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# ON THE NULLITY OF CONNECTED GRAPHS WITH LEAST EIGENVALUE AT LEAST −2

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Let $\mathcal{L}$ (resp. $\mathcal{L}^+$) be the set of connected graphs with least adjacency eigenvalue at least $-2$ (resp. larger than $-2$). The nullity of a graph $G$, denoted by $\eta(G)$, is the multiplicity of zero as an eigenvalue of the adjacency matrix of $G$. In this paper, we give the nullity set of $\mathcal{L}^+$ and an upper bound on the nullity of exceptional graphs. An expression for the nullity of generalized line graphs is given. For $G \in \mathcal{L}$, if $\eta(G)$ is sufficiently large, then $G$ is a proper generalized line graph ($G$ is not a line graph).

## 1. INTRODUCTION

Let $G$ be a simple, undirected graph with $n$ vertices. Let $A$ be the adjacency matrix of $G$, and let $D$ be the diagonal matrix of vertex degrees of $G$. The matrices $D-A$ and $D+A$ are called the *Laplacian matrix* and the *signless Laplacian matrix* of $G$, respectively. The eigenvalues of $A$ are called the *eigenvalues* of $G$. We use $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ to denote the eigenvalues of $G$. The *nullity* of $G$, denoted by $\eta(G)$, is the multiplicity of the eigenvalue 0 of $G$. Let $\mathcal{G}$ be a set of graphs. A set $N$ of nonnegative integers is said to be the *nullity set* of $\mathcal{G}$ if the nullity of each graph in $\mathcal{G}$ belongs to $N$, and for any $k \in N$ there exists at least one graph $G \in \mathcal{G}$ such that $\eta(G) = k$.

Collatz and Sinogowitz [**5**] posed the problem of characterizing all graphs with zero nullity. This question is of great interest in chemistry because, if a conjugated hydrocarbon molecule is chemically stable, then its Hückel graph has zero nullity (see [**21**]). The nullity of a graph is also important in mathematics, since it is related to the rank of the adjacency matrix. There are many results on

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the nullity of trees, unicyclic graphs and bicyclic graphs (see [**1**, **12**, **16**, **19**, **20**, **26**]).

Let $G$ be a graph with edge set $E(G)$. The line graph of $G$, denoted by $L(G)$, is the graph whose vertex set is $E(G)$ and two vertices of $L(G)$ are adjacent if the corresponding edges in $G$ are incident. For any tree $T$, Gutman and Sciriha [**17**] proved that $\eta(L(T)) = 0$ or 1. Some results on the nullity of line graphs can be found in [**2**, **13**, **14**, **17**, **18**, **22**, **24**, **25**].

Let $\mathcal{L}$ (resp. $\mathcal{L}^+$) denote the set of connected graphs with least eigenvalue at least $-2$ (resp. larger than $-2$). It is known that $L(T) \in \mathcal{L}^+$ for any tree $T$, and that graphs in $\mathcal{L}$ consist of generalized line graphs and exceptional graphs (see [**4**, **7**]). In this paper, we give the nullity set of $\mathcal{L}^+$ and an upper bound on the nullity of exceptional graphs. We also give an expression for the nullity of generalized line graphs. For $G \in \mathcal{L}$, we show that $G$ is a proper generalized line graph ($G$ is not a line graph) if $\eta(G)$ is sufficiently large.

## 2. PRELIMINARIES

A pendant double edge (2-cycle) is called a *petal*. Let $H$ be a simple undirected graph with vertex set $\{v_1, \dots, v_n\}$. Let $H(a_1, \dots, a_n)$ denote the multigraph obtained from $H$ by attaching $a_i$ petals at vertex $v_i$ ($i = 1, \dots, n$). In [**10**], the *generalized line graph* $L(H; a_1, \dots, a_n)$ is defined as follows. The vertices of $L(H; a_1, \dots, a_n)$ are the edges of $H(a_1, \dots, a_n)$ and two vertices of $L(H; a_1, \dots, a_n)$ are adjacent whenever the corresponding edges in $H(a_1, \dots, a_n)$ have exactly one vertex in common. $H(a_1, \dots, a_n)$ is called the *root graph* of $L(H; a_1, \dots, a_n)$. In particular, $L(H; 0, \dots, 0)$ is just the ordinary line graph $L(H)$. An example for the construction of a generalized line graph is depicted in Figure 1.

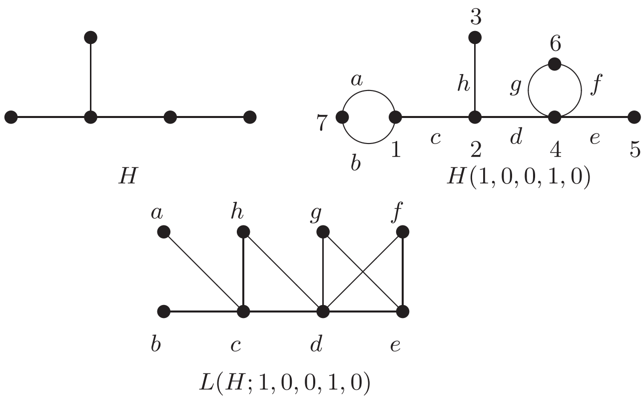


Figure 1. Construction of a generalized line graph

Let $\mathcal{L}$ (resp. $\mathcal{L}^+$) denote the set of connected graphs with least eigenvalue at least $-2$ (resp. larger than $-2$). It is known that (generalized) line graphs belong to $\mathcal{L}$. A connected graph $G \in \mathcal{L}$ is called an *exceptional graph* if $G$ is not a generalized line graph. There are 573 exceptional graphs in $\mathcal{L}^+$: 20 with 6 vertices, 110 with 7 vertices, 443 with 8 vertices. We denote these sets of exceptional graphs by $\mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8$ respectively.

A connected graph with $n$ vertices is said to be *unicyclic* if it has $n$ edges. Clearly a unicyclic graph contains a unique cycle. If a unicyclic graph has an odd cycle, then this graph is said to be *odd unicyclic*.

**Lemma 2.1** ([**11**, **28**]). *Let $G$ be a connected graph with $n$ vertices. Then $G \in \mathcal{L}^+$ if and only if one of the following holds*:
(1) $G = L(H)$, *where $H$ is a tree or an odd unicyclic graph.*
(2) $G = L(H; 1, 0, \dots, 0)$, *where $H$ is a tree.*
(3) *$G$ is one of the* 573 *graphs in* $\mathcal{E}_6 \cup \mathcal{E}_7 \cup \mathcal{E}_8$.

Let $G \in \mathcal{L}$ be a graph with $n$ vertices, and suppose that $G$ has $k$ eigenvalues larger than $-2$. The quantity $\prod_{i=1}^{k} (\lambda_i(G) + 2)$ is called the *star value* of $G$, denoted by $S_G$. If $G \in \mathcal{L}^+$, then $S_G = \prod_{i=1}^{n} (\lambda_i(G) + 2)$. The star value is a graph invariant introduced in [**7**].

**Lemma 2.2** ([**7**, **28**]). *Let $G$ be a connected graph in $\mathcal{L}^+$. The following statements hold*:
(1) *If $G \in \mathcal{E}_8$, then $S_G = 1$. If $G \in \mathcal{E}_7$, then $S_G = 2$. If $G \in \mathcal{E}_6$, then $S_G = 3$.*
(2) *If $G$ is the line graph of an odd unicyclic graph, then $S_G = 4$.*
(3) *If $G = L(H; 1, 0, \dots, 0)$, where $H$ is a tree, then $S_G = 4$.*
(4) *If $G$ is the line graph of a tree with $n$ vertices, then $S_G = n$.*

Let $G$ be a graph with vertex set $V(G)$. If $\mu$ is an eigenvalue of $G$ of multiplicity $k$, then a *star set* for $\mu$ in $G$ is a subset $X$ of $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have $\mu$ as an eigenvalue. The induced subgraph $G - X$ is called a *star complement* for $\mu$ in $G$. It is known that star sets and star complements exist for any eigenvalue of any graph (see [**8**, **23**]).

**Lemma 2.3** ([**8**]). *Let $G$ be a connected graph with least eigenvalue $-2$. Then $G$ is exceptional if and only if it has an exceptional star complement for $-2$.*

We use $m_G(\lambda)$ to denote the multiplicity of eigenvalue $\lambda$ of a graph $G$.

**Lemma 2.4** ([**6**]). *Let $G = L(H; a_1, \dots, a_n)$, where $H$ is a connected graph with $n$ vertices and $m$ edges, and $(a_1, \dots, a_n) \neq (0, \dots, 0)$. Then $m_G(-2) = m - n + \sum_{i=1}^{n} a_i$.*

**Lemma 2.5** ([**10**]). *Let $H$ be a connected graph with $n$ vertices and $m$ edges. Then*

$$m_{L(H)}(-2) = \begin{cases} m - n + 1 & \text{if } H \text{ is bipartite,} \\ m - n & \text{if } H \text{ is non-bipartite.} \end{cases}$$

The interlacing theorem for graph eigenvalues implies the following lemma.

**Lemma 2.6** ([**14**]). *Let $v$ be any vertex of a graph $G$ with at least two vertices. Then* $|\eta(G-v)-\eta(G)|\leqslant 1$.

**Lemma 2.7** ([**2**, **13**]). *Let $G$ be a graph such that the number of spanning trees of $G$ is odd. Then* $\eta(L(G))\leqslant 1$.

**Lemma 2.8** ([**24**]). *Let $T$ be a tree such that $\eta(L(T))=1$. Then the number of vertices of $T$ is even.*

**Lemma 2.9** ([**13**]). *Let $G$ be a graph with an odd number of vertices, and with an odd number of spanning trees. Then* $\eta(L(G))=0$.

For a graph $G$ with a cut vertex $u$, let $G_1$ be a component of $G-u$. We use $G_1+u$ to denote the subgraph induced by $V(G_1)\cup\{u\}$, where $V(G_1)$ is the vertex set of $G_1$.

**Lemma 2.10** ([**14**]). *Let $u$ be a cut vertex of a graph $G$, and let $G_1$ be a component of $G-u$. If $\eta(G_1)=\eta(G_1+u)+1$, then $\eta(G)=\eta(G-u)-1$.*

For an $n\times n$ Hermitian matrix $M$, let $\rho_1(M)\geqslant\rho_2(M)\geqslant\cdots\geqslant\rho_n(M)$ denote the eigenvalues of $M$.

**Lemma 2.11** ([**10**]). *Let $A$ and $B$ be $n\times n$ Hermitian matrices. Then*

$$\rho_i(A+B)\geqslant\rho_j(A)+\rho_{i-j+n}(B)\ (1\leqslant i\leqslant j\leqslant n).$$

A tree is called *starlike* if it has exactly one vertex of degree greater than 2. We denote by $T(\ell_1,\ell_2,\ldots,\ell_\Delta)$ the starlike tree with maximum degree $\Delta$ such that $T(\ell_1,\ell_2,\ldots,\ell_\Delta)-v=P_{\ell_1}\cup P_{\ell_2}\cup\cdots\cup P_{\ell_\Delta}$, where $v$ is the vertex of degree $\Delta$ and $P_{\ell_i}$ denotes a path with $\ell_i$ vertices. Let $\phi_Q(G,\lambda)$ denote the characteristic polynomial of the signless Laplacian matrix of a graph $G$.

**Lemma 2.12** ([**3**]). *Let $G=T(\ell_1,\ell_2,\ldots,\ell_\Delta)$. For $n>1$, $\phi_Q(P_n,\lambda)|\phi_Q(G,\lambda)$ if and only if* (*without loss of generality*)

$$\ell_1+\ell_2\equiv-1\mod n,\quad \ell_3,\ell_4,\ldots,\ell_\Delta\equiv 0\mod n.$$

## 3. MAIN RESULTS

For a graph $G\in\mathcal{L}$ of order $n$ with $k$ eigenvalues larger than $-2$, it is known that the star value of $G$ is $S_G=\dfrac{(-1)^n}{(n-k)!}P_G^{(n-k)}(-2)=\prod_{i=1}^{k}(\lambda_i(G)+2)$, where $P_G(x)$ is the characteristic polynomial of the adjacency matrix of $G$, and $P_G^{(q)}(x)$ is the $q$-th derivative of $P_G(x)$ (see [**7**]). Since graph eigenvalues are algebraic integers, $S_G$ is an integer. It is also known that $S_G$ is the sum of star values of star complements of $G$ for the eigenvalue $-2$ (see [**7**, Theorem 4]).

**Theorem 3.1.** *Let $G \in \mathcal{L}$ be a graph with star value $S_G$. Then $S_G$ is divisible by $2^{\eta(G)}$.*

**Proof.** Since graph eigenvalues are algebraic integers, there exists an algebraic integer $\alpha$ such that $S_G = 2^{\eta(G)}\alpha$. Hence $\alpha$ is an integer and $S_G$ is divisible by $2^{\eta(G)}$. □

Two distinct vertices $u, v$ of a graph $G$ are called *duplicate vertices* if $u, v$ are not adjacent and they have the same neighbours in $G$. Clearly duplicate vertices $u, v$ in $G$ correspond to repeated rows in the adjacency matrix of $G$, and $\eta(G) = \eta(G-u)+1$. A proper generalized line graph $G$ ($G$ is not a line graph) always has duplicate vertices, because two edges in a petal (2-cycle) of the root graph of $G$ form a pair of duplicate vertices. For instance, the generalized line graph $L(H; 1, 0, 0, 1, 0)$ shown in Fig. 1 has duplicate vertices $a, b$ and $g, f$.

It is known that $L(T) \in \mathcal{L}^+$ for any tree $T$ (cf. Lemma 2.1). The nullity of line graphs of trees is studied in [**17**, **22**, **24**, **25**]. We give the nullity set of $\mathcal{L}^+$ as follows.

**Theorem 3.2.** *The nullity set of $\mathcal{L}^+$ is $\{0,1,2\}$. Moreover, we have*
(1) *The nullity set of $\mathcal{G}_1 = \{L(H) | H$ is a tree$\}$ is $\{0,1\}$.*
(2) *The nullity set of $\mathcal{G}_2 = \{L(H) | H$ is an odd unicyclic graph$\}$ is $\{0,1\}$.*
(3) *The nullity sets of $\mathcal{E}_6$ and $\mathcal{E}_8$ are both $\{0\}$.*
(4) *The nullity set of $\mathcal{E}_7$ is $\{0,1\}$.*
(5) *The nullity set of $\mathcal{G}_3 = \{L(H; 1, 0, \ldots, 0) | H$ is a tree$\}$ is $\{1,2\}$.*
(6) *For any graph $G \in \mathcal{L}^+$, $\eta(G) = 2$ if and only if the following conditions hold:*

(6.1) *$G$ is a generalized line graph $L(H; 1, 0, \ldots, 0)$, where $H$ is a tree.*

(6.2) *Let $\widetilde{H}$ be the tree obtained from $H(1, 0, \ldots, 0)$ by deleting one edge of the petal. Then $\eta(L(\widetilde{H})) = 1$.*

(6.3) *The number of vertices of $G$ is even.*

**Proof.** It is known that the nullity set of $\mathcal{G}_1$ is $\{0,1\}$ (see [**17**]). So part (1) holds. For any graph $G \in \mathcal{G}_2$, by Lemma 2.7, we have $\eta(G) \leqslant 1$. The nullity of the line graph of an odd cycle is 0. If $U$ is the graph obtained from a triangle $C_3$ by attaching a pendant edge, then $\eta(L(U)) = 1$. Hence the nullity set of $\mathcal{G}_2$ is $\{0,1\}$, and part (2) holds. By Theorem 3.1 and Lemma 2.2, the nullity of an exceptional graph with 6 or 8 vertices is always 0. So part (3) holds. For any graph $G \in \mathcal{E}_7$, by Theorem 3.1 and Lemma 2.2, we get $\eta(G) \leqslant 1$. There are 110 graphs in $\mathcal{E}_7$, and 26 graphs in $\mathcal{E}_7$ have nullity one (these graphs are given in Theorem 3.3). So part (4) holds.

For any graph $G = L(H; 1, 0, \ldots, 0) \in \mathcal{G}_3$, where $H$ is a tree, let $\widetilde{H}$ be the tree obtained from $H(1, 0, \ldots, 0)$ by deleting one edge of the petal. Then $\eta(G) = \eta(L(\widetilde{H})) + 1$. Since the nullity set of $\mathcal{G}_1$ is $\{0,1\}$, the nullity set of $\mathcal{G}_3$ is $\{1,2\}$. Clearly $\eta(G) = 2$ if and only if $\eta(L(\widetilde{H})) = 1$. Note that $G$ and $\widetilde{H}$ have the same number of vertices. Lemma 2.8 implies that the number of vertices of $G$ is even if $\eta(L(\widetilde{H})) = 1$.

It follows from Lemma 2.1 that $\mathcal{L}^+ = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{E}_6 \cup \mathcal{E}_7 \cup \mathcal{E}_8$. Hence the nullity set of $\mathcal{L}^+$ is $\{0, 1, 2\}$. □

All graphs in $\mathcal{E}_6 \cup \mathcal{E}_7$ are listed in [**9**, Appendix Table A2]. It is known that each graph in $\mathcal{E}_7$ has an exceptional graph in $\mathcal{E}_6$ as an induced subgraph (see [**9**]). Let $G_1, G_2, \ldots, G_{110}$ denote all graphs in $\mathcal{E}_7$, where the subscript $i$ ($i = 1, \ldots, 110$) is the identification number used in [**9**, Appendix Table A2]. For a graph $G_i \in \mathcal{E}_7$, let $\begin{pmatrix} A & b \\ b^\top & 0 \end{pmatrix}$ be the adjacency matrix of $G_i$, where $A$ is the adjacency matrix of a graph in $\mathcal{E}_6$. It follows from Theorem 3.2 that $A$ is nonsingular. Hence $\eta(G_i) = 1$ if and only if $b^\top A^{-1} b = 0$. There are 20 graphs in $\mathcal{E}_6$. According to the equality $b^\top A^{-1} b = 0$, we use Maple to find all 26 graphs in $\mathcal{E}_7$ with nullity one as follows.

**Theorem 3.3.** *The exceptional graphs in $\mathcal{E}_7$ with nullity one consist of* $G_1$, $G_3$, $G_5$, $G_6$, $G_7$, $G_8$, $G_{10}$, $G_{13}$, $G_{14}$, $G_{16}$, $G_{25}$, $G_{28}$, $G_{29}$, $G_{39}$, $G_{40}$, $G_{45}$, $G_{46}$, $G_{49}$, $G_{63}$, $G_{64}$, $G_{72}$, $G_{73}$, $G_{76}$, $G_{84}$, $G_{86}$, $G_{96}$.

For a connected graph $G$, if $\lambda_2(G) = 0$, then $G$ is a complete multipartite graph (see [**10**]). The *cocktail party graph* is the unique regular graph with $2n$ vertices of degree $2n - 2$. It is a generalized line graph.

**Theorem 3.4.** *Let $G$ be an exceptional graph with eigenvalue $-2$ of multiplicity $r$. Then $\eta(G) \leqslant \min\{r + 1, 6\}$.*

**Proof.** By Lemma 2.3, $G$ has an exceptional star complement $H$ for $-2$. Lemma 2.1 implies that $H \in \mathcal{E}_6 \cup \mathcal{E}_7 \cup \mathcal{E}_8$. By Theorem 3.2 and Lemma 2.6, we have $\eta(G) \leqslant r + 1$.

Since $H \in \mathcal{E}_6 \cup \mathcal{E}_7 \cup \mathcal{E}_8$ is a star complement for $-2$ and $\lambda_1(G)$ is a positive simple eigenvalue, we get $\eta(G) \leqslant 7$. If $\eta(G) = 7$, then $G$ has three distinct eigenvalues $\lambda_1(G), 0, -2$. Since $\lambda_2(G) = 0$, $G$ is a complete multipartite graph with three distinct eigenvalues $\lambda_1(G), 0, -2$. According to the eigenvalue distribution of a complete multipartite graph (see [**27**]), $G$ is the cocktail party graph, which is not exceptional (it is a generalized line graph), a contradiction. Hence we have $\eta(G) \leqslant 6$. □

For a graph $G$ with $n$ vertices and $m$ edges, the quantity $m - n + 1$ is called the *cyclomatic number* of $G$. Clearly trees have cyclomatic number 0, unicyclic graphs have cyclomatic number 1.

**Theorem 3.5.** *For a connected graph $G$ with cyclomatic number $k$, we have that $\eta(L(G)) \leqslant k + 1$. If $\eta(L(G)) = k + 1$, then the following hold:*
(1) *For any spanning tree $T$ of $G$, we have $\eta(L(T)) = 1$.*
(2) *For any edge $e$ of $G$, if $e$ is not a bridge, then $\eta(L(G - e)) = k$.*
(3) *$G$ is bipartite and the number of vertices of $G$ is even.*

**Proof.** Let $T$ be any spanning tree of $G$. Then $T$ can be obtained from $G$ by deleting $k$ edges of $G$. By Theorem 3.2, we get $\eta(L(T)) \leqslant 1$. Lemma 2.6 implies that $\eta(L(G)) \leqslant k + 1$. If $\eta(L(G)) = k + 1$, then $\eta(L(T)) = 1$, and part (1) holds.

For any edge $e$ of $G$, if $e$ is not a bridge, then $G-e$ is a connected graph with cyclomatic number $k-1$, and $\eta(L(G-e)) \leqslant k$. If $\eta(L(G)) = k+1$, Lemma 2.6 implies that $\eta(L(G-e)) \geqslant k$, so $\eta(L(G-e)) = k$, and part (2) holds.

If $\eta(L(G)) = k+1$, by part (1), $\eta(L(T)) = 1$ for any spanning tree $T$ of $G$. Lemma 2.8 implies that the number of vertices of $G$ is even. Assume that $G$ is non-bipartite. Then $G$ has a spanning subgraph $U$ which is odd unicyclic, and $U$ can be obtained from $G$ by deleting $k-1$ edges of $G$. If $\eta(L(G)) = k+1$, then by Lemma 2.6, we get $\eta(L(U)) \geqslant 2$. Theorem 3.2 implies that $\eta(L(U)) \leqslant 1$, a contradiction. Hence $G$ is bipartite, and part (3) holds.

**Theorem 3.6.** *For a non-bipartite connected graph $G$ with cyclomatic number $k$, we have $\eta(L(G)) \leqslant k$. If $\eta(L(G)) = k$, then the following hold:*
(1) *For any spanning subgraph $U$ of $G$, if $U$ is odd unicyclic, then $\eta(L(U)) = 1$.*
(2) *The number of vertices of $G$ is even.*

**Proof.** Let $U$ be any spanning subgraph of $G$ such that $U$ is odd unicyclic. Then $U$ can be obtained from $G$ by deleting $k-1$ edges of $G$. By Theorem 3.2, we get $\eta(L(U)) \leqslant 1$. Lemma 2.6 implies that $\eta(L(G)) \leqslant k$. If $\eta(L(G)) = k$, then $\eta(L(U)) = 1$, and part (1) holds.

If $\eta(L(G)) = k$, then by part (1), $\eta(L(U)) = 1$ for any spanning subgraph $U$ of $G$ such that $U$ is odd unicyclic. Lemma 2.9 implies that the number of vertices of $G$ is even, and part (2) holds.

**Theorem 3.7.** *Let $G$ be a connected graph with cyclomatic number $k$. For any connected graph $H$, let $N$ be a graph obtained from $G \cup H$ by adding an edge between $G$ and $H$. The following statements hold:*
(1) *If $\eta(L(G)) = k+1$, then $\eta(L(N)) = \eta(L(H)) + k$.*
(2) *If $G$ is non-bipartite and $\eta(L(G)) = k$, then $\eta(L(N)) = \eta(L(H)) + k - 1$.*

**Proof.** Suppose that $N$ is the graph obtained from $G \cup H$ by adding an edge $e$ between $G$ and $H$. Then $e$ is a cut vertex of $L(N)$. Note that $G + e$ also has cyclomatic number $k$.

If $\eta(L(G)) = k+1$, then by Theorem 3.5, the number of vertices of $G$ is even. So $G+e$ has odd number of vertices. Theorem 3.5 implies that $\eta(L(G+e)) \leqslant k$. By Lemma 2.6, $\eta(L(G+e)) \geqslant k$, and so $\eta(L(G+e)) = k$. Hence we have $\eta(L(G)) = \eta(L(G+e)) + 1$. By Lemma 2.10, we get $\eta(L(N)) = \eta(L(N-e)) - 1 = \eta(L(H)) + k$. So part (1) holds.

If $G$ is non-bipartite and $\eta(L(G)) = k$, then by Theorem 3.6, the number of vertices of $G$ is even. So $G+e$ has odd number of vertices. Theorem 3.6 implies that $\eta(L(G+e)) \leqslant k-1$. By Lemma 2.6, $\eta(L(G+e)) \geqslant k-1$, and so $\eta(L(G+e)) = k-1$. Hence we have $\eta(L(G)) = \eta(L(G+e)) + 1$. By Lemma 2.10, we get $\eta(L(N)) = \eta(L(N-e)) - 1 = \eta(L(H)) + k - 1$. So part (2) holds. □

It is known that a tree with a perfect matching has Laplacian eigenvalue 2 (see [**15**, Theorem 2]). Since the Laplacian spectrum and the signless Laplacian spectrum of a bipartite graph are the same, 2 is also a signless Laplacian eigenvalue of a tree with a perfect matching. From [**3**, Lemma 2.9], we know that $\eta(L(T)) = 1$ for any tree $T$ with a perfect matching. We can obtain the following result from Theorem 3.7.

**Corollary 3.8.** *Let $T$ be a tree with a perfect matching. For any connected graph $H$, let $N$ be a graph obtained from $T \cup H$ by adding an edge between $T$ and $H$. Then $\eta(L(N)) = \eta(L(H))$.*

**Theorem 3.9.** *Let $T$ be a starlike tree. The following conditions are equivalent:*
(1) $\eta(L(T)) = 1$.
(2) 2 *is a (signless) Laplacian eigenvalue of* $T$.
(3) $T$ *has a perfect matching.*

**Proof.** From [**3**, Lemma 2.9], we know that $\eta(L(T)) = 1$ if and only if 2 is a (signless) Laplacian eigenvalue of $T$. Now $T$ has a signless Laplacian eigenvalue 2 if and only if $\phi_Q(P_2, \lambda)|\phi_Q(T, \lambda)$, where $\phi_Q(G, \lambda)$ denotes the characteristic polynomial of the signless Laplacian matrix of a graph $G$. Lemma 2.12 implies that $\phi_Q(P_2, \lambda)|\phi_Q(T, \lambda)$ if and only if $T$ has a perfect matching. □

For a graph $H$ with vertex set $\{1, 2, \ldots, n\}$, we use $H[a_1, \ldots, a_n]$ to denote the graph obtained from $H$ by attaching $a_i$ ($a_i \geqslant 0$) pendant edges at vertex $i$ ($i = 1, \ldots, n$).

**Theorem 3.10.** *Let $G = L(H; a_1, \ldots, a_n)$ be a generalized line graph. Then $\eta(G) = \eta(L(\widetilde{H})) + \sum_{i=1}^{n} a_i$, where $\widetilde{H} = H[a_1, \ldots, a_n]$.*

**Proof.** The generalized line graph $G = L(H; a_1, \ldots, a_n)$ has at least $\sum_{i=1}^{n} a_i$ pairs of duplicate vertices. For a pair of duplicate vertices $u, v$ of $G$, we have $\eta(G) = \eta(G - u) + 1$. Hence we get $\eta(G) = \eta(L(\widetilde{H})) + \sum_{i=1}^{n} a_i$, where $\widetilde{H} = H[a_1, \ldots, a_n]$. □

Let $I_a$ denote the identity matrix of order $a$.

**Theorem 3.11.** *For a connected graph $H$ with vertex set $\{1, 2, \ldots, n\}$, let $G = H[a_1, \ldots, a_n]$. Then $\eta(L(G))$ is equal to the multiplicity of the eigenvalue 0 of $Q + 2E - 2I$, where $Q$ is the signless Laplacian matrix of $H$, and $E = \operatorname{diag}(a_1, a_2, \ldots, a_n)$. Moreover, we have*
(1) *If* $\min\{a_1, \ldots, a_n\} \geqslant 1$ *and* $H$ *is non-bipartite, then* $\eta(L(G)) = 0$.
(2) *If* $\min\{a_1, \ldots, a_n\} \geqslant 1$ *and* $H$ *is bipartite, then* $\eta(L(G)) \leqslant 1$.

**Proof.** The signless Laplacian matrix of $G$ is

$$M = \begin{pmatrix} Q + E & J_1 & \cdots & J_n \\ J_1^\top & I_{a_1} & & \\ \vdots & & \ddots & \\ J_n^\top & & & I_{a_n} \end{pmatrix},$$

where $Q$ is the signless Laplacian matrix of $H$, $J_k$ is the $n \times a_k$ matrix in which each entry of the $k$-th row is 1 and all other entries are 0, $E = \mathrm{diag}(a_1, a_2, \ldots, a_n)$. From [**3**, Lemma 2.9], $\eta(L(G))$ is equal to the multiplicity of the eigenvalue 2 of $M$. So $\eta(L(G))$ is equal to the multiplicity of the eigenvalue 0 of $M - 2I$. The rank of $M - 2I$ is $\mathrm{rank}(Q + E - 2I + BB^\top) + \sum_{i=1}^{n} a_i = \mathrm{rank}(Q + 2E - 2I) + \sum_{i=1}^{n} a_i$, where $B = \begin{pmatrix} J_1 & J_2 & \cdots & J_n \end{pmatrix}$. Hence $\eta(L(G))$ is equal to the multiplicity of the eigenvalue 0 of $Q + 2E - 2I$.

If $H$ is non-bipartite, then all eigenvalues of $Q$ are positive (see [**10**]). If $\min\{a_1, \ldots, a_n\} \geqslant 1$, then by Lemma 2.11, the least eigenvalue of $Q + 2E$ is larger than 2. Hence $\eta(L(G)) = 0$, and part (1) holds.

If $H$ is a connected bipartite graph, then the least eigenvalue of $Q$ is 0, and the other eigenvalues of $Q$ are positive (see [**10**]). If $\min\{a_1, \ldots, a_n\} \geqslant 1$, then by Lemma 2.11, the least eigenvalue of $Q + 2E$ is at least 2, and the other eigenvalues of $Q + 2E$ are larger than 2. Hence $\eta(L(G)) \leqslant 1$, and part (2) holds. □

We can obtain the following result from Theorem 3.10 and Theorem 3.11.

**Corollary 3.12.** *Let $G = L(H; a_1, \ldots, a_n)$ be a connected generalized line graph, with $\min\{a_1, \ldots, a_n\} \geqslant 1$. If $H$ is non-bipartite, then $\eta(G) = \sum_{i=1}^{n} a_i$. If $H$ is bipartite, then $\sum_{i=1}^{n} a_i \leqslant \eta(G) \leqslant \sum_{i=1}^{n} a_i + 1$.*

It is known that every connected graph with least eigenvalue at least $-2$ is either a (generalized) line graph or an exceptional graph (see [**4**]).

**Theorem 3.13.** *Let $G$ be a connected graph with least eigenvalue $-2$ of multiplicity $r$. Then $\eta(G) \leqslant r + 2$, and equality holds if and only if the following hold:*
(1) *There exists a connected graph $H$ of order $n$ such that $G = L(H; a_1, \ldots, a_n)$, where $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$.*
(2) $\eta(L(\widetilde{H})) = k + 1$, *where $\widetilde{H} = H[a_1, \ldots, a_n]$, $k$ is the cyclomatic number of $H$. Moreover, $\widetilde{H}$ satisfies the conditions given in Theorem* 3.5.

**Proof.** It is known that $G$ is a (generalized) line graph or an exceptional graph. If $G$ is an exceptional graph, then by Theorem 3.4, $\eta(G) \leqslant r + 1$. If $G$ is an ordinary line graph $L(N)$, by Lemma 2.5, we get

$$r = \begin{cases} k & \text{if } N \text{ is bipartite} \\ k - 1 & \text{if } N \text{ is non-bipartite,} \end{cases}$$

where $k$ is the cyclomatic number of $N$. By Theorem 3.5 and Theorem 3.6, we have $\eta(G) \leqslant r + 1$. If $G$ is a generalized line graph and $G$ is not a line graph, then there exists a connected graph $H$ of order $n$ such that $G = L(H; a_1, \ldots, a_n)$, where $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$. By Lemma 2.4, we get $r = k - 1 + \sum_{i=1}^{n} a_i$, where $k$ is the cyclomatic number of $H$. It follows from Theorem 3.10 that $\eta(G) = \eta(L(\widetilde{H})) + \sum_{i=1}^{n} a_i$, where $\widetilde{H} = H[a_1, \dots, a_n]$. Note that $\widetilde{H}$ also has cyclomatic number $k$. Theorem 3.5 implies that $\eta(L(\widetilde{H})) \leqslant k+1$. So we get $\eta(G) = \eta(L(\widetilde{H})) + \sum_{i=1}^{n} a_i \leqslant k+1+\sum_{i=1}^{n} a_i = r+2$. Then $\eta(G) = r+2$ if and only if $\eta(L(\widetilde{H})) = k+1$. If $\eta(L(\widetilde{H})) = k+1$, then $\widetilde{H}$ satisfies the conditions given in Theorem 3.5.

## 4. CONCLUDING REMARKS

Research on graphs with least eigenvalue −2 is a classic topic in spectral graph theory. The nullity of generalized line graphs and exceptional graphs is studied in this paper. We list some problems as concluding remarks.

(1) The nullity of every exceptional graph in $\mathcal{E}_6 \cup \mathcal{E}_7 \cup \mathcal{E}_8$ is determined in this paper (cf. Theorem 3.2 and Theorem 3.3). What is the nullity set of all exceptional graphs?

(2) For a connected graph $G$ with cyclomatic number $k$, we have $\eta(L(G)) \leqslant k+1$ (cf. Theorem 3.5). We say that $G$ is a maximal graph if $\eta(L(G)) = k+1$. Do maximal graphs have further structural properties? Theorem 3.5 implies that $\eta(L(T)) = 1$ for any spanning tree $T$ of a maximal graph $G$. Given a tree $T$, there are finitely many maximal graph with $T$ as a spanning tree. Are there some interesting results on maximal graphs with a given spanning tree $T$?

(3) For a unicyclic graph $U$ with a cycle of length $g$, if $\eta((L(U)) = 2$, then $g$ is even (cf. Theorem 3.5). If $U$ is a unicyclic graph with depth one, as defined in [**18**], then $\eta((L(U)) = 2$ if and only if $U$ is a cycle of order $g$ and $g$ is divisible by 4 (see [**18**]). We conjecture that the girth of a unicyclic graph $U$ is divisible by 4 if $\eta((L(U)) = 2$.

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