Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **7** (2013), 72–82. doi:10.2298/AADM130109001K

ON DISTANCES IN SIERPIŃSKI GRAPHS: ALMOST-EXTREME VERTICES AND METRIC DIMENSION

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Sierpiński graphs S_p^n form an extensively studied family of graphs of fractal nature applicable in topology, mathematics of the Tower of Hanoi, computer science, and elsewhere. An almost-extreme vertex of S_p^n is introduced as a vertex that is either adjacent to an extreme vertex of S_p^n or is incident to an edge between two subgraphs of S_p^n isomorphic to S_p^{n-1}. Explicit formulas are given for the distance in S_p^n between an arbitrary vertex and an almost-extreme vertex. The formulas are applied to compute the total distance of almost-extreme vertices and to obtain the metric dimension of Sierpiński graphs.

1. INTRODUCTION

Sierpiński graphs S_p^n were introduced for at least three reasons. In [**18**], they were motivated by topological studies of universal spaces (cf. [**17**]) and the fact that the base-3 Sierpiński graphs S_3^n are isomorphic to the Tower of Hanoi graphs on 3 pegs. Independently, a class of graphs called WK-recursive networks was introduced in computer science in [**3**], see also [**5**]. WK-recursive networks are very similar to Sierpiński graphs, they can be obtained from Sierpiński graphs by adding a link (an open edge) to each of its extreme vertices.

Graphs S_p^n were studied by now from numerous points of view, the reader is invited to read the recent paper [**12**] about colorings of these graphs and references therein; see also [**6**] for more coloring results. Of the many other investigations, we only mention a few explicitly. An appealing application of Sierpiński graphs is

2010 Mathematics Subject Classification. 05C12.
Keywords and Phrases. Sierpiński graph, almost-extreme vertex, total distance of vertex, metric dimension.

due to Romik [**23**] who designed, based on Sierpiński labelings, a finite automaton particularly useful for the Tower of Hanoi problem. In [**19**] the structure of Sierpiński graphs was the key to determine for the first time the exact genus of infinite families of fractal graphs. Recently, the hub number of Sierpiński-like graphs was determined in [**15**].

Metric issues received a special attention on Sierpiński graphs. This is in particular due to the fact that shortest paths in the base-3 Sierpiński graphs correspond to optimal solutions in the Tower of Hanoi puzzle. In the seminal paper [**18**] a formula for the distance between vertices in S_p^n was proved, we state it as Theorem 2. Then, in [**11**], additional metric properties of these graphs were investigated, in particular establishing a connection with Stern's diatomic sequence. Parisse [**20**] followed with a paper in which he studied, among other matters, the diameter, the eccentricity, the radius, and the center of these graphs. Wiesenberger [**25**] obtained a formula for the average distance in S_p^n. The formula is far from being trivial, it extends over several lines! Very recently, Hinz and Parisse [**13**] succeeded in determining the average eccentricity and its standard deviation for all Sierpiński graphs.

The metric dimension of a graph turned out to be a natural concept while studying several different problems and was consequently also reinvented in numerous disguises. (An impressive list of its applications can be found in [**10**]) It is thus clear that this dimension presents an intrinsic graph invariant. For the first time it was independently introduced in 1974 and 1975 by Harary and Melter [**9**] and Slater [**24**], respectively. We refer to the recent semi-survey paper of Bailey and Cameron [**1**] for a great source on historical developments, connections to other invariants, non-standard terminology, and a long list of references. Another survey source for the dimension is [**7**]. Here we just recall that the metric dimension has been studied on Cartesian products of graphs [**2, 22**], distance-regular graphs [**8**], and circulant graphs [**14**].

Our paper is organized as follows. In the next section definitions, concepts, and results needed in this paper are given. Then, in Section 3, we obtain distances between almost-extreme vertices and other vertices. The advantage of the new formulas compared to Theorem 2 is that we do not need to compute the minima of related expressions. As a by-product the metric dimension of the Sierpiński graphs is determined. We point out here that in general it is very difficult to determine the exact metric dimension, see [**10**] and references therein for complexity issues on metric dimension. In the final section we use the derived formulas to compute the total distance of almost-extreme vertices.

2. PRELIMINARIES

The graphs considered are simple and connected. The *distance* $d_G(u,v)$ between vertices u and v in a graph G is the standard shortest path distance. For a vertex u of G the *total distance* $d_G(u)$ *of* u is $d_G(u) = \sum_{v \in V(G)} d_G(u,v)$. Whenever G

is clear from the context we write $d(u, v)$ and $d(u)$ instead of $d_G(u, v)$ and $d_G(u)$, respectively.

The set $\{1, 2, \ldots, n\}$ is shortly denoted by $[n]$ and the set $\{0, 1, \ldots, n-1\}$ by $[n]_0$.

Let G be a graph, then $R \subseteq V(G)$ is a *resolving set* if each vertex of G is uniquely determined by the distances to the vertices of R. More precisely, let $R = \{u_1, \ldots, u_k\}$, $k \geq 1$, then R is resolving if $(d(x, u_1), \ldots, d(x, u_k)) \neq (d(y, u_1), \ldots, d(y, u_k))$ holds for any two distinct vertices $x, y \in V(G)$. In other words, any two distinct vertices $x, y \in V(G)$ are resolved by some vertex of R, that is, there exists a vertex $u_i \in R$ such that $d(x, u_i) \neq d(y, u_i)$. The *metric dimension* of G, denoted $\mu(G)$, is the size of a minimum resolving set.

Let $p \in \mathbb{N}$, $p \geq 2$, throughout. For $n \in \mathbb{N}_0$ the *Sierpiński graph* S_p^n is defined on the vertex set $[p]^n$. Two vertices, written as $s = s_n \ldots s_1$ and $t = t_n \ldots t_1$, are adjacent if and only if they are of the form $s = \underline{s} s_\delta t_\delta^{\delta-1}$, $t = \underline{s} t_\delta s_\delta^{\delta-1}$ with $\delta \in [n]$, $\underline{s} \in [p]^{n-\delta}$, and $s_\delta \neq t_\delta$.

Note that $S_p^0 \cong K_1$, $S_p^1 \cong K_p$ for any p and that $S_2^n \cong P_{2^n}$ for every n. For S_5^3 see Figure 1. For $i \in [p]$, let iS_p^n be the subgraph of S_p^{n+1} induced by the vertices of the form $s = is_n \ldots s_1$; this subgraph is isomorphic to S_p^n.

Let $n \in \mathbb{N}$. Then S_p^n contains p *extreme vertices* of the form $i \ldots i = i^n$; they have degree $p - 1$, while all the other vertices are of degree p. We also introduce *almost-extreme vertices* of S_p^{n+1} as those vertices which are either of the form $i^n j$ or ij^n, where $i \neq j$. In Figure 1 the extreme vertices of S_5^3 are emphasized with filled circles and the almost-extreme vertices are emphasized as triangles (vertices of the form ij^2) and as diamonds (vertices of the form $i^2 j$).

Obviously, for $n \geq 2$ the graph S_p^{n+1} contains $p(p-1)$ vertices of the form $i^n j$ and also $p(p-1)$ vertices of the form ij^n. The almost-extreme vertex $i^n j$ is adjacent to the extreme vertex i^{n+1} and the almost-extreme vertex ij^n is incident with the edge between iS_p^n and jS_p^n. Thus, there are $2p(p-1)$ almost-extreme vertices. For $n = 1$ the vertices $i^n j$ and ij^n coincide, hence in S_p^2 there are exactly $p(p-1)$ almost-extreme vertices and any vertex is either extreme or almost-extreme.

The distance between a vertex of S_p^n and an extreme vertex can be computed as follows, where we use Iverson's convention that $(X) = 1$, if statement X is true, and $(X) = 0$, if X is false.

Lemma 1. **[18]** *For any $j \in [p]$ and any vertex $s = s_n \ldots s_1$ of S_p^n,*

$$d(s, j^n) = \sum_{d=1}^{n} (s_d \neq j) \cdot 2^{d-1}\,.$$

Moreover, there is exactly one shortest path between s and j^n.

An immediate consequence of Lemma 1 is that for any vertex s of S_p^n,

$$\sum_{i=1}^{p} d(s, i^n) = (p-1)(2^n - 1)\,. \tag{1}$$

(Cf. also [**20**, Proposition 2.5].) It follows that $\{i^n \mid i \in [p-1]\}$ is a resolving set for S_p^n (cf. [**21**, Lemme 3.5]): let s and t be vertices with $d(s, i^n) = d(t, i^n)$ for all $i \in [p-1]$, such that by (1) also $d(s, p^n) = d(t, p^n)$ holds; but then, by the formula in Lemma 1, $s = t$.

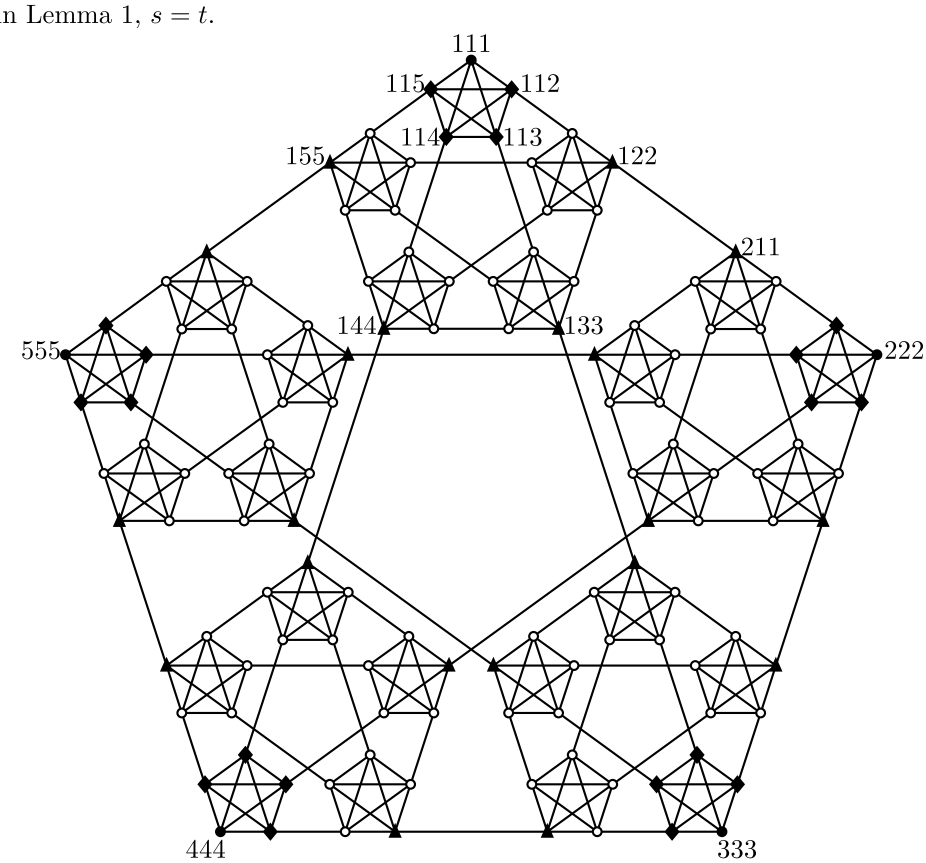

Figure 1. S_5^3 with its extreme and almost-extreme vertices emphasized

Note further that $d(i^n, j^n) = 2^n - 1$ for any $i \neq j$. More generally, the distance between arbitrary vertices of S_p^n can be determined in the following way:

Theorem 2. [**18**] *For* $i, j \in [p]$, $i \neq j$, $\delta \in [n]$, $\overline{s}$, $\overline{t} \in [p]^{\delta-1}$, *and* $\underline{s} \in [p]^{n-\delta}$, *let*

$$d_0(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d(\overline{s}, j^{\delta-1}) + 1 + d(\overline{t}, i^{\delta-1}),$$
$$\forall \ell \in [p] : \; d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d(\overline{s}, \ell^{\delta-1}) + 1 + 2^{\delta-1} + d(\overline{t}, \ell^{\delta-1}).$$

Then,

$$d(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = \min\left\{d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) \mid \ell \in [p+1]_0\right\}.$$

Remark 3. The above minimum can be equivalently written as

$$\min\left\{d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) \mid \ell \in [p+1]_0 \setminus \{i, j\}\right\}.$$

The respective paths realizing the values $d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t})$ are unique. The minimum can be obtained by at most one $\ell \in [p]$. Therefore, there are at most two shortest paths between any two vertices.

It is clear from the theorem that the distance between two vertices does not depend on a common prefix; in particular, for $i \in [p]$, $n \in \mathbb{N}_0$, and $s, t \in [p]^n$,

$$d(is, it) = d(s, t). \tag{2}$$

3. DISTANCES TO ALMOST-EXTREME VERTICES

In this section we apply Theorem 2 to the case of almost-extreme vertices and begin with the almost-extreme vertices that are adjacent to extreme vertices.

Proposition 4. *Let $i, j, k \in [p]$, $i \neq j$, $n \in \mathbb{N}_0$, and $s \in [p]^n$. Then*

$$d_{S_p^{n+1}}(is, j^n k) = d(s, j^n) + 2^n - (i = k)\,.$$

Proof. We may assume that $n \in \mathbb{N}$. By the definition of the almost-extreme vertices, $j \neq k$. Then, for $\ell \in [p] \setminus \{j\}$ and using Lemma 1,

$$\begin{aligned} d_0(is, j^n k) &= d(s, j^n) + 1 + d(j^{n-1}k, i^n) = d(s, j^n) + 2^n - (i = k) \\ &\leq 2^{n+1} - 1 \leq 1 + 2^n + d(j^{n-1}k, \ell^n) \leq d_\ell(is, j^n k)\,. \end{aligned}$$

(Here equality holds if and only if $i \neq k = \ell$, $d(s, j^n) = 2^n - 1$, and $d(s, \ell^n) = 0$, i.e. for $s = k^n$ and d_k. Only in this case there are two shortest paths between is and $j^n k$.)

Remark 5. It follows immediately from Proposition 4 that $d(is, j^n k) = d(is, j^{n+1})$ if $|\{i, j, k\}| = 3$.

This observation now allows us to approach the question of metric dimension.

Corollary 6. *For any $n \in \mathbb{N}_0$,*

$$\mu(S_p^{n+1}) = p - 1\,.$$

Moreover, if R is a minimum resolving set, then $|R \cap jS_p^n| \leq 1$ holds for any $j \in [p]$.

Proof. Let $R \subset V(S_p^{n+1})$. Assume that $R \cap jS_p^n = \emptyset = R \cap kS_p^n$ for some $j \neq k$. It then follows from Remark 5 that for each $r \in R$ we have $d(r, j^n k) = d(r, j^{n+1})$, such that R cannot be a resolving set for S_p^{n+1}. Hence each resolving set must contain at least $p - 1$ elements. Since we have seen earlier that (any) $p - 1$ extreme vertices form a resolving set, we deduce that $\mu(S_p^{n+1}) = p - 1$ and, with recourse to the pigeonhole principle, that no jS_p^n can contain more than one element of a minimal resolving set. □

The first assertion of Corollary 6 has been found independently and at the same time by Aline Parreau [**21**, Théorème 3.6].

We now turn to the other class of almost-extreme vertices of S_p^{n+1}. To facilitate the formulation of a formula for $d(is, jk^n)$, we call $s \in [p]^n$ *special* (with respect to $i, j, k \in [p]$, $|\{i, j, k\}| = 3$, i.e. if $p \geq 3$), if there is a $\delta \in [n]$ such that $s = \underline{s}k\overline{s}$ with $\underline{s} \in ([p] \setminus \{j, k\})^{n-\delta}$ and $\overline{s} \in [p]^{\delta-1}$. Then the following holds.

Proposition 7. *Let $i, j, k \in [p]$, $i \neq j$, $j \neq k$, $n \in \mathbb{N}$, and $s \in [p]^n$. Then*

$$d_{S_p^{n+1}}(is, jk^n) = \begin{cases} d(s, k^n) + 2^n + 1, & \text{if } s \text{ is special,} \\ d(s, j^n) + 2^n - (i = k)(2^n - 1), & \text{otherwise.} \end{cases}$$

Proof. We have

$$d_0(is, jk^n) = d(s, j^n) + 1 + d(i^n, k^n) = d(s, j^n) + 2^n - (i = k)(2^n - 1)$$

and for $\ell \in [p] \setminus \{i, j\}$,

$$d_\ell(is, jk^n) = d(s, \ell^n) + 1 + 2^n + d(\ell^n, k^n)\,.$$

This is strictly larger than $d_0(is, jk^n)$, if $\ell \neq k$. So we may assume that $k \neq i$ and have to compare $d_0(is, jk^n)$ with

$$d_k(is, jk^n) = d(s, k^n) + 1 + 2^n$$

i.e. we look at the sign of

$$\begin{aligned} \rho(s) &:= d_0(is, jk^n) - d_k(is, jk^n) \\ &= d(s, j^n) - d(s, k^n) - 1 \\ &= \sum_{d=1}^{n} ((s_d = k) - (s_d = j)) \cdot 2^{d-1} - 1\,. \end{aligned}$$

Now $\sum_{d=1}^{n} \tau_d \cdot 2^{d-1} \geq 1$, if and only if $\tau = \tau_n \ldots \tau_1 \in \{-1, 0, 1\}^n$ has the special form $0^{n-\delta}1\overline{\tau}$ with $\overline{\tau} \in \{-1, 0, 1\}^{\delta-1}$ for some $\delta \in [n]$ (with equality if and only if $\overline{\tau} = (-1)^{\delta-1}$). This is equivalent to s being special. (Note that there are two shortest paths if and only if $s = \underline{s}kj^{\delta-1}$.) □

Proposition 7 is illustrated in Figure 2 on S_6^4. The subgraph $1S_6^3$ is drawn explicitly and special vertices with respect to $i = 1$, $j = 2$, $k = 5$ are drawn with filled circles.

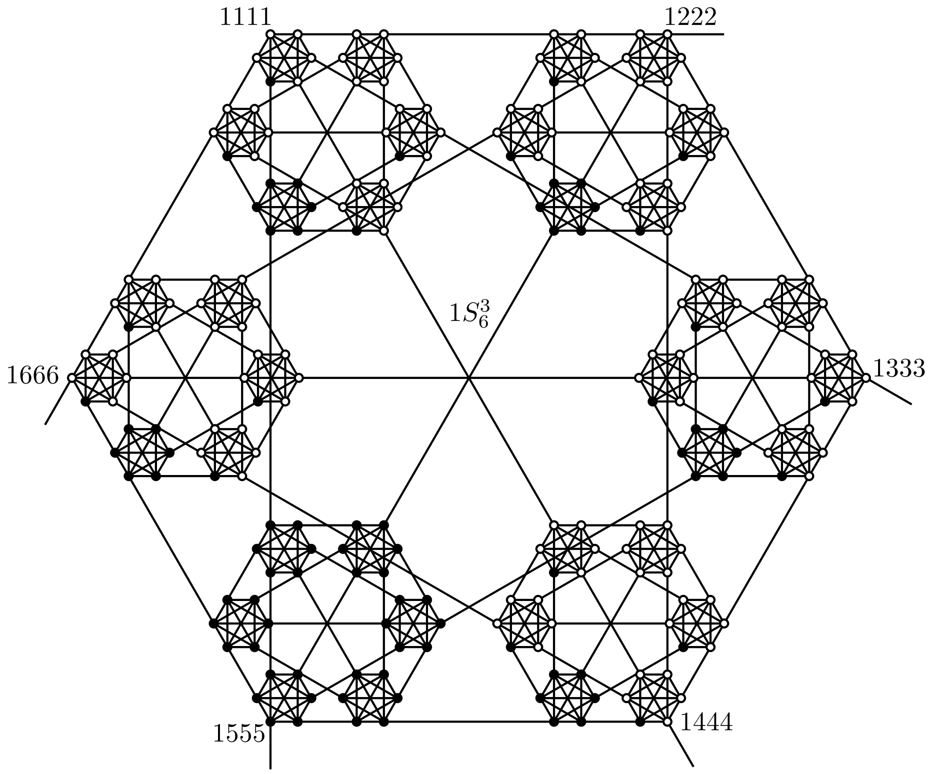

Figure 2. Illustration of Proposition 7 on S_6^4

4. TOTAL DISTANCE OF ALMOST-EXTREME VERTICES

The total distance of a vertex in particular plays an important role in mathematical chemistry, cf. [**16**], because it is a building block for the extensively investigated Wiener index of a graph. In this section we determine the total distance of almost-extreme vertices of Sierpiński graphs. To make the paper self-contained we first reprove the following result that can be found in [**25**] as well as in the proof of [**20**, Corollary 2.6].

Lemma 8. *For any $n \in \mathbb{N}$ and each $i \in [p]$,*

$$d_{S_p^n}(i^n) = p^{n-1}(p-1)(2^n - 1).$$

Proof. Since for every $d \in [p]$ there are $p^{n-1}(p-1)$ vertices $s = s_n \dots s_1$ with $s_d \neq i$, $i \in [p]$, it follows by Lemma 1

$$\sum_{s \in [p]^n} d(s, i^n) = \sum_{s \in [p]^n} \sum_{d=1}^{n} (s_d \neq i) \cdot 2^{d-1} = \sum_{d=1}^{n} \left(\sum_{s \in [p]^n} (s_d \neq i) \right) \cdot 2^{d-1}$$

$$= p^{n-1}(p-1)\sum_{d=1}^{n} 2^{d-1} = p^{n-1}(p-1)\left(2^n - 1\right) .$$

Theorem 9. *Let $j, k \in [p]$, $j \neq k$, and $n \in \mathbb{N}_0$. Then*

$$d_{S_p^{n+1}}(j^n k) = \frac{p-1}{p}(2p)^{n+1} - \left(1 + \frac{1}{p(p-1)}\right)p^{n+1} + \frac{p}{p-1}.$$

Proof. Set $x_0 = 1$ and $x_{n+1} = d_{S_p^{n+1}}(j^n k)$, $n \geq 0$. Then, using (2), Proposition 4, and Lemma 8,

$$\begin{aligned} x_{n+1} &= \sum_{i\in[p]}\sum_{s\in[p]^n} d(is, j^n k) \\ &= \sum_{s\in[p]^n} d(js, j^n k) + \sum_{s\in[p]^n} d(ks, j^n k) + \sum_{i\in[p]\setminus\{j,k\}}\sum_{s\in[p]^n} d(is, j^n k) \\ &= x_n + \frac{2p-1}{p}p^n(2^n - 1) + (p-2)\left(\frac{2p-1}{p}(2p)^n - \frac{p-1}{p}p^n\right) \\ &= x_n + \frac{(2p-1)(p-1)}{p}(2p)^n - \left(1 + \frac{(p-1)^2}{p}\right)p^n. \end{aligned}$$

A straightforward calculation leads to the desired result.

Remark 10. The expression of Theorem 9 can be further transformed as follows:

$$\begin{aligned} d_{S_p^{n+1}}(j^n k) &= \frac{p-1}{p}(2p)^{n+1} - \left(1 + \frac{1}{p(p-1)}\right)p^{n+1} + \frac{p}{p-1} \\ &= p^n(p-1)2^{n+1} - p^n(p-1) + p^n(p-1) - p^{n+1} - \frac{p^n}{p-1} + \frac{p}{p-1} \\ &= p^n(p-1)(2^{n+1}-1) - p\cdot\frac{p^n - 1}{p-1} \\ &= d_{S_p^{n+1}}(j^{n+1}) - \sum_{\ell=1}^{n} p^{\ell} . \end{aligned}$$

This alternative way to calculate $d_{S_p^{n+1}}(j^n k)$ can be interpreted as $d_{S_p^{n+1}}(j^{n+1})$ minus the additional step to all the vertices reachable directly from $j^n k$ and there are $p + p^2 + p^3 + \cdots + p^n$ such vertices.

Based on (2), Lemma 8, and Proposition 7, the corresponding result for the other almost-extreme vertices reads as follows.

Theorem 11. *Let $j, k \in [p]$, $j \neq k$, and $n \in \mathbb{N}_0$. Then*

$$d_{S_p^{n+1}}(jk^n) = \frac{p^2-2}{p(p+2)}(2p)^{n+1} - \frac{p-2}{2p}p^{n+1} - \frac{p}{2(p+2)}(p-2)^{n+1} .$$

Proof. Let us first calculate

$$d_0(jk^n) := \sum_{is\in[p]^{n+1}} d_0(is, jk^n) = d(k^n) + d(j^n) + p^n + (p-2)(d(j^n) + (2p)^n)$$
$$= (2p-3)(2p)^n - (p-2)p^n\,.$$

However, if $p \geq 3$, this value over-estimates $d(jk^n)$, because we did not take into account the smaller distance between is and jk^n if s is special with respect to i, j, k. We therefore have to calculate the sum $\mathrm{P} := \sum \rho(s)$ over all such special s and, for symmetry reasons, a fixed $i \in [p] \setminus \{j, k\}$ with ρ defined as in the proof of Proposition 7. We get

$$\mathrm{P} = \sum_{\delta=1}^{n}\left((p-2)^{n-\delta}p^{\delta-1}(2^{\delta-1}-1) + \sum_{\overline{s}\in[p]^{\delta-1}}\sum_{d=1}^{\delta-1}((\overline{s}_d = k) - (\overline{s}_d = j))\cdot 2^{d-1}\right).$$

The sum inside the large brackets is zero, because $\overline{s}_d$ is equal to k as often as it is equal to j. Therefore,

$$\mathrm{P} = \sum_{\delta=1}^{n}(p-2)^{n-\delta}(2p)^{\delta-1} - \sum_{\delta=1}^{n}(p-2)^{n-\delta}p^{\delta-1} = \frac{1}{p+2}(2p)^n - \frac{1}{2}p^n + \frac{p}{2(p+2)}(p-2)^n\,.$$

The statement of the theorem now follows from $d(jk^n) = d_0(jk^n) - (p-2)\mathrm{P}$. □

Note that for $n = 2$, both kinds of almost-extreme vertices coincide and their total distances must be equal. Indeed, for $n = 2$, Theorems 9 and 11 both give the value $d_{S_p^2}(jk) = p(3p-4)$. We also add that the expression of Theorem 11 can be rewritten as follows:

$$d_{S_p^{n+1}}(jk^n) = \frac{1}{2}p^n(p-2)(2^{n+1}-1) + \frac{p}{2}\sum_{\ell=0}^{n}(2p)^{n-\ell}(p-2)^{\ell}\,.$$

In this case, however, we have no interpretation for the formula such as in Remark 10.

For the classical case $p = 3$, where S_3^n is isomorphic to the Hanoi graph H_3^n with extreme vertices mapped onto perfect ones and almost-extreme vertices being transformed into vertices of the same form, we finally obtain from Lemma 8 and Theorems 9 and 11:

Corollary 12. *Let $i, j, k \in [p]$, $j \neq k$, and $n \in \mathbb{N}_0$. Then*

$$d_{S_3^n}(i^n) = \frac{2}{3}3^n(2^n-1) = d_{H_3^n}(i^n)\,.$$
$$d_{S_3^{n+1}}(j^nk) = \frac{2}{3}\cdot 6^{n+1} - \frac{7}{6}\cdot 3^{n+1} + \frac{3}{2} = d_{H_3^{n+1}}(j^nk)\,,$$
$$d_{S_3^{n+1}}(jk^n) = \frac{7}{15}\cdot 6^{n+1} - \frac{1}{6}\cdot 3^{n+1} - \frac{3}{10} = d_{H_3^{n+1}}(jk^n)\,.$$

Acknowledgements. We are utmost grateful to the three referees for their numerous constructive suggestions. One of the referees suggested a thorough reorganization of the paper with more compact statements and shorter proofs which we followed with pleasure. We are also grateful to Andreas M. Hinz and to Daniele Parisse for valuable discussions.

This work has been financed by ARRS Slovenia under the grant P1-0297 and within the EUROCORES Programme EUROGIGA/GReGAS of the European Science Foundation. The first author is also with the Institute of Mathematics, Physics and Mechanics, Ljubljana.

REFERENCES

1. R. F. Bailey, P. J. Cameron: *Base size, metric dimension and other invariants of groups and graphs.* Bull. Lond. Math. Soc., **43** (2011), 209–242.
2. J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood: *On the metric dimension of Cartesian products of graphs.* SIAM J. Discrete Math., **21** (2007), 423–441.
3. G. Della Vecchia, C. Sanges: *A recursively scalable network VLSI implementation.* Future Generation Comput. Syst., **4** (1988), 235–243.
4. F. Foucaud, R. Klasing, A. Kosowski, A. Raspaud: *On the size of identifying codes in triangle-free graphs.* Discrete Appl. Math., **160** (2012), 1532–1546.
5. J.-S. Fu: *Hamiltonian connectivity of the WK-recursive network with faulty nodes.* Inform. Sci., **178** (2008), 2573–2584.
6. H.-Y. Fu: *$\{P_r\}$-free colorings of Sierpiński-like graphs.* Ars Combin., **105** (2012), 513–524.
7. W. Goddard, O. R. Oellermann: *Distance in graphs.* Structural Analysis of Complex Networks, Birkhäuser/Springer, New York, 2011, 49–72.
8. J. Guo, K. Wang, F. Li: *Metric dimension of some distance-regular graphs.* J. Comb. Optim., in press, DOI 10.1007/s10878-012-9459-x.
9. F. Harary, R. A. Melter: *On the metric dimension of a graph.* Ars Combin., **2** (1976), 191–195.
10. M. Hauptmann, R. Schmied, C. Viehmann: *Approximation complexity of metric dimension problem.* J. Discrete Algorithms, **14** (2012), 214–222.
11. A. M. Hinz, S. Klavžar, U. Milutinović, D. Parisse, C. Petr: *Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence.* European J. Combin., **26** (2005), 693–708.
12. A. M. Hinz, D. Parisse: *Coloring Hanoi and Sierpiński graphs.* Discrete Math., **312** (2012), 1521–1535.
13. A. M. Hinz, D. Parisse: *The average eccentricity of Sierpiński graphs.* Graphs Combin., **28** (2012), 671–686.
14. M. Imran, A. Q. Baig, S. A. U. H. Bokhary, I. Javaid: *On the metric dimension of circulant graphs.* Appl. Math. Lett., **25** (2012), 320–325.
15. C.-H. Lin, J.-J. Liu, Y.-L. Wang, W. C.-K. Yen: *The hub number of Sierpiński-like graphs.* Theory Comput. Syst., **49** (2011), 588–600.

16. X. Lin: *On the Wiener index and circumference.* MATCH Commun. Math. Comput. Chem., **67** (2012), 331–336.

17. S. Lipscomb: *Fractals and Universal Spaces in Dimension Theory.* Springer, Berlin, 2009.

18. S. Klavžar, U. Milutinović: *Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem.* Czechoslovak Math. J., **47** (122) (1997), 95–104.

19. S. Klavžar, B. Mohar: *Crossing numbers of Sierpiński-like graphs.* J. Graph Theory, **50** (2005), 186–198.

20. D. Parisse: *On some metric properties of the Sierpiński graphs S_k^n.* Ars Combin., **90** (2009), 145–160.

21. A. Parreau: *Problèmes d'identification dans les graphes.* Ph.D. Thesis, Université de Grenoble, July 2012.

22. J. Peters-Fransen, O. R. Oellermann: *The metric dimension of Cartesian products of graphs.* Util. Math., **69** (2006), 33–41.

23. D. Romik: *Shortest paths in the Tower of Hanoi graph and finite automata.* SIAM J. Discrete Math., **20** (2006), 610–622.

24. P. Slater: *Leaves of trees.* Congr. Numer., **14** (1975), 549–559.

25. H. L. Wiesenberger: *Stochastische Eigenschaften von Hanoi- und Sierpiński-Graphen.* Diploma Thesis, Ludwig-Maximilians-Universität München, 2010.

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(Received July 20, 2012)
(Revised January 9, 2013)

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