EXTENDING BICOLORINGS FOR STEINER TRIPLE SYSTEMS

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In memory of Lucia Gionfriddo

We initiate the study of extended bicolorings of Steiner triple systems (STS) which start with a \( k \)-bicoloring of an STS(\( v \)) and end up with a \( k \)-bicoloring of an STS(\( 2v + 1 \)) obtained by a doubling construction, using only the original colors used in coloring the subsystem STS(\( v \)). By producing many such extended bicolorings, we obtain several infinite classes of orders for which there exist STSs with different lower and upper chromatic number.

1. INTRODUCTION

A Steiner triple system (STS) is a pair \((V, B)\) where \( V \) is a \( v \)-set and \( B \) is a collection of 3-subsets of \( V \) called triples such that every 2-subset of \( V \) is contained in exactly one triple, see [4]. A coloring of an STS \((V, B)\) is a mapping \( \phi : V \rightarrow C \); the elements of \( C \) are called colors. If \(|C| = k\), we have a \( k \)-coloring. For each \( c \in C \), the set \( \phi^{-1}(c) = \{ x : \phi(x) = c \} \) is a color class. A coloring \( \phi \) of \((V, B)\) is a bicoloring if \(|\phi(B)| = 2\) for all \( B \in B \). Here \( \phi(B) = \bigcup_{x \in B} \phi(x) \). Thus in a bicoloring of \((V, B)\), every triple has two elements in one color class and one in another class, so there are no monochromatic triples nor polychromatic triples (i.e. triples receiving three colors). A strict \( k \)-bicoloring is one in which exactly \( k \) colors are used. From now on we assume that all our bicolorings are strict, unless the contrary is explicitly stated.
Considerations of bicolorings of Steiner triple systems arose from the theory of mixed hypergraphs pioneered by Voloshin \[22, 23\]. In a mixed hypergraph setting, there are two kinds of edges: $C$-edges which must contain two vertices colored with the same color, and $D$-edges which must contain two vertices of different colors. Requiring all edges of a Steiner system to be both, $C$-triples and $D$-triples leads to the concept of bicolorings. In the literature, often the terms BSTS, BSQS, or bi-STS coloring are used instead of bicoloring (cf. \[6, 14, 15, 16, 17, 18, 19\]). We can also find results related to particular color patterns for different designs in \[1, 5, 7, 9, 10, 11, 12, 13, 20\].

The minimum (maximum) possible number $k$ in a strict $k$-bicoloring of an STS is called the lower (upper) chromatic number of the STS. However, not every STS has a bicoloring. The smallest such example occurs for STSs of order 15: of which starts with an STS($v$)

$K$ is a 1-factorization of the complete graph $v$.

The independence number is at most $\left\lfloor\frac{v}{3}\right\rfloor$.

We can also find results related to particular color patterns for different designs in some other settings.

The question that we want to address is the following. Given an STS($v$) with a bicoloring $C = C(n_1, \ldots, n_k)$, when does there exist an STS(2$v$ + 1) obtained by a doubling construction which admits a bicoloring $C' = C'(n_1', \ldots, n_k')$? In other words, when can we color the elements of $X$ with the original $k$ colors of the $k$-bicoloring $C$ without introducing an extra color as above? If such a coloring exists, we call it an extended bicoloring of $C$. Thus extended bicolorings may exist only for orders $2v + 1 \equiv 3$ or $7 \pmod{12}$ as $v \equiv 1$ or $3 \pmod{6}$.

The importance of extended bicolorings lies in the fact that they enable one
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to construct STSs with different lower and upper chromatic numbers; there are
only scarce results in the literature on the latter (cf., e.g., [14]). The extendibility
of partial colorings is a relevant issue in graph theory (see [2, 21]), both for its the-
oretical interest and practical applications. Here we initiate the study of extending
bicolorings in Steiner triple systems, with the aim to derive consequences in the
coloring theory of mixed hypergraphs. In this way the present work relates several
intensively studied areas.

2. EXTENDED BICOLORINGS

Let \( S = (V, B) \) be an STS(\( v \)) which is \( k \)-bicolorable with \( C = C(n_1, \ldots, n_k) \)
and let \( S' = (X, C) \) be an STS(\( 2v + 1 \)) obtained from \( S \) by a doubling construc-
tion. We are trying to investigate the conditions under which there exists an
extended bicoloring of \( S' \), say \( C' = C'(n_1', \ldots, n_k') \) where the elements of the subsystem \( (V, B) \)
are colored as in \( C \), and the elements of \( Y = X \setminus V \) are colored with the same colors
as those used in \( C \). If \( c_i = n_i' - n_i, 1 \leq i \leq k \) are the numbers of vertices in \( Y \)
colored with the color \( i \in C \) then clearly, \( \sum_{i=1}^{k} c_i = v + 1 \) (it may happen that \( c_j = 0 \)
for some \( j \in \{1, \ldots, k\} \)). Beside this obvious condition, the following is a necessary
condition for the existence of an extended \( k \)-bicoloring of \( S' \).

**Theorem 1.** Let \( S = (V, B) \) be an STS(\( v \)) which is \( k \)-bicolorable with \( C = C(n_1, \ldots, n_k) \)
and let \( S' = (X, C) \) be an STS(\( 2v + 1 \)) obtained from \( S \) by a doubling construc-
tion. With the notation as above,

\[
\sum_{i=1}^{k} c_i^2 + 2 \sum_{i=1}^{k} n_i c_i = (v + 1)^2.
\]

**Proof.** The number of pairs of elements of \( Y \) equals \( \binom{v + 1}{2} \). Clearly, the number
of monochromatic pairs among these is \( \sum_{i=1}^{k} \binom{c_i}{2} \). On the other hand, the number of
two-colored pairs among these is \( \sum_{i=1}^{k} n_i c_i \); indeed, if \( a_i \in V \) is colored with color \( j \),
then any pair \( \{x, y\} \) in the 1-factor \( F_i \) is either monochromatic or else one of \( x, y \)
is colored with color \( j \). Consequently, the number of two-colored pairs in \( F_i \) is \( n_j \). Thus we have

\[
\sum_{i=1}^{k} \binom{c_i}{2} + \sum_{i=1}^{k} n_i c_i = \binom{v + 1}{2}
\]

whence (1) follows easily. \( \square \)

Solutions of (1) will be called solutions with respect to \( C \). We stress that
condition (1) given by Theorem 1 is only necessary for the existence of an extended
bicoloring. It certainly is not sufficient: in [6] condition (1) was determined for \( v = 2^h - 1 \) and all of its solutions were determined for \( h \leq 10 \), nevertheless these solutions do not lead to any extended bicolorings.

**Corollary 2.** Let \( S' \) be a \( k \)-bicolorable \( STS(2v + 1) \) obtained by a doubling construction from a \( k \)-bicolorable \( STS(v) \) with the coloring \( C = C(n_1, \ldots, n_k) \), and let \((c_1, \ldots, c_k)\) be a solution to (1) with respect to \( C \).

1. If there is a \( c_j = 0 \), then all \( c_i \)'s are even.
2. If there is \( j \) such that \( c_j > \frac{v+1}{2} \), then there exists no extended bicoloring of \( C \).

**Proof.** 1. If there is a \( j \) such that \( c_j = 0 \), then in any factor corresponding to an element \( a_i \in V \) colored with the color \( j \), all pairs must be monochromatic which implies that every \( c_i \) has to be even.

2. If \( c_j > \frac{v+1}{2} \) for some \( j \), then in all factors associated with elements of \( V \) colored with color \( j \), there must exist monochromatic pairs of color \( j \), and thus monochromatic triples, which is a contradiction.

We illustrate the use of Corollary 2 on the example of a (potential) extended bicoloring of an \( STS(19) \). First notice that no extended bicolorings of \( STS(v) \) exist for \( v = 7 \) or \( v = 15 \), as shown in [6]. The unique \( STS(9) \) admits a bicoloring \( C = C(1,4,4) \), and no other bicolorings (see [3] or [17]). The following are all solutions with respect to \( C \): (a) \((3,2,5)\), (b) \((3,5,2)\), (c) \((5,0,5)\), (d) \((5,5,0)\), (e) \((8,0,2)\), and (f) \((8,2,0)\). Corollary 2.1 eliminates solutions (c), (d), (e) and (f) from contention. Concerning (a), since \( c_1 = 3 \) and \( c_3 = 5 \), it must be that in the four 1-factors associated with elements colored with color 2, there are exactly two 2-colored pairs colored with colors 1 and 2 and with 2 and 3, one monochromatic pair of color 1, and two monochromatic pairs of color 3. Since \( c_1 = 3 \), this is easily seen to be impossible, so solution (a) cannot lead to an extended bicoloring. The same reasoning applies to the solution (b). Thus there exist no extended bicolorings of any \( STS(19) \) (obtained from an \( STS(9) \) by a doubling construction, of course). Thus the smallest \( w \) for which an \( STS(w) \) may admit an extended bicoloring is \( w = 27 \) (where the \( STS(27) \) is obtained from an \( STS(13) \) by a doubling construction).

We remark that due to the above, any uniquely 3-colorable \( STS(19) \), or any 3- and 4-colorable \( STS(19) \) cannot contain a sub-\( STS(9) \). It was shown in [14] that there exist uniquely 3-bicolorable, uniquely 4-bicolorable, and also 3- and 4-bicolorable \( STS(19) \).

**Theorem 3.** Let \( S \) be a \( k \)-bicolorable \( STS(v) \) with the \( k \)-bicoloring \( C = C(n_1, \ldots, n_k) \), and suppose there exist \( i, j, i \neq j \), such that \( n_i + n_j = \frac{v+1}{2} \equiv 0 \pmod{2} \). Then there exists an \( STS(2v + 1) \), \( S' \), obtained by a doubling construction from \( S \) such that \( S' \) has an extended \( k \)-bicoloring \( C' \).

**Proof.** Since \( v+1 \equiv 0 \pmod{4} \), we may use as the 1-factorization \( F = \{F_1, \ldots, F_v\} \) in the doubling construction the following 1-factorization. Write \( Y = Y_1 \cup Y_2 \) where \( |Y_i| = \frac{v+1}{2} \), take \( F_1, \ldots, F_{\frac{v+1}{2}} \) to be the 1-factors of any 1-factorization of the com-
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Since there exists an

Theorem 5. There exists an STS(27), (W, C) obtained by a doubling construction from an STS(13), (V, B), which has an extended 3-bicoloring C = C(2, 5, 6). For this system, \( \chi = 3 \) and \( \bar{\chi} = 4 \).

Proof. All solutions \((e_1, e_2, e_3)\) with respect to the coloring C (cf. Theorem 1) are as follows: (a) \((4, 4, 6)\), (b) \((7, 1, 6)\), (c) \((4, 7, 3)\), (d) \((7, 7, 0)\), (e) \((10, 1, 3)\), (f) \((10, 4, 0)\). By Corollary 2, solutions (d), (e), and (f) cannot lead to an extended bicoloring of C. Concerning solution (c), there are three monochromatic pairs of elements of color 3. Two of these pairs may occur in the two 1-factors corresponding to the vertices of V of color 1 but the third pair cannot occur in a 1-factor corresponding to a vertex of V of color 2 (as there are 7 vertices of \( W \setminus V \) of color 2, and that would force a monochromatic triple of color 2), nor clearly in a 1-factor

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As shown earlier, there exist no extended bicolorings of STS(w) for \( w \leq 19 \). Since \( w \equiv 3 \) or 7 \((\text{mod} \ 12)\), the smallest \( w \) for which there might exist an extended bicoloring is \( w = 27 \). Such an extended bicoloring does indeed exist.

Theorem 5. There exists an STS(27), (W, C) obtained by a doubling construction from an STS(13), (V, B), which has an extended 3-bicoloring C = C(2, 5, 6). For this system, \( \chi = 3 \) and \( \bar{\chi} = 4 \).

Proof. All solutions \((e_1, e_2, e_3)\) with respect to the coloring C (cf. Theorem 1) are as follows: (a) \((4, 4, 6)\), (b) \((7, 1, 6)\), (c) \((4, 7, 3)\), (d) \((7, 7, 0)\), (e) \((10, 1, 3)\), (f) \((10, 4, 0)\). By Corollary 2, solutions (d), (e), and (f) cannot lead to an extended bicoloring of C. Concerning solution (c), there are three monochromatic pairs of elements of color 3. Two of these pairs may occur in the two 1-factors corresponding to the vertices of V of color 1 but the third pair cannot occur in a 1-factor corresponding to a vertex of V of color 2 (as there are 7 vertices of \( W \setminus V \) of color 2, and that would force a monochromatic triple of color 2), nor clearly in a 1-factor.
corresponding to a vertex of color 3. Thus solution (c) does not lead to an extended bicoloring of $C$ either.

On the other hand, each of the first two solutions, namely $(4, 4, 6)$ and $(7, 1, 6)$, lead to an extended bicoloring $C' = C''(6, 9, 12)$. The 1-factorizations $F$ used in the respective doubling constructions are given in the Appendix [8]. Our STS(27), $(W, C)$, besides having an extended 3-bicoloring with respect to $C$, is also 4-bicolorable with the coloring $C''' = C''(2, 5, 6, 14)$. At the same time, a 5-bicoloring of $(W, C)$ is impossible due to [20], since $27 < 2^5 - 1$. Thus $\chi = 3, \bar{\chi} = 4$, which completes the proof. □

Concerning order 31, an inspection of the tables in [3] shows that there exists no extended bicoloring for this order: there exists no 3-bicoloring of an STS(15), and no 4-bicoloring of STS(31) whatsoever. However, the next admissible order 39 shows a quite different behaviour.

**Theorem 6.** There exist STS(39) admitting extended bicolorings obtained from extended bicolorings of STS(19) of type $C_1 = C(4, 6, 9)$ and $C_2 = C(1, 2, 8, 8)$. More specifically, there exist STS(39) with $(\chi, \bar{\chi})$ equal to either $(3, 4)$, or $(4, 5)$, or $(3, 5)$.

**Proof.** It was shown in [14] that there exist STS(19) (a) admitting only the 3-bicoloring $C(4, 6, 9)$, (b) admitting only the 4-bicoloring $C(1, 2, 8, 8)$, and (c) admitting both, the 3-bicoloring $C(4, 6, 9)$ and the 4-bicoloring $C(1, 2, 8, 8)$. By Theorem 3, both $C_1$ and $C_2$ are extendable bicolorings (since we have $4 + 6 = 10$ and $2 + 8 = 10$, respectively). Starting with an STS(19) of type (a), (b), or (c), we obtain an STS(39) of the respective kind as claimed. □

In what follows we discuss in somewhat less detailed manner the existence of extended bicolorings for those STS($v$) of orders $43 \leq v \leq 99$ which can be obtained by a doubling construction.

**Theorem 7.** For an STS($v$), $v \in \{51, 63, 67, 75\}$, there exists no extended bicoloring.

**Proof.** (i) For an STS(25), the only types of bicoloring that are possible are $C_1 = C(5, 10, 10)$ and $C_2 = C(1, 4, 8, 12)$. While there exist 12 solutions with respect to $C_1$, none satisfies the condition of Corollary 2 and thus cannot lead to an extended bicoloring. There exist no solutions with respect to $C_2$, and so no STS(51) can have an extended bicoloring.

(ii) By [6], no STS(63) obtained by doubling from an STS(31) can have an extended bicoloring.

(iii) None of the bicolorings of any STS(33) or STS(37) (cf. [3]) yields a solution with respect to such a coloring, thus there is no extended bicoloring of any STS(67) or STS(75). □

**Theorem 8.** For an STS($v$), $v \in \{43, 55, 79, 87, 91, 99\}$, there exist extended bicolorings. More specifically, there exists an extended 3-bicoloring of an STS(43), extended 4-bicolorings of an STS($v$) for $v \in \{55, 87, 91\}$, and extended 4- and 5-bicolorings of an STS($v$) for $v \in \{79, 99\}$. 
Proof. (i) A bicolorable STS(21) can only be 3-bicolorable, with colorings \(C_1 = C(5, 6, 10)\) or \(C_2 = C(4, 8, 9)\) (see [3] or [14]). Both are extendable to a 3-bicoloring \(C = C(10, 16, 17)\) of an STS(43) for which we have \(\chi = 3\) and \(\bar{\chi} = 4\) (there exists no 5-bicolorable STS(43), cf. [3]). The 1-factorization \(F\) in the corresponding doubling construction is given in the Appendix [8].

(ii) By Theorem 3, the 4-bicoloring \(C(1, 4, 10, 12)\) of an STS(27) is extendable to a 4-bicoloring \(C(1, 12, 18, 24)\) of an STS(55), also the 4-bicoloring \(C(1, 8, 12, 18)\), and the 4-bicoloring \(C(2, 6, 13, 18)\), respectively, of an STS(39) is extendable to a 4-bicoloring \(C(1, 18, 28, 32)\), and to a 4-bicoloring \(C(6, 13, 22, 38)\) of an STS(79), respectively; finally, the 4-bicolorings \(C(1, 10, 12, 20)\), and \(C(4, 4, 17, 18)\), respectively, of an STS(43) is extendable to a 4-bicoloring \(C(1, 20, 32, 34)\), and to a 4-bicoloring \(C(4, 17, 26, 40)\), respectively. (The two essential colors are indicated in bold.)

(iii) Extendability of the 5-bicolorings \(C(1, 2, 8, 8, 20)\) and \(C(1, 4, 4, 10, 20)\) of an STS(43) follows from the more general Theorem 4.

(iv) There are only two possible types of a 4-bicoloring of an STS(45), namely \(C_1 = C(2, 8, 14, 21)\) and \(C_2 = C(4, 6, 13, 22)\). There are 12 solutions with respect to \(C_1\) but none of them leads to an extended bicoloring. Similarly, there are 12 solutions with respect to \(C_2\) but only one of them, namely \((c_1, c_2, c_3, c_4) = (4, 8, 12, 22)\), leads to an extended 4-bicoloring. The corresponding 1-factorization \(F\) in the doubling construction that leads to this extended bicoloring is given in the Appendix [8].

(v) Although there exist 3-, 4-, and 5-bicolorable STS(49), none of the 3-bicolorings is extendable. On the other hand, 4-bicolorings \(C(2, 8, 18, 21)\) and \(C(5, 6, 14, 24)\) as well as the 5-bicoloring \(C(1, 4, 4, 20, 20)\) are all extendable. This is shown by examining all solutions with respect to the particular bicoloring \(C\). Due to the considerable number of these solutions (84, 29 and 27, respectively), we omit the details. The 1-factorizations occurring in the doubling constructions leading to the respective extended bicolorings are given in the Appendix [8]. □

Theorem 9. There exist extended 4-bicolorings for each \(w \in \{127, 151, 159, 175\}\); there exist extended 5-bicolorings for each \(w \in \{103, 111, 127, 135, 151, 159, 175\}\).

Proof. Below we list 4- and 5-bicolorings (known to exist by [3]) to which it is possible to apply Theorem 3; the two essential colors are in bold.

<table>
<thead>
<tr>
<th>Order (2v + 1)</th>
<th>Extendable colorings of an STS(v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>103</td>
<td>((1, 2, 8, 16, 24))</td>
</tr>
<tr>
<td>111</td>
<td>((1, 2, 8, 20, 24))</td>
</tr>
<tr>
<td>127</td>
<td>((2, 14, 18, 29), (4, 9, 22, 28), (2, 5, 6, 20, 30))</td>
</tr>
<tr>
<td>135</td>
<td>((1, 2, 16, 16, 32))</td>
</tr>
<tr>
<td>151</td>
<td>((4, 12, 26, 33), (1, 4, 10, 28, 32))</td>
</tr>
<tr>
<td>159</td>
<td>((4, 14, 25, 36), (6, 10, 29, 34), (1, 4, 12, 26, 36), (1, 2, 16, 24, 36))</td>
</tr>
<tr>
<td>175</td>
<td>((4, 17, 26, 40), (2, 5, 10, 34, 36))</td>
</tr>
</tbody>
</table>

We summarize our results as follows.
Theorem 10. Let \( \Omega = \{27, 39, 43, 55, 79, 87, 91, 99, 103, 111, 127, 135, 151, 159, 175\} \). For each \( v \in \Omega \), there exists an STS\( (v) \) with an extended bicoloring, and thus for all \( v \in \Omega \), we have \( \chi \neq \bar{\chi} \).

Proof. For each \( v \in \Omega \) we have an extended \( k \)-coloring for some \( k \), and also (at least) a \((k+1)\)-bicoloring (with \( n_{k+1} = v + 1 \)). \( \square \)

Corollary 11. For each \( v \in \Omega' = \{27, 39, 43, 91, 99, 103, 127, 135, 151\} \), there exists an infinite class of STS\( (w) \), where \( w = 2^t(v + 1) - 1, t \geq 1 \), such that \( \chi \neq \bar{\chi} \).

Proof. Apply repeatedly the doubling construction to the appropriate STS\( (v) \). \( \square \)

4. CONCLUSION

In this paper, we have investigated extended bicolorings with the explicit aim to prove the existence of STSs with \( \chi \neq \bar{\chi} \), that is, with different lower and upper chromatic number. We established the existence of extended bicolorings and of such STSs for several infinite classes of orders \( 2v + 1 \equiv 3 \) or \( 7 \pmod{12} \), by utilizing the doubling construction. The problem of determining for which orders \( v \equiv 1 \) or \( 3 \pmod{6} \) does there exist an STS\( (v) \) with different lower and upper chromatic number is certainly worthwhile. Another interesting question is, how large can the difference \( \bar{\chi} - \chi \) be? It is also a legitimate question to ask whether an analogue of extended bicolorings may exist for other recursive constructions, such as the known \( v \to 2v + t \) rules where \( t > 1 \) (cf. [4]). For example, it is possible to use the \( v \to 2v + 5 \) rule starting with an STS\( (7) \) and ending up with an STS\( (19) \) to show that the 3-bicoloring \((1, 2, 4)\) for STS\( (7) \) can be extended to a 3-bicoloring \((4, 6, 9)\) for an STS\( (19) \)? The next example answers this question.

Example 12. The following STS\( (19) \) with a sub-STS\( (7) \) has a 3-bicoloring and these triples: \{0, 1, 9\}, \{2, 3, 9\}, \{0, 2, 10\}, \{1, 3, 10\}, \{0, 3, 15\}, \{1, 2, 15\}, \{9, 10, 15\}, \{0, 4, 11\}, \{0, 5, 12\}, \{0, 6, 13\}, \{2, 5, 14\}, \{1, 5, 14\}, \{0, 8, 16\}, \{1, 4, 12\}, \{2, 7, 13\}, \{1, 6, 14\}, \{1, 7, 13\}, \{18, 9, 17\}, \{2, 4, 13\}, \{2, 14, 18\}, \{2, 12, 17\}, \{3, 14, 17\}, \{3, 8, 11\}, \{4, 5, 9\}, \{4, 3, 12\}, \{4, 7, 10\}, \{5, 6, 10\}, \{5, 7, 12\}, \{5, 8, 13\}, \{6, 7, 9\}, \{6, 8, 12\}, \{7, 8, 15\}, \{9, 11, 16\}, \{9, 12, 17\}, \{9, 13, 18\}, \{8, 9, 14\}, \{7, 10, 11\}, \{10, 12, 16\}, \{10, 13, 17\}, \{10, 14, 18\}, \{11, 12, 18\}, \{11, 13, 15\}, \{11, 14, 17\}, \{3, 12, 13\}, \{12, 14, 15\}, \{13, 14, 16\}, \{16, 15, 16\}, \{4, 15, 17\}, \{5, 15, 18\}, \{2, 16, 17\}, \{1, 16, 18\}, \{16, 17, 18\}, \{17, 18\}.

The first seven triples are those of an STS\( (7) \) on \{0, 1, 2, 3, 9, 10, 15\}: the three color classes are \{0, 1, 2, 3, 4, 5, 6, 7, 8\}, \{9, 10, 11, 12, 13, 14\} and \{15, 16, 17, 18\}.

Even if the answer in this case proved to be affirmative, and may prove so in similar cases, it is not immediately clear that this will have as a consequence the existence of STSs with \( \chi \neq \bar{\chi} \). Thus the doubling construction appears to offer most benefits from the stated applications point of view. Nevertheless, it seems to us worthwhile to study “extended” bicolorings for recursive rules for STSs other than doubling.

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