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ON THE SPECTRUM OF THE FINITE LAPLACE TRANSFORM WITH SOME APPLICATIONS

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This paper is devoted to the computation of the spectrum of the finite Laplace transform (FLT) and its applications. For this purpose, we give two different practical methods. The first one uses a discretization of the FLT. The second one is based on the Gaussian quadrature method. The spectrum of the FLT is then used to invert the Laplace transform of time limited functions as well as the Laplace transform of essentially time limited functions. Several numerical results are given to illustrate the results of this work.

1. INTRODUCTION

The finite Laplace transform (FLT) defined as

$$\mathcal{L}_{0,b}f(x) = \int_0^b e^{-xy} f(y)\mathrm{d}y, \quad b > 0,$$

plays an important role in solving boundary value problems for ordinary and partial differential equations, see [**7**]. It is also used to solve a weakly singular integral equation in transfer theory, see [**18**]. Several applications of this transform in linear control problems have been studied in [**15**]. The finite Laplace transform was studied first by Debnath and Thomas in [**8**]. They have given some properties of the FLT. Some other properties can be found in [**20**]. In [**4**], the authors have considered the following integral transform

$$\mathcal{L}_{a,b}f(x) = \int_a^b e^{-xy} f(y)\mathrm{d}y, \tag{1}$$

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from $L^2[a, b]$ to $L^2([0, +\infty[)$ which is also the finite Laplace transform. Here $0 < a < b$ are two positive real numbers. They have proved that the differential operator $\widetilde{D}$ given by

$$\widetilde{D}f = -\left((x^2-1)(\alpha^2-x^2)f'\right)' + 2(x^2-1)f$$

commutes with the Stieltjes transform

$$\mathcal{S}_\alpha f(x) = (\mathcal{L}_{a,b})^* \mathcal{L}_{a,b} f(x) = \int_1^\alpha \frac{f(y)}{x+y} \mathrm{d}y,$$

where $\alpha = \dfrac{b}{a}$. In [**3**], the authors have studied the spectral properties of $\widetilde{D}$.

Note here that the parameter a, introduced in (1), cannot be equal to 0. In this paper, we complete the work given in [**3**]. We suppose that $a = 0$, we study the spectrum of the operator $\mathcal{L}_{0,b}$ defined from $L^2[0, b]$ into itself and we give some of its applications. The eigenfunctions of FLT and their corresponding eigenvalues are computed by two methods. The first one is based on the discretization of $\mathcal{L}_{0,b}$ following a suitable set of orthogonal polynomials while the second method is based on the use of a Gaussian quadrature method. Finally, we use such eigenfunctions and eigenvalues to invert the finite Laplace transform as well as the Laplace transform over the set of essentially time limited functions. In the following, the Laplace transform is given by

$$\mathcal{L}f(x) = \int_0^{+\infty} e^{-xy} f(y) \mathrm{d}y,$$

see [**7**]. Note here that a large number of different methods for the inversion of the Laplace transform can be found in the literature, see for example the extensive list of papers collected in [**16, 17**]. A comparison of methods of inversion of the Laplace transform is given in a survey by Davies and Martin in [**5**].

The outline of this paper is as follows. In section 2, we provide two methods of computation of the eigenfunctions of the finite Laplace transform and their corresponding eigenvalues. The first method is based on the discretization of $\mathcal{L}_{0,b}$. The second method is based on the Gaussian quadrature method. Section 3 is devoted to some applications of the results given in section 2. The first application is the inversion of the finite Laplace transform while the second one is the inversion of the Laplace transform of functions essentially time limited to $[0, b]$. In section 4, we illustrate such methods by several numerical results.

2. ON THE COMPUTATION OF THE SPECTRUM OF THE FINITE LAPLACE TRANSFORM OPERATOR

It is easy to see that $\mathcal{L}_{0,b}$ observed as a map from $L^2[0, b]$ to itself is a Hilbert-Schmidt operator, with the Hilbert-Schmidt norm

$$\|\mathcal{L}_{0,b}\|_{HS} = \left(\int_0^b \int_0^b e^{-2xy} \mathrm{d}x\, \mathrm{d}y\right)^{1/2} \leq b.$$

Let $(\mu_n)_{n\geq 0}$ denote the infinite set of the eigenvalues of $\mathcal{L}_{0,b}$, arranged in the decreasing order of their magnitude, that is

$$|\mu_0| > |\mu_1| > \cdots > |\mu_n| > \cdots$$

In the following, we denote by φ_n the n^{th} eigenfunction of $\mathcal{L}_{0,b}$ associated with μ_n, that is

$$\mathcal{L}_{0,b}(\varphi_n)(x) = \int_0^b e^{-xy}\varphi_n(y)\mathrm{d}y = \mu_n\varphi_n(x), \tag{2}$$

and we adopt the following normalization of φ_n

$$\|\varphi_n\chi_{[0,b]}\|_2 = \left(\int_0^b |\varphi_n(x)|^2\mathrm{d}x\right)^{1/2} = |\mu_n|.$$

Note here that both φ_n and μ_n depend on b, but for simplicity of notation we have omitted this. In accordance with the previous statements, we state the following result about the eigenfunctions of $\mathcal{L}_{0,b}$.

Proposition 1. *The set $B = \{\varphi_n, n \in \mathbb{N}\}$ is an orthogonal basis of $L^2[0,b]$.*

The following lemma gives the derivative of μ_n with respect to b.

Lemma 1. *Let ψ_n be the function given by $\psi_n(x) = \varphi_n(bx)$. Then*

$$\frac{\partial\mu_n}{\partial b} = \frac{(\psi_n(1))^2}{\mu_n}. \tag{3}$$

Proof. We adopt the techniques used in [**19**] to prove a similar result for the eigenvalue of the finite Hankel transform. Let us introduce the following changes in the integral equation (2):

$$y = bu, \quad x = bv, \quad \psi_n(x) = \varphi_n(bx),$$

to obtain the equivalent integral equation

$$b\int_0^1 e^{-b^2uv}\psi_n(u)\mathrm{d}u = \mu_n\psi_n(v). \tag{4}$$

By differentiating both members of the previous equality with respect to b, we obtain

$$\int_0^1 e^{-b^2uv}\psi_n(u)\mathrm{d}u - 2b^2v\int_0^1 ue^{-b^2uv}\psi_n(u)\mathrm{d}u + b\int_0^1 e^{-b^2uv}\frac{\partial\psi_n(u)}{\partial b}\mathrm{d}u = \frac{\partial\mu_n}{\partial b}\psi_n(v) + \mu_n\frac{\partial\psi_n(v)}{\partial b}. \tag{5}$$

Differentiating (4) with respect to v, we get

$$-b^3\int_0^1 ue^{-b^2uv}\psi_n(u)\mathrm{d}u=\mu_n\frac{\partial\psi_n(v)}{\partial v}. \tag{6}$$

Using (5) and (6) together with (4), we obtain

$$\frac{\mu_n}{b}\psi_n(v)+2\frac{\mu_n}{b}v\frac{\partial\psi_n(v)}{\partial v}+b\int_0^1 e^{-b^2uv}\frac{\partial\psi_n(u)}{\partial b}\mathrm{d}u=\frac{\partial\mu_n}{\partial b}\psi_n(v)+\mu_n\frac{\partial\psi_n(v)}{\partial b}. \tag{7}$$

Multiplying both sides of (7) by $\psi_n(v)$ and integrating over $(0,1)$, one finds

$$\begin{aligned}\frac{\mu_n}{b}\|\psi_n\chi_{(0,1)}\|_2^2+2\frac{\mu_n}{b}\int_0^1 v\psi_n(v)\frac{\partial\psi_n(v)}{\partial v}\mathrm{d}v+b\int_0^1\psi_n(v)\int_0^1 e^{-b^2uv}\frac{\partial\psi_n(u)}{\partial b}\mathrm{d}u\,\mathrm{d}v\\ =\frac{\partial\mu_n}{\partial b}\|\psi_n\chi_{(0,1)}\|_2^2+\mu_n\int_0^1\psi_n(v)\frac{\partial\psi_n(v)}{\partial b}\mathrm{d}v.\end{aligned} \tag{8}$$

By using Fubini's theorem and (4), the equality (8) can be simply written as follows

$$\frac{\mu_n}{b}\|\psi_n\chi_{(0,1)}\|_2^2+2\frac{\mu_n}{b}I=\frac{\partial\mu_n}{\partial b}\|\psi_n\chi_{(0,1)}\|_2^2, \tag{9}$$

where $I=\int_0^1 v\psi_n(v)\frac{\partial\psi_n(v)}{\partial v}\mathrm{d}v$. Integrating I by parts one can easily check that

$$I=\frac{1}{2}\left((\psi_n(1))^2-\|\psi_n\chi_{(0,1)}\|_2^2\right). \tag{10}$$

Remark also that $\|\psi_n\chi_{(0,1)}\|_2^2=\frac{(\mu_n)^2}{b}$. Thanks to (9) and (10) and the normalization of ψ_n, one obtains (3). □

Let $(P_k)_{k\geq 0}$ be the set of polynomials given by:

$$P_k(x)=\frac{\sqrt{2k+1}}{\sqrt{b^{2k+1}}k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[x^k(x-b)^k\right],\quad k\geq 0. \tag{11}$$

The following properties of $\{P_k,k\in\mathbb{N}\}$ will be used in the following (see [**11**], for example):

P_1 : The set $\{P_k,k\in\mathbb{N}\}$ is an orthonomal basis of $L^2(0,b)$.

P_2 : For all $k\in\mathbb{N}$ we have

$$P_{k+1}(x)=(\alpha_k x+\beta_k)P_k(x)-\gamma_kP_{k-1}(x),$$

where

$$\alpha_k=\frac{2}{b}\frac{\sqrt{(2k+3)(2k+1)}}{k+1},\ \beta_k=\frac{\sqrt{(2k+3)(2k+1)}}{k+1},\ \gamma_k=\frac{k}{k+1}\sqrt{\frac{2k+3}{2k-1}}. \tag{12}$$

Note that from the theory of orthogonal polynomials it follows that $\forall\, n \geq 0$, $P_n(x)$ has n distinct zeros inside $[0, b]$. Moreover, these n different zeros are simply given as the eigenvalues of a tridiagonal symmetric matrix D of order n, given by

$$D = [d_{i,j}]_{1\leq i,j\leq n}\,, \quad d_{i,i} = -\frac{\beta_{i-1}}{\alpha_{i-1}}, \qquad (13)$$
$$d_{i,i+1} = d_{i+1,i} = \frac{-1}{\alpha_{i-1}}, \quad d_{i,j} = 0 \text{ if } j \neq i-1, i, i+1,$$

where the α_i and β_i are given by (12).

2.1. Matrix representation of $\mathcal{L}_{0,b}$

In this section, we follow the techniques introduced in [**10, 11**] to compute the spectrum of the finite Fourier transform. For the sake of completeness of the paper, we describe briefly such techniques. First, we introduce the finite moments

$$M_{ij} = \int_0^b x^i P_j(x)\mathrm{d}x, \quad i, j \in \mathbb{N},$$

of the polynomials given by (11). Using a similar method to [**10**], we can easily get:

(m_1) For $i < j$, $M_{ij} = 0$.

(m_2) For $j \leq i$,

$$M_{ij} = \frac{b^{i+1/2}\sqrt{2j+1}\,(i!)^2}{(i+j)!\,(i-j)!}.$$

(m_3) For $j \leq i$,

$$|M_{ij}| \leq \frac{b^{i+1/2}}{\sqrt{2i+1}}. \qquad (14)$$

To proceed further, we first expand φ_n into its Fourier series following the polynomials P_k

$$\varphi_n(x) = \sum_{k=0}^{+\infty} \eta_k^n P_k(x), \quad x \in [0, b], \qquad (15)$$

where

$$\eta_k^n = \int_0^b \varphi_n(x) P_k(x)\mathrm{d}x. \qquad (16)$$

As in [**10**], the following lemma gives the decay rate of the coefficients η_k^n given in (15).

Lemma 2. *Under the previous notations and assumptions, for any positive integer $k \geq [2eb^2]$, we have the following upper bound of η_k^n:*

$$|\eta_k^n| \leq \frac{be^{b^2}}{\sqrt{2k\pi}} \frac{1}{2^k}. \tag{17}$$

Here $[x]$ denotes the integer part of x.

Proof. By combining (16) and (2) with (14) and Hölder's inequality, we obtain

$$\begin{aligned}
|\eta_k^n| &= \left| \frac{1}{\mu_n} \int_0^b \left(\int_0^b e^{-xy} \varphi_n(y) \mathrm{d}y \right) P_k(x) \,\mathrm{d}x \right| \\
&= \frac{1}{|\mu_n|} \left| \int_0^b \left(\int_0^b \sum_{j\geq 0} \frac{(-1)^j}{j!} x^j y^j \varphi_n(y) \mathrm{d}y \right) P_k(x) \,\mathrm{d}x \right| \\
&\leq \frac{1}{|\mu_n|} \sum_{j\geq 0} \frac{1}{j!} \left| \int_0^b y^j \varphi_n(y) \mathrm{d}y \right| \left| \int_0^b x^j P_k(x) \,\mathrm{d}x \right| \\
&\leq \frac{1}{|\mu_n|} \sum_{j\geq k} \frac{1}{j!} \frac{b^{j+1/2}}{\sqrt{2j+1}} \left(\int_0^b y^{2j} \mathrm{d}y \right)^{1/2} \|\varphi_n \chi_{[0,b]}\|_2 \leq b \sum_{j\geq k} \frac{b^{2j}}{j!} \;\leq\; be^{b^2} \frac{b^{2k}}{k!}.
\end{aligned}$$

From Stirling's formula, [**1**] we have that

$$\Gamma(s+1) \geq \sqrt{2\pi} s^{s+1/2} e^{-s}, \quad \forall\, s > 0,$$

hence (17) follows. □

To proceed further, we need the following lemma

Lemma 3. *Let $k; \ell \geq 0$ be two integers and let b be a positive real number. The coefficients $a_{k\ell} = \langle \mathcal{L}_{0,b}(P_k), P_\ell \rangle$ satisfy the following two conditions:*

(c_1) $a_{k\ell} = \sum_{n\geq \nu} \frac{(-1)^n}{n!} M_{n\ell} M_{nk}$.

(c_2) $|a_{kl}| \leq \frac{1}{\nu!} |M_{\nu\ell} M_{\nu k}|$, *where* $\nu = \max(k, \ell)$.

Proof. Note that the proof of this lemma is based on some techniques similar to those used in [**10**]. We have that

$$\mathcal{L}_{0,b}(P_k)(x) = \int_0^b e^{-xy} P_k(y) \mathrm{d}y = \int_0^b \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n y^n P_k(y) \mathrm{d}y.$$

It is well known, see [**2**], that

$$\max_{x\in[0,b]} \mid P_k(x) \mid \leq \sqrt{\frac{b}{2}}\sqrt{\frac{2k+1}{2}}.$$

Hence, $\forall x \in [0,b]$, the series $\sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n y^n P_k(y)$ converges uniformly in $[0,b]$. Consequently, we have

$$\mathcal{L}_{0,b}(P_k)(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n \int_0^b y^n P_k(y)\mathrm{d}y = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n M_{nk}.$$

Since, for $j < k$, we have $M_{j,k} = 0$, then, for all $k, \ell \geq 0$,

$$a_{k\ell} = \sum_{n\geq\nu} \frac{(-1)^n}{n!} M_{n\ell} M_{nk}.$$

Moreover, we have $|a_{k\ell}| \leq \frac{1}{\nu!} |M_{\nu l} M_{\nu k}|$, which completes the proof of the previous lemma. □

As in [**10, 11**], we can easily check that the spectrum of $\mathcal{L}_{0,b}$ coincides with the spectrum of $A = (a_{k\ell})_{k,\ell\geq 0}$, where $a_{k\ell}$ is given by (c_1). Moreover, the coefficients of the Fourier series given by (15) are nothing but the components of the n^{th} eigenvector of A. In practice, one can get a highly accurate approximation $\widetilde{\mu}_n$ of μ_n by using a submatrix of A of order $K > 0$. The eigenvector $(\tilde{\eta}_k^n)_{0\leq k\leq K}$ is taken as a good approximation in the L^2-norm of $(\eta_k^n)_{k\geq 0}$. Consequently

$$\tilde{\varphi}_n(x) = \sum_{k=0}^{K} \tilde{\eta}_k^n P_k(x), \quad x \in [0,b] \tag{18}$$

is the approximation of the exact eigenfunction $\varphi_n(x)$ on the interval $[0,b]$.

To approximate the spectrum of the operator $\mathcal{L}_{0,b}$, by its matrix representation, we need the following perturbation result on the spectrum of matrices, cited in [**12**].

Theorem 1 (Weyl's perturbation theorem)**.** *Let A and B be two Hermitian matrices of order n. Let $\sigma(A) = \{\alpha_0 \geq \cdots \geq \alpha_{n-1}\}$ and $\sigma(B) = \{\beta_0 \geq \cdots \geq \beta_{n-1}\}$ denote the spectrums of A and B, respectively. Then, we have*

$$\max_{0\leq j\leq n-1} |\alpha_j - \beta_j| \leq \|A - B\|. \tag{19}$$

Remark 1. The previous theorem holds for more general hermitian operators on a Hilbert space.

As a consequence of (c_2), we deduce that the coefficients a_{kl} decay exponentially to 0 as $k+l \to +\infty$. Hence, by using the previous theorem, we conclude that, for any positive integer K we get highly accurate approximation of the first K eigenvalues of $\mathcal{L}_{0,b}$ by considering the first K eigenvalues of the appropriate submatrix of A given by $A_K = (a_{k\ell})_{0\leq k,\ell\leq K}$.

2.2. Gaussian quadrature method

In this paragraph, we use the polynomials $(P_k)_k$ and their properties to construct a quadrature method for the eigenvalue problem (2). Note that the Gaussian quadrature method of order $2n$, associated with the orthogonal $P_n(x)$, given by (11) over $[0, b]$ is given by

$$\int_0^1 f(x)\, x \,\mathrm{d}x \approx \sum_{k=1}^{n} \omega_k f(x_k), \quad 1 \leq k \leq n,$$

where the nodes x_k are the eigenvalues of the matrix D given by (13) and the different quadrature weights $(\omega_k)_{1\leq k\leq n}$ are simply given by the following practical formula

$$\omega_k = -\frac{k_{n+1}}{k_n} \frac{1}{P_{n+1}(x_k) P_n'(x_k)}, \quad 1 \leq k \leq n.$$

Here, $k_n = \dfrac{\sqrt{2k+1}(2k)!}{\sqrt{b^{2k+1}}(k!)^2}$ is the highest coefficient of $P_n(x)$. For more details on the Gaussian quadrature method, the reader is referred to [**9**].

The following theorem provides a discretization formula for the eigenproblem (2).

Theorem 2. *Let ϵ be an arbitrary real number satisfying $0 < \epsilon < 1$. For a fixed positive integer n, let $N_{\varepsilon,n} = \min\left\{ m \in \mathbb{N}, \dfrac{2^{2m} b^{4m+3/2}}{(2m+1)!\binom{2m}{m}^2} < \varepsilon\, |\mu_n| \right\}$. Then for any integer $N \geq N_{\varepsilon,n}$, we have*

$$\sup_{x\in[0,b]} \left| \varphi_n(x) - \frac{1}{\mu_n} \sum_{p=1}^{N} \omega_p \exp(-x x_p) \varphi_n(x_p) \right| \leq \varepsilon.$$

Here, x_p, $1 \leq p \leq n$ are different zeros of the orthogonal polynomial $P_n(x)$ and $\varphi_n(\cdot)$, μ_n are as given by (2).

As a consequence of the previous theorem, we obtain the following discretization scheme for the eigenvalue problem (2),

$$\sum_{j=1}^{N} \omega_j e^{-x_i y_j} \widetilde{\varphi}_n(y_j) = \widetilde{\mu}_n \widetilde{\varphi}_n(x_i), \quad 1 \leq i, j \leq N,$$

where x_i, y_j and ω_i denote the different nodes and weights of our proposed quadrature method. If A_K denotes the square matrix of order K, defined by

$$A_K = \left(\omega_j e^{-x_i y_j}\right)_{1\leq i,j\leq K},$$

then the set of the eigenvalues of A_K is an approximation of a finite subset of the eigenvalues of the operator $\mathcal{L}_{0,b}$ given by (2). Moreover, for any integer $0 \le n \le K$, the eigenvector $\widetilde{U}_n$ corresponding to the approximate eigenvalue $\widetilde{\mu}_n$ is given by $\widetilde{U}_n = (\widetilde{\varphi}_n(x_i))_{1\le i\le K}$. Finally, to provide approximate values $\varphi_n(x)$ of $\widetilde{\varphi}_n(x)$ along the interval $[0, b]$, we use the following interpolation formula,

$$\widetilde{\varphi}_n(x) = \frac{1}{\mu_n} \sum_{j=1}^{K} \omega_j e^{-xy_j} \widetilde{\varphi}_n(y_j), \quad 0 \le x \le b. \tag{20}$$

Remark 2. As stated by the previous theorem, the interpolation formula (20) is highly accurate. As an example, for $b = 5$, $n = 0$ and $K = 40$, we have found that

$$\frac{1}{K}\left(\sum_{i=0}^{K} (\varphi_n(x_i) - \widetilde{\varphi}_n(x_i))^2\right)^{\frac{1}{2}} = 2.038073914\text{E} - 02,$$

where $x_i = \dfrac{ib}{K}$.

Now, using similar techniques as in [**2, 10, 11**], we assert that the first K eigenvalues of $\mathcal{L}_{0,b}$ are well approximated by the eigenvalues of A_K. This is given by the following theorem.

Theorem 3. *Let $\sigma(\mathcal{L}_{0,b}) = (\mu_n)_{n\ge 0}$ and $\sigma(A_N) = (\widetilde{\mu}_n)_{0\le n\le N-1}$ denote the spectrum of the finite Laplace operator $\mathcal{L}_{0,b}$ and the matrix A_N, where N is a positive integer larger then $N_{\epsilon,n}$. Then we have*

$$\sup_{0\le n\le N-1} |\mu_n - \widetilde{\mu}_n| \le \varepsilon b\sqrt{N}.$$

3. APPLICATIONS

Our goal here is to invert the Laplace transform of time limited functions as well as essentially time limited functions by the use of the eigenfunctions φ_n of the finite Laplace transform. Suppose that the Laplace transform $\mathcal{L}f$ of an L^2-unit norm function f, is known at least on $[0, b]$. We find f.

3.1. Inversion of the Laplace transform of time limited functions

Suppose that f is time-limited on $[0, b]$. That is, $f(x) = f(x)\chi_{[0,b]}(x)$. To find the unknown function f, we expand it into its Fourier series with respect to $\{\varphi_n, n \in \mathbb{N}\}$,

$$f(y) = \sum_{k\in\mathbb{N}} a_k(f)\varphi_k(y), \quad y \in [0, b], \tag{21}$$

where $a_k(f) = \frac{1}{\mu_k}\int\limits_0^b \varphi_k(y)f(y)\mathrm{d}y$ are the Fourier coefficients of f to be determined in the following. Combining (21) and (2), we obtain

$$\mathcal{L}_{0,b}f(y) = \sum_{k\in\mathbb{N}} a_k(f)\mu_k\varphi_k(y), \quad y \in [0,b]. \tag{22}$$

Multiplying both sides of (22) by $\varphi_k(y)$ and integrating over [0,b], we conclude that

$$a_k(f) = \frac{1}{\mu_k}\int_0^b \varphi_k(y)\mathcal{L}_{0,b}f(y)\mathrm{d}y, \quad k \in \mathbb{N}.$$

Once $(a_k(f))_k$ are known, we use (21) to obtain the unknown f. Note here that in practice the series given by (21) is truncated to an order N to obtain the following function

$$f_N(y) = \sum_{k=0}^{N} a_k(f)\varphi_k(y), \quad y \in [0,b].$$

Then, an error bound of the approximation of f by f_N, over $[0,b]$ is given by the following proposition.

Proposition 2. *Under the above notations and assumption, we have*

$$\|f - f_N\|_2^2 = \sum_{k=N+1} |a_k(f)|^2\mu_k^2 \leq (\mu_{N+1})^2.$$

The approximation error in the previous proposition can be assessed via the following lemma [**13, 14**].

Lemma 4. *Let T be the integral operator given by*:

$$Tx(t) = \int_a^b k(t,s)x(s)\mathrm{d}s,$$

where $k(.,.)$ is symmetric, positive definite and has p^{th} order continuous partial derivatives. Then, the n^{th} eigenvalue λ_n of T satisfies $\lambda_n = o\Big(\frac{1}{n^{p+1}}\Big)$.

3.2. Inversion of the Laplace transform of essentially time limited functions

In the sequel, we suppose that f is time-limited to $[0,b]$ at level ϵ_b. That is $f \in E(\epsilon_b) = \{f \in L^2\left([0,+\infty[\right),\ \|f\|_2 = 1,\ \|f\chi_{]b,+\infty[}\|_2 \leq \epsilon_b\}$. Remark that, by the use the Hölder's inequality, one gets for $x > 0$,

$$\begin{aligned} |\mathcal{L}f(x) - \mathcal{L}_{0,b}f(x)| &= \int_b^{+\infty} e^{-xy}f(y)\mathrm{d}y \\ &\leq \left(\int_b^{+\infty} e^{-2xy}dy\right)^{1/2}\left(\int_b^{+\infty} |f(y)|^2dy\right)^{1/2} \leq \frac{e^{-2xb}}{\sqrt{2x}}. \end{aligned} \tag{23}$$

Note that

$$\mathcal{L}_{0,b} f(x) = \int_0^b e^{-xy} f_b(y) \mathrm{d}y, \tag{24}$$

where $f_b(y) = f(y)\chi_{[0,b]}(y)$. Our aim is to find f_b and to show that f_b is a good approximation of f. As in 3.1., we expand f_b into its Fourier series following $\{\varphi_n, n \in \mathbb{N}\}$

$$f_b(y) = \sum_{k=0}^{N} a_k(f_b)\varphi_k(y), \quad y \in [0, b]. \tag{25}$$

By using (24), one gets

$$a_k(f_b) = \frac{1}{\mu_k} \int_0^b \varphi_k(y) \mathcal{L}_{0,b} f_b(y) \mathrm{d}y, \quad k \in \mathbb{N}.$$

Once $(a_k(f_b))_k$ are known, we use (25) to obtain the unknown f_b. Note here that in practice the series given in (25) is truncated to an order N and the function f_b is approximated by $f_{b,N}(y) = \sum_{k=0}^{N} a_k(f_b)\varphi_k(y), \quad y \in [0, b]$.

Note that the error of such approximation is given by

$$\|f_b - f_{b,N}\|_2^2 = \sum_{k=N+1} |a_k(f)|^2 \mu_k^2 \le (\mu_{N+1})^2.$$

Under the assumption that $f \in E(\epsilon_b)$ we get the following estimate of approximation error for f.

Proposition 3. *Under the above notations and assumptions, we have*

$$\|f - f_{b,N}\|_2 \le \epsilon_b + |\mu_{N+1}|.$$

4. NUMERICAL EXAMPLES

To illustrate the results of section 2, we have considered different values of the bandwidth b. Also, we have applied the Gaussian quadrature based method for the computation of the spectrum and the eigenfunctions of the finite Laplace transform $\mathcal{L}_{0,b}$ with $N = 40$ quadrature points. Table 1 shows the obtained eigenvalues μ_n with different values of the parameter b. Moreover, we have used (20) with a maximum truncation order $K = N$ to obtain accurate approximations to the normalized $\varphi_n(x)$ along the interval $[0, b]$. Table 2 lists, for $b = 5$, the approximate values $\widetilde{\varphi}_0(x)$ of $\varphi_0(x)$, for different values of x as well as the different approximation errors in absolute value $|\varphi_0(x) - \widetilde{\varphi}_0(x)|$. Also, in figure 1 and 2, we have plotted the graphs of the first three normalized eigenfunctions of the finite Laplace operator $\mathcal{L}_{0,b}$ corresponding respectively to the parameter $b = 1$ and $b = 5$ (φ_0 = solid, φ_1 = dot and φ_2 = dash).

	$b=1$	$b=5$	$b=10$
n	μ_n	μ_n	μ_n
0	0.809579221e-00	1.343346838e-00	1.440931952e-00
2	0.216334666e-02	2.168106787e-01	0.386895185e-01
5	0.929136341e-08	3.806005533e-03	0.247988181e-02
10	0.299404378e-18	5.676369190e-07	0.101780894e-04
15	0.798452541e-30	1.167244175e-11	0.176316362e-07

Table 1. Values of the eigenvalues μ_n of the finite Laplace transform $\mathcal{L}_{0,b}$ corresponding to the different values of parameter $b = 2, 5, 10$.

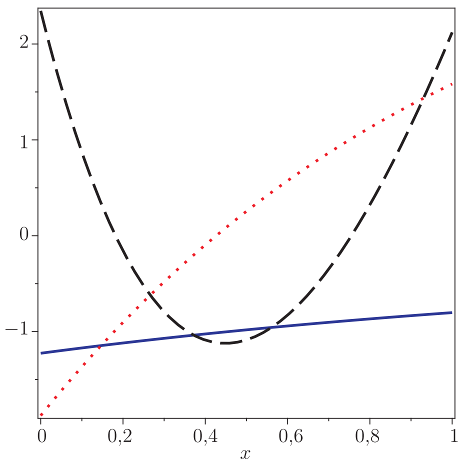

Figure 1

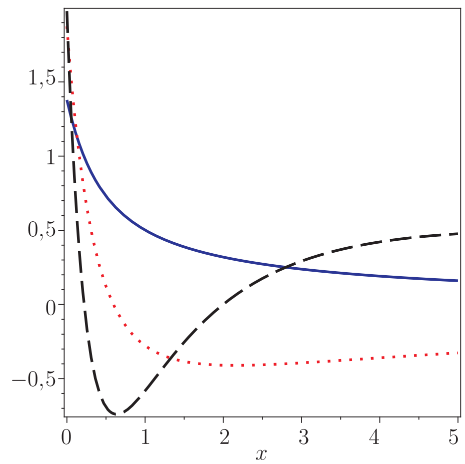

Figure 2

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