

STABILITY ESTIMATES FOR DISCRETE HARMONIC FUNCTIONS ON PRODUCT DOMAINS

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We study the Dirichlet problem for discrete harmonic functions in unbounded product domains on multidimensional lattices. First we prove some versions of the Phragmén-Lindelöf theorem and use Fourier series to obtain a discrete analog of the three-line theorem for the gradients of harmonic functions in a strip. Then we derive estimates for the discrete harmonic measure and use elementary spectral inequalities to obtain stability estimates for Dirichlet problem in cylinder domains.

1. INTRODUCTION

We consider functions defined on subsets of the multidimensional lattice $(\delta\mathbb{Z})^m$ in \mathbb{R}^m . The usual $(2m + 1)$ -point discretization of the Laplace operator is denoted by Δ_m or $\Delta_{\delta,m}$ to emphasize the mesh of the lattice, while the accurate definition is given below. Then we study the following Dirichlet problem

$$\Delta_m u = 0, \quad u = f \text{ on } \partial D, \quad u \in H_b(D),$$

where $H_b(D)$ is some class of functions of bounded growth in D , and D is an unbounded connected (on the lattice) subset of $(\delta\mathbb{Z})^m$. Our main question is for which $H_b(D)$ the problem above has a unique solution. Moreover, when the solution is unique we estimate how the error in the boundary data affects the error of the solution. Such estimates are called conditional stability estimates, we suppose a priori that solution belongs to $H_b(D)$. Since our problem is linear, stability estimate reduces to a bound of some norm of the solution $u \in H_b(D)$ by some norm of its boundary values f .

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First, we prove that if $D = \Omega \times \mathbb{R}^k$, where Ω is a bounded domain in \mathbb{R}^n , $u(x, y)$ is a discrete harmonic function in $D \cap (\delta\mathbb{Z})^{n+k}$ that satisfies

$$|u(x, y)| \leq C \exp(c\|y\|_1)$$

for some $c = c(\Omega, k)$, and $u = 0$ on ∂D then $u = 0$ (here and in what follows $\|y\|_1 = |y_1| + \dots + |y_k|$, and $\|y\|_\infty = \max\{|y_1|, \dots, |y_k|\}$ where $y = (y_1, \dots, y_k) \in \mathbb{R}^k$). We refer to this statement as a discrete version of the Phragmén-Lindelöf theorem, it implies the uniqueness in the Dirichlet problem in the class of functions of limited growth.

We consider more carefully the case $\Omega = [0, 1]$ and solve the Dirichlet problem using Fourier analysis when the boundary data is in ℓ^2 . We obtain

$$\|u(x, \cdot)\|_{\ell^2} \leq \|f\|_{\ell^2}.$$

We also use this technique to show that gradients of discrete harmonic functions satisfy the following three-line inequality that resembles three-line theorem of Hadamard,

$$(1) \quad \|\nabla u(\delta k, \cdot)\|_{\ell^2(\mathbb{Z}^k)} \leq (\|\nabla u(0, \cdot)\|)^{1-\frac{k}{M}} (\|\nabla u(\delta M, \cdot)\|)^{\frac{k}{M}},$$

where $(M+1)\delta = 1$. Both the Phragmén-Lindelöf theorem and Hadamard's three line theorem are classical results in complex analysis (for example see [18]). We discuss discrete version of their multidimensional generalizations, corresponding continuous results are known and we provide the references throughout the text.

Finally, to obtain conditional stability estimates for Dirichlet problem with partial boundary data (see Theorem 5), we study the discrete harmonic measure in the truncated cylinder $\Omega \times [-N, N]$. We also use elementary properties of the spectrum of the discrete Dirichlet problem for the Laplacian on Ω and some comparison results that can be found in T. BIYIKOGLU, J. LEYDOLD, P. F. STADLER [2] and D. CVETKOVIĆ, P. ROWLINSON, S. SIMIĆ [8].

The article is organized as follows. In the next section we give necessary definitions and results for discrete harmonic functions, including basic properties of the eigenvalues and eigenfunctions of the discrete Laplace operator with Dirichlet boundary condition. We also prove a simple version of the Phragmén-Lindelöf theorem for product domains. In Section 3 we use Fourier analysis to study discrete harmonic functions in a strip, in particular we obtain the logarithmic convexity inequality (1). Our main stability result for the Dirichlet problem in an infinite cylinder is proved in the last section, it follows from estimates of discrete harmonic measure and a more accurate version of the Phragmén-Lindelöf theorem.

2. PRELIMINARIES

2.1. Discrete harmonic functions

The theory of discrete harmonic functions on the lattices dates back to at least as early as 1920s, when fundamental works of H. B. PHILLIPS and N. WIENER

[17], and R. COURANT, K. FRIEDRICHS, and H. LEWY [6] were published. In the middle of the last century an important contribution to the theory of discrete harmonic functions was done by H. A. HEILBRONN [10] and R. J. DUFFIN [9]. One of the original motivations for the study of discrete harmonic functions is that such functions converge to continuous ones. For example to obtain a solution of the Dirichlet problem one may solve discrete problems in lattice domains and pass to the limit as the mesh size of the lattice goes to zero, we refer the reader to the classical works mentioned above and to the article of I. G. PETROWSKY [16].

Suppose that $u(x)$ is a function defined on a subset of the lattice $(\delta\mathbb{Z})^m$. Then the δ -discrete Laplacian of u is defined by

$$\Delta_\delta u(x) = \Delta_{\delta,m} u(x) = \delta^{-2} \left(\sum_{j=1}^m (u(x + \delta e_j) + u(x - \delta e_j)) - 2mu(x) \right),$$

where e_1, e_2, \dots, e_n is the standard coordinate basis for \mathbb{Z}^m and $-\Delta_\delta$ coincides with the combinatorial Laplacian of the lattice where the conductance associated to each edge equals δ^{-2} . This is the discrete version of the Laplace-Beltrami operator in Riemannian manifolds. We refer the reader to T. BIYIKOGLU, J. LEYDOLD, P. F. STADLER [2] for the details. Potential theory on finite networks is an active area of investigation, see for example [1] and references therein.

Definition. A function u is called δ -discrete harmonic at a point x of the lattice $(\delta\mathbb{Z})^m$ if it is define at x together with all its neighbors and satisfies the equation

$$\Delta_\delta u(x) = 0.$$

So the value of a discrete harmonic function at a lattice point is the average of its values at the $2m$ neighboring points.

Discrete harmonic functions share many properties of continuous ones. For example results on the maximum principle, solution to the Dirichlet problem, Green's function, and Liouville's theorem can be found in the very first articles on the subject, see also Y. COLIN DE VERDIÈRE [5] and C. KISELMAN [12] for more recent surveys and more general discrete structures. On the other hand not all results about continuous harmonic functions are easily generalized to the discrete case. For example zero sets of discrete harmonic functions are difficult to compare to those of continuous ones. For any finite square there exists a discrete harmonic polynomial that vanishes at each lattice point of this square. We study growth properties of discrete harmonic functions in cylinders and strips and provide accurate estimates that show to which extend continuous theorems can be generalized to solutions of the discrete equation that arises in the simplest numerical scheme.

We consider discrete harmonic functions on subsets of $(\delta\mathbb{Z})^m$, $D^\delta \subset (\delta\mathbb{Z})^m$ is called a (discrete) domain if it is connected, i.e., for any two points x and y in D^δ there exists a sequence $\{x_0, x_1, \dots, x_s\}$ such that $x_0 = x$, $x_s = y$, $x_j \in D^\delta$, x_j and x_{j+1} are neighboring points of the lattice $(\delta\mathbb{Z})^m$.

A point $x \in (\delta\mathbb{Z})^m \setminus D^\delta$ is called a boundary point of D^δ if at least one of the $2m$ neighbors of x is in D^δ . We denote the set of boundary points of D^δ by ∂D^δ , we also use the notation $\overline{D^\delta} = D^\delta \cup \partial D^\delta$. A domain is called finite if it contains only finite number of points, otherwise it is called infinite.

Definition. A function u defined on $D^\delta \cup \partial D^\delta$ is called δ -discrete subharmonic (superharmonic) in D^δ if $\Delta_\delta u \geq 0$ (≤ 0) in D^δ .

Clearly, a function is δ -discrete harmonic in D^δ if it is both δ -discrete subharmonic and superharmonic. The following maximum principle holds (see for example [12]).

Theorem. If u is δ -discrete subharmonic in a finite domain D then

$$\max_{\overline{D}} u = \max_{\partial D} u.$$

Simple examples show that the maximum principle does not hold for infinite domains.

2.2. Eigenvalues and eigenfunctions for the discrete Laplacian

In order to prove a version of the Phragmén-Lindelöf theorem for discrete subharmonic functions in cylindrical domains, we need some basic facts about eigenfunctions and eigenvalues of the discrete Dirichlet problem for the Laplacian on the base of the cylinder. For more general theory of graph spectra we refer the reader to D. CVETKOVIĆ, P. ROWLINSON, S. SIMIĆ [8] and F. R. K. CHUNG [4, ch 1].

Throughout the paper Ω denotes a bounded domain in \mathbb{R}^n $n \geq 1$, with Lipschitz boundary and $\Omega^\delta = \Omega \cap (\delta\mathbb{Z})^n$. We always assume that $\delta < \delta_0$ is small enough such that Ω^δ is a discrete connected set. We study δ -discrete harmonic functions that are defined on the product domain $D^\delta(\Omega) = \overline{\Omega}^\delta \times (\delta\mathbb{Z})^k$ and vanish on the boundary. We consider the eigenvalues $\{\lambda_j(\Omega)\}$ of the continuous n -dimensional Dirichlet problem for the Laplacian on Ω and the eigenvalues of the corresponding discrete operators. We denote the eigenvalue for the discrete Dirichlet problem on $\overline{\Omega}^\delta$ by $\lambda_j^\delta(\Omega^\delta)$ and we use the notation λ_j^δ when it does not lead to confusion. It is known (see for example [2] or [8]) that the eigenvalues of the following problem

$$\begin{cases} -\Delta_{\delta,n} f = \lambda f & \text{in } \Omega^\delta \\ f = 0 & \text{on } \partial\Omega^\delta \end{cases}$$

are positive, $0 < \lambda_1^\delta < \lambda_2^\delta \leq \dots \leq \lambda_{K^\delta}^\delta$, the first eigenvalue is simple and the corresponding eigenfunction f_1^δ can be chosen strictly positive in Ω^δ . The last statement is an analog of the classical result on the first eigenfunction of Dirichlet problem for the Laplacian, see R. COURANT, D. HILBERT [7, §6, ch VI]. For the discrete operator it follows from the Perron-Frobenius theorem on positive matrices, see for example [2, Corollary 2.23]. Clearly K^δ is finite in the discrete case and equals the number of points of Ω^δ .

It is also known that $\lambda_k^\delta(\Omega^\delta) \rightarrow \lambda_k(\Omega)$ as $\delta \rightarrow 0$. We don't discuss the limits arguments in this article, but we indicate which of our estimates survive the limit passage as $\delta \rightarrow 0$.

The eigenvalues $\lambda_k^\delta(\Omega^\delta)$ are given by the following minimax principle, see [2, Corollary 2.6],

$$\lambda_k^\delta(\Omega^\delta) = \min_{w \in W_k} \max_{0 \neq g \in w} \frac{\langle g, L_{\Omega^\delta}^\delta g \rangle}{\langle g, g \rangle},$$

where W_k denotes the set of subspaces of dimension at least k and $L_{\Omega^\delta}^\delta$ is the δ -discrete Laplacian of Ω with Dirichlet boundary condition. This readily implies that if $\Omega' \supset \Omega$ then

$$(2) \quad \lambda_k^\delta(\Omega') \leq \lambda_k^\delta(\Omega).$$

We denote by N_Ω^δ the counting function, $N_\Omega^\delta(\lambda)$ equals the number of eigenvalues $\lambda_k^\delta(\Omega)$ that are less than or equal to λ . Then (2) implies

$$(3) \quad N_{\Omega'}^\delta(\lambda) \geq N_\Omega^\delta(\lambda).$$

2.3. Eigenvalues for the cube

We need some estimates of the growth of the eigenvalues $\lambda_j^\delta(\Omega)$ to prove a precise version of the Phragmén-Lindelöf theorem in the last section of this article. We obtain them by comparing the eigenvalues to those of a large cube Q containing Ω . The latter can be found explicitly. Let $Q_R = (0, R)^n$, where $R \in \mathbb{N}$ and let $M = 1/\delta \in \mathbb{N}$. We consider the following problem

$$\begin{cases} -\Delta_{\delta,n} f = \lambda f & \text{in } Q_R^\delta \\ f = 0 & \text{on } \partial Q_R^\delta \end{cases}$$

This is an eigenvalue problem for a matrix of the size $(R\delta^{-1} - 1)^n \times (R\delta^{-1} - 1)^n$. Let $J = \{1, 2, \dots, R\delta^{-1} - 1\}$, for any $\bar{k} \in J^n$, $\bar{k} = (k_1, \dots, k_n)$ the function

$$f_{\bar{k}}(x_1, \dots, x_n) = \prod_{j=1}^n \sin \frac{k_j \pi}{R} x_j$$

is an eigenfunction and the corresponding eigenvalue is

$$\lambda_{\bar{k}}^\delta = 2\delta^{-2} \left(n - \sum_{j=1}^n \cos \frac{k_j \pi \delta}{R} \right).$$

Using the elementary inequality $1 - \cos x \geq 2\pi^{-2}x^2$, when $x \in (0, \pi)$ we obtain

$$\lambda_{\bar{k}}^\delta \geq 4R^{-2} \sum_{j=1}^n k_j^2.$$

For the details of eigenvalues and eigenfunctions for the cube we refer the reader to F. R. K. CHUNG [4, ch 1].

The following inequality for the counting function for the cube follows

$$(4) \quad N_{Q_R}^\delta(\lambda) \leq C_n(R)(\lambda^{n/2} + 1),$$

where the constant does not depend on δ . This inequality is an illustration of the Weyl's asymptotic for the counting function for eigenvalues of Dirichlet problem for the Laplacian.

2.4 Phragmén-Lindelöf theorems in cylindrical domains

Let Ω be a bounded subdomain of \mathbb{R}^n and $D^\delta = \Omega^\delta \times (\delta\mathbb{Z})^k$. Clearly,

$$\Delta_{\delta, n+k} u(x, y) = \Delta_{\delta, n} u(x, y) + \Delta_{\delta, k} u(x, y),$$

where the first Laplacian is obtained with respect to x -variables and the second with respect to y -variables. Let f_1^δ be the first eigenfunction of the Dirichlet problem for the Laplacian in Ω^δ defined above. As we noted, f_1^δ is strictly positive on Ω^δ , and we have the following positive harmonic function in D^δ

$$u^\delta(x, y) = f_1^\delta(x) \cosh b_\delta y_1 \cosh b_\delta y_2 \dots \cosh b_\delta y_k,$$

where b_δ is the only positive solution of

$$(5) \quad \cosh \delta b_\delta = 1 + \frac{1}{2k} \delta^2 \lambda_1^\delta.$$

In the discrete setting the function f_1^δ is strictly positive; this makes the proof of our first theorem of Phragmén-Lindelöf type more simple than the proof of a similar result for continuous functions, see for example I. MIYAMOTO [15], F. T. BRAUN [3] and D. V. WIDDER [19].

Theorem 1. *Let v be a δ -discrete subharmonic function in D^δ such that $v \leq 0$ on $\partial\Omega^\delta \times (\delta\mathbb{Z})^k$. Let λ_1^δ be the first eigenvalue of the δ -discrete Dirichlet problem for the Laplacian in Ω and b_δ be the positive solution to the equation (5). Suppose that*

$$v(x, y) \leq o(1) \exp(b_\delta \|y\|_1), \quad \text{when } \|y\|_1 \rightarrow \infty.$$

Then $v \leq 0$ on D^δ .

Proof. We want to compare $v(x, y)$ to a multiple of $u^\delta(x, y)$ on $\overline{\Omega}^\delta \times [-N, N]^k$. On the part of the boundary $\partial\Omega^\delta \times (\delta\mathbb{Z})^k$ we have $v \leq 0$ and $u^\delta = 0$ because $f_1^\delta = 0$ on $\partial\Omega^\delta$. On the other part of the boundary, $\|y\|_1 \geq N$ and

$$v(x, y) \leq C_N \exp(b_\delta \|y\|_1) \leq \frac{2^k C_N}{\min_{\Omega^\delta} f_1^\delta} u^\delta(x, y),$$

where $C_N \rightarrow 0$ as $N \rightarrow \infty$.

The maximum principle for subharmonic functions implies that

$$v(x, y) \leq \frac{2^k C_N}{\min_{\Omega^\delta} f_1^\delta} u^\delta(x, y), \quad \text{where } x \in \Omega^\delta, \ y \in (\delta\mathbb{Z})^k, \ \|y\|_\infty \leq N.$$

Now if we fix (x, y) and let N grow to infinity, we obtain $v(x, y) \leq 0$. □

The theorem holds for subharmonic functions with all estimates from above only. If we have a discrete harmonic function h and apply the above statement to h and $-h$ we obtain the uniqueness for the Dirichlet problem in D^δ in the class of functions

$$H_b(D^\delta) = \{u : D^\delta \rightarrow \mathbb{R} : |u(x, y)| = o(\exp(b_\delta \|y\|_1)), \|y\|_1 \rightarrow \infty\}.$$

Corollary. *Let u and v be δ -discrete harmonic functions on D^δ , $u, v \in H_b(D^\delta)$. If $u = v$ on $\partial(\Omega^\delta) \times (\delta\mathbb{Z})^k$ then $u = v$ on D^δ .*

Proof. Let $g = u - v$. Then g is δ -discrete harmonic in D^δ and $g = 0$ on $\partial(\Omega^\delta) \times (\delta\mathbb{Z})^k$. Moreover $|g(x, y)| \leq |u(x, y)| + |v(x, y)|$ and therefore

$$|g(x, y)| \leq C_N \exp(b_\delta \|y\|_1), \quad \text{when } \|y\|_1 \geq N,$$

where $C_N \rightarrow 0$ as $N \rightarrow \infty$. Then $g \leq 0$ on D^δ by Theorem 1. In the same way we obtain $-g \leq 0$ and thus $u = v$. □

We note that $b_\delta \rightarrow \sqrt{\lambda_1(\Omega)/k}$ when $\delta \rightarrow 0$, however Theorem 1 does not survive a limit argument as $\delta \rightarrow 0$. In the last section we provide an estimate for δ -discrete harmonic functions in truncated cylinders that allows us to prove a more accurate version of the Phragmén-Lindelöf theorem.

3. DISCRETE HARMONIC FUNCTIONS ON STRIPS

In this section we study quantitative uniqueness for discrete harmonic functions and their gradients on strips $S = (0, 1) \times \mathbb{R}^n$. We remark that eigenvalues of the discrete Dirichlet problem for the Laplacian on $[0, 1]^\delta$ are

$$\lambda_l^\delta = 2\delta^{-2}(1 - \cos 2\pi l\delta).$$

In particular the Phragmén-Lindelöf theorem proved in the last section implies the uniqueness in the Dirichlet problem for discrete harmonic functions that satisfy

$$(6) \quad |u(x, y)| = o(\exp(b_\delta \|y\|_1)), \quad \|y\|_1 \rightarrow \infty,$$

where

$$\cosh \delta b_\delta = \frac{n+1}{n} - \frac{1}{n} \cos 2\pi\delta.$$

3.1. Tempered harmonic functions in a strip

Now we consider *tempered* harmonic functions in the strip and use the Fourier representation to solve the Dirichlet problem.

Definition. Let u be a δ -discrete function on S^δ , u is said to be tempered if

$$\sum_{k=0}^{1/\delta} \sum_{j \in \mathbb{Z}^n} |u(\delta k, \delta j)|^2 < \infty.$$

Theorem 2. Let u be a δ -discrete harmonic function in S^δ such that (6) holds. Suppose that

$$\sum_{j \in \mathbb{Z}^n} |u(0, \delta j)|^2 < \infty \quad \text{and} \quad \sum_{j \in \mathbb{Z}^n} |u(1, \delta j)|^2 < \infty.$$

Then $\{u(\delta k, \delta j)\}_{j \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$ for each $k = 1, 2, \dots, L-1$, i.e, u is tempered, moreover

$$\sum_{j \in \mathbb{Z}^n} |u(\delta k, \delta j)|^2 \leq \sum_{j \in \mathbb{Z}^n} |u(0, \delta j)|^2 + \sum_{j \in \mathbb{Z}^n} |u(1, \delta j)|^2.$$

Proof. Let

$$\varphi_0(t) = \sum_{j \in \mathbb{Z}^n} u(0, \delta j) e^{2\pi i j \cdot t}, \quad \text{and} \quad \varphi_L(t) = \sum_{j \in \mathbb{Z}^n} u(1, \delta j) e^{2\pi i j \cdot t}$$

for $t \in [0, 1]^n$. Then $\varphi_0, \varphi_L \in L^2([0, 1]^n)$.

For each $t \in [0, 1]^n$ we define $q(t)$ such that $q(t) \geq 1$ and

$$q(t) + q(t)^{-1} = 2(n+1) - 2 \sum_{\ell=1}^n \cos 2\pi t_\ell.$$

More precisely $q(t) = \lambda(t) + \sqrt{\lambda^2(t) - 1}$ and then $q(t)^{-1} = \lambda(t) - \sqrt{\lambda^2(t) - 1}$, where

$$\lambda(t) = n + 1 - \sum_{\ell=1}^n \cos 2\pi t_\ell.$$

Now for $k = 1, \dots, L-1$ we consider

$$\varphi_k(t) = \frac{q(t)^k - q(t)^{-k}}{q(t)^L - q(t)^{-L}} \varphi_L(t) + \frac{q(t)^{L-k} - q(t)^{-L+k}}{q(t)^L - q(t)^{-L}} \varphi_0(t).$$

Since $q \geq 1$, we have

$$q(t)^k - q(t)^{-k} \leq q(t)^L - q(t)^{-L}, \quad \text{and} \quad q(t)^{L-k} - q(t)^{-L+k} \leq q(t)^L - q(t)^{-L}.$$

Then $\varphi_k \in L^2([0, 1]^n)$ and $\|\varphi_k\|_2 \leq \|\varphi_0\|_2 + \|\varphi_L\|_2$. Thus

$$\varphi_k(t) = \sum_{j \in \mathbb{Z}^n} v(k, j) e^{2\pi i j \cdot t},$$

where $\{v(k, j)\}_{j \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$. Remark that

$$q(t) = \frac{1 + q^2(t)}{2\lambda(t)} \quad \text{and therefore} \quad q^k(t) = \frac{q^{k-1}(t) + q^{k+1}(t)}{2\lambda(t)}.$$

Then

$$\varphi_k(t) = \frac{\varphi_{k-1}(t) + \varphi_{k+1}(t)}{2\lambda(t)}$$

and

$$\varphi_k = \frac{1}{2(n+1)} \left[\varphi_{k+1} + \varphi_{k-1} + \varphi_k \left(\sum_{\ell=1}^n e^{2\pi i t_\ell} + e^{-2\pi i t_\ell} \right) \right].$$

Hence the Fourier coefficients $v(k, j)$ satisfy

$$v(k, j) = \frac{1}{2(n+1)} \left[v(k+1, j) + v(k-1, j) + \sum_{\ell=1}^n (v(k, j - e_\ell) + v(k, j + e_\ell)) \right].$$

It means that v is a discrete harmonic function on $[1, L-1] \times \mathbb{Z}^n$. We have that $v(0, j) = u(0, \delta j)$ and $v(L, j) = u(1, \delta j)$. Note also that

$$|v(k, J)|^2 \leq \sum_{j \in \mathbb{Z}^n} |v(k, j)|^2 = \|\varphi_k\|_{L^2([0,1]^n)}^2 \leq \left(\|\varphi_0\|_{L^2([0,1]^n)} + \|\varphi_L\|_{L^2([0,1]^n)} \right)^2.$$

Thus $v(k, J)$ is bounded, in particular $|v(k, j)| = o(\exp(b_\delta \|y\|_1))$ when $\|y\|_1 \rightarrow \infty$. By Corollary in 2.4, we have $v(k, j) = u(\delta k, \delta j)$ and $\{u(\delta k, \delta j)\}_{j \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$ with the required estimate.

REMARK. We have also proved that if u is a δ -discrete harmonic function on S^δ that is square-summable along the hyperplanes $\{\delta k\} \times (\delta \mathbb{Z})^n$ then there exist two functions $a_1, a_2 \in L^2([0, 1]^n)$ such that

$$(7) \quad u(\delta k, \delta j) = \int_{[0,1]^n} \left(a_1(t)q(t)^k + a_2(t)q(t)^{-k} \right) e^{-2\pi j \cdot t} dt,$$

where $q(t) \geq 1$ and is defined by

$$q(t) + q^{-1}(t) = 2(n+1) - 2 \sum_{\ell=1}^n \cos 2\pi t_\ell.$$

Reviewing the computations in the proof of the lemma, we see that

$$a_1(t) = \frac{\varphi_L(t) - q(t)^{-L} \varphi_0(t)}{q(t)^L - q(t)^{-L}}, \quad a_2(t) = \frac{q(t)^L \varphi_0(t) - \varphi_L(t)}{q(t)^L - q(t)^{-L}}.$$

Thus the theorem provides a constructive procedure for solution of the Dirichlet problem for tempered harmonic function in a strip as well as a stability estimate for this procedure.

3.2. Three line theorem for discrete harmonic functions

In this subsection we prove a three line theorem for the gradients of discrete harmonic functions, the corresponding continuous result and its connections to the interpolation theory can be found in S. JANSON and J. PEETRE [11].

Definition. Let $u(x, y)$ be a δ -discrete function on a subdomain of the lattice $(\delta\mathbb{Z})^{n+1}$, its discrete partial derivatives are defined by

$$u_x(x, y) = \delta^{-1} (u(x + \delta, y) - u(x, y)) \quad \text{and}$$

$$u_{y_i}(x, y) = \delta^{-1} (u(x, y + \delta e_i) - u(x, y)).$$

For the case of the strip $S = [0, 1] \times \mathbb{R}^n$ all discrete partial derivatives in y -variables are defined on the same domain, while u_x is defined on $[0, 1 - \delta] \times \mathbb{R}^n$.

Definition. The discrete gradient of a discrete function $u(x, y)$ on a subdomain of the lattice $(\delta\mathbb{Z})^{n+1}$ is defined as

$$\nabla u(x, y) = (u_x(x, y), u_{y_1}(x, y), u_{y_2}(x, y), \dots, u_{y_n}(x, y))$$

Theorem 3. Let u be a δ -discrete harmonic function in $[0, 1] \times \mathbb{R}^n$, $\delta^{-1} = M + 1$ for some positive integer M . Suppose that u satisfies (6) and

$$\{u(0, \delta j)\}_{j \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n), \quad \{u(1, \delta j)\}_{j \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n).$$

Let further

$$m(k) = \delta^2 \|u_x(\delta k, \delta j)\|_{\ell^2(\mathbb{Z}^n)}^2 + \delta^2 \sum_{\ell=1}^n \|u_{y_\ell}(\delta k, \delta j)\|_{\ell^2(\mathbb{Z}^n)}^2 \quad \text{for } k = 0, 1, \dots, M.$$

Then

$$m(k) \leq (m(0))^{1-\frac{k}{M}} (m(M))^{\frac{k}{M}}.$$

Proof. Using (7) and the definition of the discrete partial derivatives, we get

$$u_x(\delta k, \delta j) = \delta^{-1} \int_{[0, 1]^n} (a_1(t)q(t)^k (q(t) - 1) + a_2(t)q(t)^{-k} (q(t)^{-1} - 1)) e^{-2\pi j \cdot t} dt,$$

and

$$\|u_x(\delta k, \delta j)\|_{\ell^2(\mathbb{Z}^n)}^2 = \delta^{-2} \|a_1(t)q(t)^k (q(t) - 1) + a_2(t)q(t)^{-k} (q(t)^{-1} - 1)\|_{L^2([0, 1]^n)}^2.$$

Further,

$$u_{y_\ell}(\delta k, \delta j) = \delta^{-1} \int_{[0, 1]^n} (a_1(t)q(t)^k + a_2(t)q(t)^{-k}) e^{-2\pi j \cdot t} (e^{-2\pi i t_\ell} - 1) dt,$$

$$\|u_{y_\ell}(\delta k, \delta j)\|_{\ell^2(\mathbb{Z}^n)}^2 = \delta^{-2} \|(a_1(t)q(t)^k + a_2(t)q(t)^{-k})(e^{-2\pi i t_\ell} - 1)\|_{L^2([0, 1]^n)}^2.$$

Then, adding up the identities above, we get

$$(8) \quad m(k) = \delta^2 \|u_x(\delta k, \delta j)\|_{\ell^2(\mathbb{Z}^n)}^2 + \delta^2 \sum_{\ell=1}^n \|u_{y_\ell}(\delta k, \delta j)\|_{\ell^2(\mathbb{Z}^n)}^2$$

$$= \|a_1(t)q(t)^k (q(t) - 1) + a_2(t)q(t)^{-k} (q(t)^{-1} - 1)\|_{L^2([0, 1]^n)}^2$$

$$+ \sum_{\ell=1}^n \|a_1(t)q(t)^k (e^{-2\pi i t_\ell} - 1) + a_2(t)q(t)^{-k} (e^{-2\pi i t_\ell} - 1)\|_{L^2([0, 1]^n)}^2.$$

We note that $q(t)$ is real and by the definition $q(t) + q(t)^{-1} = 2(n+1) - 2 \sum_{\ell=1}^n \cos 2\pi t_\ell$, therefore

$$(9) \quad (q(t) - 1)(q(t)^{-1} - 1) = 2 \sum_{\ell=1}^n \cos 2\pi t_\ell - 2n = - \sum_{\ell=1}^n (e^{-2\pi i t_\ell} - 1)(e^{2\pi i t_\ell} - 1).$$

Finally,

$$(10) \quad \delta^2 m(k) = \|a_1(t)q(t)^k(q(t) - 1)\|_{L^2([0,1]^n)}^2 + \|a_2(t)q(t)^{-k}(q(t)^{-1} - 1)\|_{L^2([0,1]^n)}^2 \\ + \sum_{\ell=1}^n \|a_1(t)q(t)^k (e^{-2\pi i t_\ell} - 1)\|_{L^2([0,1]^n)}^2 + \|a_2(t)q(t)^{-k} (e^{-2\pi i t_\ell} - 1)\|_{L^2([0,1]^n)}^2.$$

Each term in the right hand side of the last formula can be written in the form $s(k) = \|b(t)q(t)^{\pm k}\|_2^2$ for some $b \in L^2([0,1]^n)$ and $q(t)^{\pm k} \in L^\infty([0,1]^n)$. By Hölder's inequality, we have

$$s(k) = \|b(t)q(t)^{\pm k}\|_{L^2([0,1]^n)}^2 \leq \left(\int_{[0,1]^n} |b(t)|^2 dt \right)^{1 - \frac{k}{M}} \left(\int_{[0,1]^n} |b(t)|^2 q^2(t) dt \right)^{\frac{k}{M}} \\ \leq (s(0))^{1 - \frac{k}{M}} (s(M))^{\frac{k}{M}}.$$

Applying the same computation for each term and using the lemma below we conclude the proof of the theorem.

Lemma 1. *If each function $m_\ell : [0, 1, \dots, M] \rightarrow \mathbb{R}_+$ satisfies the inequality*

$$m(k) \leq [m(0)]^{1 - \frac{k}{M}} [m(M)]^{\frac{k}{M}}$$

then the sum $m(k) = \sum_{\ell} m_\ell(k)$ satisfies the same inequality.

Proof. It suffices to prove the statement when $m(k) = m_1(k) + m_2(k)$ is the sum of two functions. Let $\alpha = k/M$, then we have

$$m(k) = m_1(k) + m_2(k) \leq m_1(0)^{1-\alpha} m_1(M)^\alpha + m_2(0)^{1-\alpha} m_2(M)^\alpha \\ = m(0)^{1-\alpha} m(M)^\alpha \left[\left(\frac{m_1(0)}{m(0)} \right)^{1-\alpha} \left(\frac{m_1(M)}{m(M)} \right)^\alpha + \left(\frac{m_2(0)}{m(0)} \right)^{1-\alpha} \left(\frac{m_2(M)}{m(M)} \right)^\alpha \right].$$

And the lemma follows from the elementary inequality

$$x^{1-\alpha} y^\alpha + (1-x)^{1-\alpha} (1-y)^\alpha \leq 1$$

when $x, y \in [0, 1]$ and $\alpha \in [0, 1]$.

REMARK. The proof of Theorem 3 above is similar to that of the continuous three-line theorem, see [11]. In the continuous case the passage from (8) to (10) is trivial, in discrete case we fortunately have the identity (9).

For continuous harmonic functions similar three balls or three spheres theorems can be obtained, see for example J. KOREVAAR and J. L. H. MEYERS [13] and E. MALINNIKOVA [14]. There are no trivial generalizations of those results as a harmonic function can vanish on any finite square without being identically zero.

4. HARMONIC MEASURE AND STABILITY ESTIMATES

In this section we study δ -discrete harmonic functions that are defined on the cylinder $D^\delta(\Omega) = \Omega^\delta \times (\delta\mathbb{Z})$. Discrete harmonic measure on truncated cylinder is estimated first, then we apply these estimates to give a more precise version of the Phragmén-Lindelöf theorem and prove some stability results.

4.1. Discrete harmonic measure

Let now $\mathcal{H}_0(D^\delta)$ denote the space of δ -discrete harmonic functions on $D^\delta(\Omega)$ that vanish on the boundary. Such function is uniquely determined by its values on two layers $\Omega^\delta \times \{a\}$ and $\Omega^\delta \times \{b\}$ (where it may attain arbitrary values) and the dimension of $\mathcal{H}_0(D^\delta)$ equals $2K^\delta$, where K^δ is the number of points in Ω^δ .

We note that for a function $u(x) = u(x', x_{n+1})$ on $D^\delta(\Omega)$ we have

$$\begin{aligned} \Delta_{\delta, n+1} u(x', x_{n+1}) &= \\ \Delta_{\delta, n} u(x', x_{n+1}) + \delta^{-2} (u(x', x_{n+1} + \delta) + u(x', x_{n+1} - \delta) - 2u(x', x_{n+1})). \end{aligned}$$

Let $\{f_k^\delta\}_{k=1}^{K^\delta}$ be a sequence of eigenfunctions of the Dirichlet problem for the Laplacian in Ω^δ , discussed in 2.2. Then it is easy to check that the following functions form a basis for $\mathcal{H}_0(D^\delta)$

$$u_k^\delta(x) = f_k^\delta(x') \cosh(a_k^\delta x_{n+1}), \quad v_k^\delta(x) = f_k^\delta(x') \sinh(a_k^\delta x_{n+1}), \quad k = 1, 2, \dots, K^\delta,$$

where a_k^δ is the only positive solution of

$$\cosh \delta a_k^\delta = 1 + \frac{1}{2} \delta^2 \lambda_k^\delta.$$

Now we calculate the discrete harmonic measure of the bases of a truncated cylinder. Let g_N^δ be the δ -discrete harmonic function on $D_N^\delta(\Omega) = \overline{\Omega}^\delta \times ([-N, N] \cap (\delta\mathbb{Z}))$ defined by its boundary values

$$\begin{cases} g_N^\delta(x', \pm N) = 1 & x' \in \Omega^\delta \\ g_N^\delta(x', x_{n+1}) = 0 & x' \in \partial\Omega^\delta, -N \leq x_{n+1} \leq N. \end{cases}$$

Lemma 2. *The harmonic measure $g_N^\delta(x) = g_N^\delta(x', x_{n+1})$ is given by*

$$g_N^\delta(x', x_{n+1}) = \sum_{k=1}^{K^\delta} d_k^\delta f_k^\delta(x') \frac{\cosh(a_k^\delta x_{n+1})}{\cosh a_k^\delta N},$$

where $d_k^\delta = \sum_{x' \in \Omega^\delta} f_k^\delta(x')$.

Proof. Clearly g_N^δ is an even function with respect to x_{n+1} and therefore it can be written as

$$(11) \quad g_N^\delta(x', x_{n+1}) = \sum_{k=1}^{K^\delta} C_k f_k^\delta(x') \cosh(a_k^\delta x_{n+1}),$$

where the coefficients C_k satisfy the linear system of equations

$$1 = \sum_{k=1}^{K^\delta} C_k f_k^\delta(x') \cosh(a_k^\delta N),$$

for each $x' \in \Omega^\delta$. Since functions $\{f_k^\delta\}_{k=1}^{K^\delta}$ form an orthonormal basis, we obtain

$$(12) \quad C_k \cosh a_k^\delta N = \sum_{x'} f_k^\delta(x') = d_k^\delta.$$

Substituting (12) in (11) we get the required formula. \square

We conclude this subsection by one auxiliary inequality. We note that the values of the function $g_N^\delta(x', x_{n+1})$ on the middle hyperplane $\{x_{n+1} = 0\}$ are given by

$$g_N^\delta(x', 0) = \sum_{k=1}^{K^\delta} d_k^\delta f_k^\delta(x') \frac{1}{\cosh a_k^\delta N}.$$

Then a linear combination of the values of u on $\Omega^\delta \times \{0\}$ admits the following estimate

$$(13) \quad \begin{aligned} \sum_{x'} w(x') g_N^\delta(x', 0) &= \sum_{x'} \sum_{k=1}^{K^\delta} d_k^\delta w(x') f_k^\delta(x') \frac{1}{\cosh a_k^\delta N} \\ &\leq \sum_{k=1}^{K^\delta} \frac{|d_k^\delta|}{\cosh a_k^\delta N} \left(\sum_{x'} |w(x')|^2 \right)^{1/2}, \end{aligned}$$

we applied the Cauchy-Schwarz inequality and used that eigenfunctions f_k^δ are normalized by $\sum_{x'} |f_k^\delta(x')|^2 = 1$.

4.2. Phragmén-Lindelöf theorem, improved version

Now we prove a version of the Phragmén-Lindelöf theorem for δ -discrete subharmonic functions in truncated cylinder $D_N^\delta(\Omega)$. We want to show that if a subharmonic function is positive inside the cylinder, say at some points on the section $\Omega^\delta \times \{0\}$, then it grows at least exponentially. Moreover, we can give estimates on the truncated cylinders and not only asymptotic result as in Theorem 1. We use the following notation $u^+ = \max\{0, u\}$.

Theorem 4. Let u be a δ -discrete subharmonic function on $D_N^\delta(\Omega)$ such that $u(x', x_{n+1}) = 0$ when $x' \in \partial\Omega^\delta$ and u satisfies the following positivity condition on $\Omega \times \{0\}$

$$\sum_{x' \in \Omega^\delta} u^+(x', 0)^2 = A^2 K^\delta > 0.$$

Then

$$(14) \quad \max_{\Omega^\delta \times [-N, N]} u(x', x_{n+1}) \geq \frac{A}{2} \left(\sum_k \exp(-a_k^\delta N) \right)^{-1},$$

where $a_k^\delta = \delta^{-1} \cosh^{-1} \left(1 + \frac{1}{2} \delta^2 \lambda_k^\delta \right)$. In particular, there exists a constant C_Ω that depends only on Ω such that

$$(15) \quad \max_{\Omega^\delta \times [-N, N]} u(x', x_{n+1}) \geq C_\Omega A \exp(-a_1^\delta N)^{-1},$$

for any $N \in \mathbb{N}$ and any $\delta < \delta_0$.

The inequality (14) is more precise than (15). We write the constant explicitly and, as soon as λ_k^δ are known, the right hand side of (14) can be estimated. Clearly, the right hand side of (14) is of order $\exp(a_1^\delta N)$ when $N \rightarrow \infty$. This is expressed accurately in inequality (15). The constant C_Ω is not explicit, but it depends neither on N nor on δ , so we can also fix N and let δ go to zero to get estimates of continuous functions that can be approximated by discrete subharmonic ones.

Proof. Let $M_N = \max_{|x_{n+1}|=N} u(x', x_{n+1})$. Then by the maximum principle,

$$u(x', x_{n+1}) \leq M_N g_N^\delta(x', x_{n+1}) \quad \text{on } \Omega^\delta \times [-N, N],$$

where g_N^δ is the harmonic measure from Lemma 2, clearly $g_N^\delta \geq 0$. Taking the linear combination over $x' \in \Omega^\delta$ with non-negative coefficients $w(x') = u^+(x', 0)$ and using (13), we obtain

$$\sum_{x'} u^+(x', 0)^2 = \sum_{x'} u^+(x', 0) u^+(x', 0) \leq M_N \sum_{k=1}^{K^\delta} \frac{|d_k^\delta|}{\cosh a_k^\delta N} \left(\sum_{x'} |u^+(x', 0)|^2 \right)^{1/2}.$$

Then we have

$$M_N \geq \left(\sum_{x'} u^+(x', 0)^2 \right)^{1/2} \left(\sum_{k=1}^{K^\delta} \frac{|d_k^\delta|}{\cosh a_k^\delta N} \right)^{-1} = A (K^\delta)^{1/2} \left(\sum_{k=1}^{K^\delta} \frac{|d_k^\delta|}{\cosh a_k^\delta N} \right)^{-1}.$$

Applying the Cauchy-Schwarz inequality, we get

$$|d_k^\delta| = \left| \sum_{x'} f_k^\delta(x') \right| \leq \left(\sum_{x'} (f_k^\delta(x'))^2 \right)^{1/2} \left(\sum_{x'} 1 \right)^{1/2} \leq (K^\delta)^{1/2}.$$

Now, we combine the last two inequalities and obtain

$$M_N \geq A \left(\sum_{k=1}^{K^\delta} \frac{1}{\cosh a_k^\delta N} \right)^{-1}.$$

Then (14) follows from the following inequality

$$\sum_{k=1}^{K^\delta} \frac{1}{\cosh a_k^\delta N} \leq 2 \sum_{k=1}^{K^\delta} \exp(-a_k^\delta N).$$

To prove (15) we may assume that δ is small (otherwise we have an upper bound for K^δ). We partition the eigenvalues λ_k^δ into two groups. We choose a positive number c and define $I_1 = \{k : \lambda_k^\delta < c\delta^{-2}\}$ and $I_2 = \{k : \lambda_k^\delta \geq c\delta^{-2}\}$. Let also $c_0 = \cosh^{-1}(1+c)$, then

$$\sum_{k \in I_2} \exp(-a_k^\delta N) \leq \sum_{k \in I_2} \exp(-\delta^{-1}c_0 N) \leq K^\delta \exp(-\delta^{-1}c_0 N) \leq C_0 \exp(-a_1^\delta N),$$

when δ is small enough, since $K^\delta \leq C\delta^{-n}$ and $a_1^\delta \rightarrow (\lambda_1(\Omega))^{1/2}$ as $\delta \rightarrow 0$.

For the second part of the sum we have $\delta\sqrt{\lambda_k^\delta} < c$. We consider the function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by

$$\cosh \alpha(s) = 1 + \frac{1}{2}s^2.$$

Then $a_k^\delta = \delta^{-1}\alpha(\delta\sqrt{\lambda_k^\delta})$ and a simple calculation gives

$$\alpha'(s) = \frac{2}{\sqrt{4+s^2}}.$$

Denoting the minimum of the derivative of α on $[0, c]$ by d , we obtain

$$a_k^\delta \geq a_1^\delta + d \left((\lambda_k^\delta)^{1/2} - (\lambda_1^\delta)^{1/2} \right).$$

Now we partition I_1 further into $J_\ell = \{k : \ell \leq (\lambda_k^\delta)^{1/2} - (\lambda_1^\delta)^{1/2} < \ell + 1\}$, $\ell = 0, 1, \dots$ and let $|J_\ell|$ denote the cardinality of J_ℓ . We consider any cube Q such that $\Omega \subset Q$ and apply inequalities (3) and (4) to obtain

$$|J_\ell| \leq N_\Omega^\delta \left(\left((\lambda_1^\delta)^{\frac{1}{2}} + \ell + 1 \right)^2 \right) \leq N_Q^\delta \left(\left((\lambda_1^\delta)^{\frac{1}{2}} + \ell + 1 \right)^2 \right) \leq C_\Omega (\ell + 1)^n,$$

for each $\ell = 0, 1, \dots$. Finally, we get

$$\begin{aligned} \sum_{k \in I_1} \exp(-a_k^\delta) &\leq \sum_{\ell=0}^{\infty} \sum_{k \in J_\ell} \exp(-a_k^\delta N) \leq \sum_{\ell=0}^{\infty} \exp(-(a_1^\delta + \ell d)N) |J_\ell| \\ &\leq C_\Omega \exp(-a_1^\delta N) \sum_{\ell=0}^{\infty} (\ell + 1)^n \exp(-\ell d N). \end{aligned}$$

The last sum is finite and can be bounded by a constant independent of $N \in \mathbb{N}$ and δ . This concludes the proof of the theorem. \square

One of the differences between the continuous and discrete cases lies in the formulas connecting eigenvalues λ and corresponding numbers a . For the continuous case one has $a(\lambda) = \sqrt{\lambda}$ while for the discrete case the formula becomes

$$a^\delta(\lambda) = \delta^{-1} \cosh^{-1} \left(1 + \frac{1}{2} \delta^2 \lambda \right).$$

This function resembles $\sqrt{\lambda}$ on the interval $[0, c\delta^{-2}]$ but grows as $\log \lambda$ when $\lambda \rightarrow \infty$. To deal with the discrete case we have partitioned the set of eigenvalues into two parts.

4.3. Stability estimates for solution of the Dirichlet problem

A standard argument shows that estimates of the harmonic measure imply conditional stability estimates for harmonic function. We apply it for truncated cylinders and prove the following

Theorem 5. *Let h be a δ -discrete harmonic function on $D_N^\delta(\Omega)$ with boundary values f on $\partial\Omega^\delta \times [-N, N]$ and such that $|h(x', \pm N)| \leq M_N$. Then*

$$(16) \quad \max_{x'} |h(x', 0)| \leq \max |f| + C_\Omega (M_N + \max |f|) \exp(-a_1^\delta N).$$

In particular, if h is harmonic in $D^\delta(\Omega)$,

$$|h(x', x_{n+1})| = o(\exp(a_1^\delta |x_{n+1}|)) \quad \text{when } |x_{n+1}| \rightarrow \infty$$

and h is bounded on the boundary $\partial\Omega \times (\delta\mathbb{Z})$ then h is bounded by the same constant in $D^\delta(\Omega)$.

Proof. Let v_N be the δ -discrete harmonic function in the truncated cylinder $D_N^\delta(\Omega) = (\Omega \times (-N, N))^\delta$ that solves the following Dirichlet problem

$$\Delta_{n+1, \delta} v = 0, \quad v(x', \pm N) = 0, \quad x' \in \Omega^\delta, \quad \text{and } v(x', x_{n+1}) = f(x', x_{n+1}), \quad x' \in \partial\Omega^\delta.$$

By the maximum principle for the bounded domain $D_N^\delta(\Omega)$, $|v(x)| \leq \max |f|$. Then $u = h - v$ is δ -discrete harmonic function on $D_N^\delta(\Omega)$ that vanishes on the part $\partial\Omega^\delta \times [-N, N]$ of the boundary and satisfies

$$\max_{\Omega^\delta \times [-N, N]} |u(x', x_{n+1})| \leq \max |f| + M_N.$$

We compare it to a multiple of the harmonic measure g_N^δ and use the estimate

$$|g_N^\delta(x', 0)| \leq C_\Omega \exp(-a_1^\delta N)$$

that follows from the proof of Theorem 4. Then we obtain

$$|u(x', 0)| \leq C_\Omega (M_N + \max |f|) \exp(-a_1^\delta N).$$

This implies (16). The second statement of the theorem follows from (16).

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