$ ; (iii) for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $q(x, y) \leq \delta$  and  $q(x, z) \leq \delta$  imply  $d(y, z) \leq \varepsilon$ .

Obviously, each metric  $d$  on a set  $X$  is a w-distance for the metric space  $(X, d)$ .

A weakly contractive mapping on a metric space  $(X, d)$  is a mapping  $T : X \rightarrow X$  such that there exist a w-distance  $q$  on  $(X, d)$  and  $c \in (0, 1)$  with the property that  $q(Tx, Ty) \leq cq(x, y)$  for all  $x, y \in X$ .

SUZUKI and TAKAHASHI proved in Theorem 4 of [30] the following.

**Theorem 1.** *A metric space  $(X, d)$  is complete if and only if every weakly contractive mapping on  $(X, d)$  has a (unique) fixed point in  $X$ .*

In order to extend Theorem 1 to partial metric spaces, we first adapt the notion of w-distance to the partial metric framework as follows.

Let  $(X, p)$  be a partial metric space. Consider the following conditions for a function  $q : X \times X \rightarrow [0, \infty)$ :

(W1)  $q(x, z) \leq q(x, y) + q(y, z)$  for all  $x, y, z \in X$ ;

(W2)  $q(x, \cdot) : X \rightarrow [0, \infty)$  is lower semicontinuous for  $(X, \tau_{p^s})$  for all  $x \in X$ ;

(W3) for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $q(x, y) \leq \delta$  and  $q(x, z) \leq \delta$  imply  $p(y, z) \leq \varepsilon$ .

(W3<sup>+</sup>) for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $q(x, y) \leq \delta$  and  $q(x, z) \leq \delta$  imply  $p(y, z) \leq \varepsilon + p(y, y)$ .

If  $q$  satisfies conditions (W1), (W2) and (W3), then  $q$  is called a  $w_0$ -distance on  $(X, p)$ . If  $q$  satisfies conditions (W1), (W2) and (W3<sup>+</sup>), then  $q$  is called a w-distance on  $(X, p)$ .

REMARK 1. Since condition (W3) implies condition (W3<sup>+</sup>), it follows that every  $w_0$ -distance on a partial metric space  $(X, p)$  is a  $w$ -distance on  $(X, p)$ .

**Proposition 1.** *Every partial metric  $p$  on a set  $X$  is a  $w_0$ -distance on  $(X, p)$ .*

**Proof.** Let  $p$  be a partial metric on a set  $X$ . Then  $p$  obviously satisfies condition (W1). On the other hand, condition (W2) follows from Lemma 2.2 of [23]. Finally, given  $\varepsilon > 0$ , take  $\delta = \varepsilon/2$ . If  $p(x, y) \leq \delta$  and  $p(x, z) \leq \delta$ , we obtain  $p(y, z) \leq p(y, x) + p(x, z) \leq \varepsilon$ , and hence  $p$  satisfies condition (W3).

REMARK 2. It follows from Remark 1 and Proposition 1 that every partial metric  $p$  on a set  $X$  is a  $w$ -distance on  $(X, p)$ .

REMARK 3. In checking that a metric  $d$  on a set  $X$  is a  $w$ -distance on the metric space  $(X, d)$ , condition (iii) in the definition of  $w$ -distance for a metric space follows as an immediately consequence of the triangle inequality of  $d$ . Similarly, the proof of Proposition 1 above shows that it suffices to use the inequality  $p(y, z) \leq p(y, x) + p(x, z)$  to deduce that the partial metric  $p$  is a  $w_0$ -distance for the partial metric space  $(X, p)$ . However, it is easy to check that given a partial metric space  $(X, p)$ , the function  $q : X \times X \rightarrow [0, \infty)$  defined by  $q(x, y) = p(x, y) - (p(x, x)/2)$ , is also a  $w_0$ -distance on  $(X, p)$ . In this case, it is necessary to use the triangle inequality (iv) in the definition of a partial metric, to verify that  $q$  satisfies condition (W3), as well as conditions (W1) and (W2).

The proof of the next result is immediate, so it is omitted.

**Proposition 2.** *Let  $(X, p)$  a partial metric space. A function  $q : X \times X \rightarrow [0, \infty)$  is a  $w$ -distance on  $(X, p)$  if and only if it is a  $w$ -distance on the metric space  $(X, p^s)$ .*

The following is an example of a  $w$ -distance on a partial metric space  $(X, p)$  which is not  $w_0$ -distance on  $(X, p)$ .

EXAMPLE 1. Let  $X = [0, \infty)$  and let  $p(x, y) = \max\{x, y\}$  for all  $x, y \in X$ . Then  $(X, p)$  is a (complete) partial metric space. By Proposition 2,  $p^s$  is a  $w$ -distance on  $(X, p)$ . However  $p^s$  is not a  $w_0$ -distance on  $(X, p)$ : Indeed, for each  $n \in \mathbb{N}$  take  $x_n = y_n = n$  and  $z_n = n + 1/n$ , then  $p^s(x_n, y_n) = 0$  and  $p^s(x_n, z_n) = 1/n$  for all  $n \in \mathbb{N}$ , but  $p(y_n, z_n) = n + 1/n$ . This shows that  $p^s$  does not satisfy (W3) with respect to  $(X, p)$ .

REMARK 4. The fact that the self-distance in a partial metric space can be different from zero allows us to construct an interesting type of  $w_0$ -distances for certain partial metric spaces, which illustrates the differences between  $w$ -distances for metric spaces and for partial metric spaces. Indeed, let  $X$  be a set such that  $|X| \geq 2$ . If  $d$  is a metric on  $X$ , then the function  $q : X \times X \rightarrow [0, \infty)$  defined by  $q(x, y) = d(y, y)$  (i.e.,  $q(x, y) = 0$ ) for all  $x, y \in X$ , is not  $w$ -distance for  $(X, d)$  because, clearly, condition (iii) in the definition of a  $w$ -distance for metric spaces is not satisfied. However, if  $p$  is a partial metric on  $X$  such that, for each  $x, y \in X$ ,  $p(x, y) \leq p(x, x) + p(y, y)$ , then it is not hard to prove that the function  $q : X \times X \rightarrow [0, \infty)$  defined by  $q(x, y) = p(y, y)$  is a  $w_0$ -distance, and hence a  $w$ -distance, on  $(X, p)$ . Instances of this kind of partial metric spaces are the partial metric space of Example 1, and the partial metric space  $(\mathbb{N}, p)$  where  $p(n, n) = 1/n$  and  $p(n, m) = (1/n) + (1/m)$  if  $n \neq m$ ,  $n, m \in \mathbb{N}$ .

Generalizing in a natural way the notion of weakly contractive mapping on a metric space, we say that a selfmapping  $T$  on a partial metric space  $(X, p)$  is a weakly contractive (resp. 0-weakly contractive) mapping on  $(X, p)$  if there exist a w-distance (resp.  $w_0$ -distance)  $q$  on  $(X, p)$  and  $c \in (0, 1)$  with the property that  $q(Tx, Ty) \leq cq(x, y)$  for all  $x, y \in X$ . Therefore, from Proposition 2 we deduce the following.

**Proposition 3.** *A selfmapping  $T$  on a partial metric space  $(X, p)$  is a weakly contractive mapping if and only if it is a weakly contractive mapping on the metric space  $(X, p^s)$ .*

**Theorem 2.** *A partial metric space  $(X, p)$  is complete if and only if every weakly contractive mapping on  $(X, p)$  has a (unique) fixed point in  $X$ .*

**Proof.** Since  $(X, p)$  is complete if and only if  $(X, p^s)$  is complete, then the result follows from Theorem 1 and Proposition 3.  $\square$

Next we give an example of a weakly contractive mapping on a 0-complete non-complete partial metric space  $(X, p)$  that has no fixed point in  $X$ .

EXAMPLE 2. Let  $X = (1, \infty)$  and let  $p$  the partial metric on  $X$  defined by  $p(x, y) = \max\{x, y\}$  for all  $x, y \in X$ . Clearly  $(X, p)$  is a 0-complete non-complete partial metric space. Now let  $T : X \rightarrow X$  defined by  $Tx = (x + 1)/2$ . Then  $T$  has no fixed point in  $X$ . However, we have

$$p^s(Tx, Ty) = |Tx - Ty| = \frac{1}{2}|x - y| = \frac{1}{2}p^s(x, y),$$

for all  $x, y \in X$ , so  $T$  is a weakly contractive mapping on  $(X, p)$  because  $p^s$  is a w-distance on  $(X, p)$  by Proposition 2.

Our next result shows that the mapping  $T$  of Example 2 is not 0-weakly contractive on the 0-complete partial metric space  $(X, p)$  of Example 2.

**Theorem 3.** *A partial metric space  $(X, p)$  is 0-complete if and only if every 0-weakly contractive mapping on  $(X, p)$  has a (unique) fixed point in  $X$ .*

**Proof.** Suppose that  $(X, p)$  is 0-complete and let  $T$  be a 0-weakly contractive mapping on  $(X, p)$ . Then, there exist a  $w_0$ -distance  $q$  on  $(X, p)$  and a positive constant  $c < 1$ , such that

$$q(Tx, Ty) \leq cq(x, y),$$

for all  $x, y \in X$ . We shall prove that  $T$  has a unique fixed point in  $X$ . Indeed, fix  $x_0 \in X$ , and let  $x_n = T^n x_0 = T x_{n-1}$  for all  $n \in \mathbb{N}$ . Then, we have

$$q(x_n, x_{n+1}) = q(T^n x_0, T^{n+1} x_0) \leq cq(T^{n-1} x_0, T^n x_0) \leq \dots \leq c^n q(x_0, x_1).$$

for all  $n \in \mathbb{N}$ . Therefore

$$q(x_n, x_{n+k}) \leq [c^n + \dots + c^{n+k-1}] q(x_0, x_1) \leq \frac{c^n}{1-c} q(x_0, x_1)$$

for all  $n, k \in \mathbb{N}$ . Given  $\varepsilon > 0$  let  $\delta = \delta(\varepsilon)$  for which condition (W3) is satisfied. Since there is  $n_\delta \in \mathbb{N}$  such that  $q(x_{n_\delta}, x_n) < \delta$  and  $q(x_{n_\delta}, x_m) < \delta$  for all  $m, n > n_\delta$ , it follows that  $p(x_n, x_m) < \varepsilon$  for all  $m, n > n_\delta$ , so  $(x_n)_{n \in \omega}$  is a 0-Cauchy sequence in  $(X, p)$ , and thus there is  $z \in X$  such that  $p^s(z, x_n) \rightarrow 0$ .

We show that  $q(x_n, z) \rightarrow 0$ . Indeed, choose an arbitrary  $\varepsilon > 0$ . There is  $n_\varepsilon \in \mathbb{N}$  such that  $q(x_n, x_{n+k}) < \varepsilon$  whenever  $n > n_\varepsilon$  and  $k \in \mathbb{N}$ . By condition (W2), for each  $n > n_\varepsilon$  there exists  $n_k > n$  such that

$$q(x_n, z) < q(x_n, x_{n_k}) + \frac{1}{n},$$

so  $q(x_n, z) < \varepsilon + 1/n$  whenever  $n > n_\varepsilon$ . We conclude that  $q(x_n, z) \rightarrow 0$ . Hence  $q(x_n, Tz) \rightarrow 0$  because

$$q(x_{n+1}, Tz) = q(Tx_n, Tz) \leq cq(x_n, z),$$

for all  $n \in \omega$ . Now it follows from condition (W3) that  $p(z, Tz) = 0$ , i.e.,  $z = Tz$ . Finally, let  $u \in X$  with  $u = Tu$ . Thus

$$q(u, z) = q(Tu, Tz) \leq cq(u, z),$$

so  $q(u, z) = 0$ . Again, by condition (W3),  $p(u, z) = 0$ , i.e.,  $u = z$ .

To prove the converse we shall adapt the technique of the “if” part of the proof of Theorem 4 of [30]. Suppose that  $(X, p)$  is not 0-complete. Then, there exists a 0-Cauchy sequence  $(x_n)_{n \in \mathbb{N}}$  in  $(X, p)$  which does not converges for  $\tau_{p^s}$ . We may assume, without loss of generality, that  $p(x_n, x_{n+1}) < 2^{-(n+1)}$  and  $p(x_{n+1}, x_{n+1}) \leq 2p(x_n, x_n)/3$  for all  $n \in \mathbb{N}$ .

Let  $\alpha_n = \lim_{m \rightarrow \infty} p(x_n, x_m)$  for all  $n \in \mathbb{N}$ . It is easy to check that, in fact, each  $\alpha_n$  exists and that  $\alpha_n \rightarrow 0$ . Moreover  $\alpha_n > 0$  for all  $n \in \mathbb{N}$ . Hence, there is a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  of  $(x_n)_{n \in \mathbb{N}}$  such that  $\alpha_{n_k} > 3\alpha_{n_{k+1}}$  for all  $k \in \mathbb{N}$ .

Put  $y_k = x_{n_k}$  for all  $k \in \mathbb{N}$  and  $F = \{y_k : k \in \mathbb{N}\}$ .

Now define a function  $q : X \times X \rightarrow [0, \infty)$  by  $q(x, y) = 1$  if  $x \notin F$  or  $y \notin F$ , and  $q(x, y) = p(x, y)$ , otherwise. We show that  $q$  is a  $w_0$ -distance on  $(X, p)$ . First note that  $q(x, y) \leq 1$  for all  $x, y \in X$ .

To verify condition (W1), take  $x, y, z \in X$ . If  $x, y, z \in F$ , then  $q(x, z) = p(x, z) \leq p(x, y) + p(y, z) = q(x, y) + q(y, z)$ . Otherwise, condition (W1) is trivially satisfied because in that case  $q(x, y) + q(y, z) \geq 1$ . To verify condition (W2), let  $(z_n)_{n \in \mathbb{N}}$  be a sequence in  $X$  that converges to  $z \in X$  for  $\tau_{p^s}$ , and take  $x \in X$ . We may assume, without loss of generality, that  $z_n \notin F$  for all  $n \in \mathbb{N}$ . Then  $q(x, z_n) = 1$ , and thus  $q(x, z) \leq q(x, z_n)$  for all  $n \in \mathbb{N}$ . Consequently  $q(x, \cdot)$  is lower semicontinuous on  $(X, \tau_{p^s})$ . To verify condition (W3), let  $\varepsilon > 0$  be arbitrary. Take  $\delta = \min\{\varepsilon/2, 1/2\}$ . Then, for  $x, y, z \in X$  with  $q(x, z) \leq \delta$  and  $q(x, y) \leq \delta$ , we deduce that  $x, y, z \in F$ , so  $q(x, z) = p(x, z)$  and  $q(x, y) = p(x, y)$ . Therefore  $p(x, z) \leq \varepsilon/2$  and  $p(x, y) \leq \varepsilon/2$ , and thus  $p(y, z) \leq \varepsilon$ .

Next we prove that  $q(Tx, Ty) \leq 2q(x, y)/3$  for all  $x, y \in X$ , where  $Tx = y_1$  if  $x \notin F$  and  $Ty_n = y_{n+1}$  for all  $n \in \mathbb{N}$ . Indeed, since  $q(x, y) = q(y, x)$  for all  $x, y \in X$ , it suffices to discuss the following three cases.

Case 1.  $x, y \notin F$ . Then

$$q(Tx, Ty) = q(y_1, y_1) = p(y_1, y_1) \leq p(y_1, y_2) < \frac{1}{2} = \frac{1}{2}q(x, y) < \frac{2}{3}q(x, y).$$

Case 2.  $x \notin F$  and  $y \in F$ . Then  $y = y_k$  for some  $k \in \mathbb{N}$ , so

$$\begin{aligned} q(Tx, Ty) &= q(y_1, y_{k+1}) = p(y_1, y_{k+1}) \\ &\leq \sum_{n=1}^k p(y_n, y_{n+1}) < \sum_{n=1}^k 2^{-(n+1)} < \frac{1}{2} = \frac{1}{2}q(x, y) < \frac{2}{3}q(x, y). \end{aligned}$$

Case 3.  $x, y \in F$ . Then  $x = y_j, y = y_k$  for some  $j, k \in \mathbb{N}$ , and assume, without loss of generality, that  $j < k$ . Since

$$p(y_j, y_k) \geq p(y_j, y_n) - p(y_n, y_k),$$

for all  $n \in \mathbb{N}$ , it follows, taking limits as  $n \rightarrow \infty$ , that

$$p(y_j, y_k) \geq \alpha_{n_j} - \alpha_{n_k} \geq \alpha_{n_j} - \alpha_{n_{j+1}} \geq 2\alpha_{n_{j+1}}.$$

Moreover, since

$$p(y_{j+1}, y_{k+1}) \leq p(y_{j+1}, y_n) + p(y_n, y_{k+1}),$$

for all  $n \in \mathbb{N}$ , it follows, taking limits as  $n \rightarrow \infty$ , that

$$p(y_{j+1}, y_{k+1}) \leq \alpha_{n_{j+1}} + \alpha_{n_{k+1}} \leq \alpha_{n_{j+1}} + \alpha_{n_{j+2}} < \frac{4}{3}\alpha_{n_{j+1}}.$$

Therefore

$$\begin{aligned} q(Tx, Ty) &= q(Ty_j, Ty_k) = q(y_{j+1}, y_{k+1}) = p(y_{j+1}, y_{k+1}) \\ &< \frac{4}{3}\alpha_{n_{j+1}} \leq \frac{4}{3} \cdot \frac{1}{2}p(y_j, y_k) = \frac{2}{3}q(x, y). \end{aligned}$$

We have shown that  $T$  is a 0-weakly contractive mapping on  $(X, p)$  without fixed point in  $X$ . This concludes the proof.  $\square$

We finish the paper with two examples that illustrate Theorems 2 and 3, respectively.

EXAMPLE 3. Let  $X = [0, 1]$  and let  $p$  be the partial metric on  $X$  defined by  $p(x, y) = \max\{x, y\}$  for all  $x, y \in X$ . Then  $(X, p)$  is a complete partial metric space. Let  $T : X \rightarrow X$  be the mapping defined by  $T1 = 0$  and  $Tx = x/2$  otherwise. Then  $T$  is weakly contractive with the  $w$ -distance  $q$  on  $X$  defined by  $q(1, 1) = 0$  and  $q(x, y) = e^{\max\{x, y\}} - 1$  otherwise. Indeed, condition (W1) is clearly satisfied. Since  $p^s$  is the usual metric on  $X$ , it easily follows that  $q(x, \cdot)$  is lower semicontinuous for  $(X, \tau_{p^s})$ . Therefore, condition (W2) is satisfied. Finally, for each  $\varepsilon > 0$ , we choose  $\delta = e^{\varepsilon/2} - 1 > 0$ ; thus, for  $x, y \in X$ , with  $x < 1$  or  $y < 1$ , we have  $q(x, y) = e^{\max\{x, y\}} - 1 \leq \delta$  and  $q(x, z) = e^{\max\{x, z\}} - 1 \leq \delta$  imply  $p(y, z) = \max\{y, z\} \leq \varepsilon$ ; and for  $x = y = 1$ , we have  $q(x, y) = 0$  and  $p(x, y) \leq p(x, y) + \varepsilon$ .

Thus, condition  $(W3^+)$  is satisfied on  $(X, p)$  (note that, in particular, condition  $(W3)$  is satisfied on  $(X \setminus \{1\}, p)$ ). Hence  $q$  is a  $w$ -distance on  $(X, p)$ . (It is interesting to observe that  $(X, q)$  is, in fact, a non-complete partial metric space.)

Now, for each  $x, y \in X$  with  $y \leq x < 1$  we have

$$\begin{aligned} q(Tx, Ty) &= e^{\max\{x/2, y/2\}} - 1 = e^{x/2} - 1 \leq e^{x/2} \cosh\left(\frac{x}{2}\right) - 1 \\ &= e^{x/2} \left( \frac{e^{x/2} + e^{-x/2}}{2} \right) - 1 = \frac{e^x + 1}{2} - 1 = \frac{e^x - 1}{2} = \frac{1}{2}q(x, y), \end{aligned}$$

whereas  $q(T1, T1) = 0$ , and for  $x = 1 > y$ ,

$$q(T1, Ty) = q(0, \frac{y}{2}) = e^{y/2} - 1 < \frac{1}{2}(e - 1) = \frac{1}{2}q(1, y).$$

We have shown that  $T$  is a weakly contractive mapping on  $(X, p)$ . Therefore, conditions of Theorem 2 are satisfied.

**EXAMPLE 4.** Let  $X = [0, \infty) \cap \mathbb{Q}$ , where by  $\mathbb{Q}$  we denote the set of rational numbers, and let  $p$  be the partial metric on  $X$  defined by  $p(x, y) = \max\{x, y\}$  for all  $x, y \in X$ . Then  $(X, p)$  is a 0-complete non-complete partial metric space. Let  $c \in (0, 1)$  and  $T : X \rightarrow X$  be the mapping defined by  $Tx = cx$ . Now let  $q : X \times X \rightarrow [0, \infty)$  defined by  $q(x, y) = y$  for all  $x, y \in X$ . Then, compare Remark 4,  $q$  is a  $w_0$ -distance on  $(X, p)$ . Moreover  $T$  is 0-weakly contractive for  $q$  because for each  $x, y \in X$ , we have  $q(Tx, Ty) = Ty = cy = cq(x, y)$ . Therefore, conditions of Theorem 3 are satisfied. Note that we can not apply Theorem 2 to this example because  $(X, p)$  is not complete.

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