EXPONENTIAL FUNCTIONS OF DISCRETE FRACTIONAL CALCULUS

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In this paper, exponential functions of discrete fractional calculus with the nabla operator are studied. We begin with proving some properties of exponential functions along with some relations to the discrete Mittag-Leffler functions. We then study sequential linear difference equations of fractional order with constant coefficients. A corresponding characteristic equation is defined and considered in two cases where characteristic real roots are same or distinct. We define a generalized Casoratian for a set of discrete functions. As a consequence, for solutions of sequential linear difference equations, their nonzero Casoratian ensures their linear independence.

1. INTRODUCTION

The purpose of this paper is to introduce the exponential functions of discrete fractional calculus and to analyze them. We shall study sequential linear difference equations of fractional order with constant coefficients and prove that their general solutions are linear combinations of exponential functions in some cases. We shall employ backward difference or nabla operator, and the Riemann-Liouville definition of the fractional difference.

Discrete fractional calculus has generated interest in recent years. Some of the work has employed the forward or delta difference. We refer the reader to [**2, 7, 12, 13, 14, 15, 16**], for example, and more recently [**8, 10, 11**]. Probably more work has been developed for the fractional backward or nabla difference and we refer the reader to [**1, 4, 5, 6, 9, 17, 18**].

In this paper, we are concerned with the solution, so called exponential function, of the following initial value problem (IVP)

$$\nabla_0^\alpha y(t) = ay(t), \quad t = 1, 2, \ldots \tag{1.1}$$

$$\nabla_0^{-(1-\alpha)} y(t) \mid_{t=0} = y(0) = 1, \tag{1.2}$$

where $0 < \alpha < 1$ and $|a| < 1$. The unique solution, represented by $\widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}})$, of the IVP (1.1)-(1.2) was derived using $\mathcal{N}$-transform in the paper [4].

In Section 2, we present the basic definitions and identities of discrete fractional calculus. We prove many properties of the exponential function $\widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}})$ along with some relations to the discrete Mittag-Leffler functions. In Section 3, we define a generalized Casoratian for a set of discrete functions. In Section 4, we use the results of Section 3 to show that the set $\{\widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}), \widehat{e}_{\alpha,\alpha}(b, t^{\overline{\alpha}})\}$ is linearly independent under certain assumptions on the real numbers a and b. We define the characteristic equation and show that how the roots of the characteristic equation help us to determine the basis of the solution space which consists of all solutions of the corresponding sequential fractional difference equation.

2. PRELIMINARIES

Let a be any real number and α be any positive real number such that $0 < n - 1 \leq \alpha < n$ where n is an integer.

The $\alpha - th$ order fractional sum of f is defined by

$$\nabla_a^{-\alpha} f(t) = \sum_{s=a}^{t} \frac{(t - \rho(s))^{\overline{\alpha - 1}}}{\Gamma(\alpha)} f(s), \tag{2.1}$$

where $t \in \mathbb{N}_a = \{a, a + 1, a + 2, \ldots\}$, $\rho(t) = t - 1$ is backward jump operator of the time scale calculus and the raising factorial power function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t + \alpha)}{\Gamma(t)}.$$

We note that the Gamma function is not defined at zero and negative integers. Therefore we consider a map $t \to t^{\overline{\alpha}}$ from the set $\{t \in \mathbb{R} : t \text{ and } t + \alpha \text{ do not belong to } \mathbb{Z}^- \cup \{0\}\}$ to the set of real numbers $\mathbb{R}$.

The $\alpha - th$ order fractional difference (a Riemann-Liouville fractional difference) of f is defined by

$$\nabla_a^\alpha f(t) = \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \nabla^n \sum_{s=a}^{t} \frac{(t - \rho(s))^{\overline{n - \alpha - 1}}}{\Gamma(n - \alpha)} f(s),$$

where $f : \mathbb{N}_a \to \mathbb{R}$.

The proofs of the following three lemmas can be found in [6] and [17], respectively.

Lemma 2.1 (Power Rule). *Let $\nu > 0$ and μ be two real numbers so that $\frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)}$ is defined. Then the following holds*

$$\nabla_a^{-\nu}(t - a + 1)^{\overline{\mu}} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \nu + 1)}(t - a + 1)^{\overline{\nu + \mu}},$$

for every $t \in \mathbb{N}_a$.

Lemma 2.2. *For any $\alpha > 0$, the following equality holds:*

$$\nabla_{a+1}^{-\alpha}\nabla f(t) = \nabla\nabla_a^{-\alpha}f(t) - \frac{(t - a + 1)^{\overline{\alpha - 1}}}{\Gamma(\alpha)}f(a),$$

where f is defined on $\mathbb{N}_a$.

In the next lemma we use the notation defined by GRAY and ZHANG in [**17**]

$$(2.2) \qquad \overset{h(t)}{\nabla_a^{-\alpha}}f(t) = \sum_{s=a}^{h(t)} \frac{(h(t) - \rho(s))^{\overline{\alpha - 1}}}{\Gamma(\alpha)}f(s),$$

where $h : \mathbb{N}_a \to \mathbb{N}_a$.

Lemma 2.3 (Leibniz Rule). *For any $\alpha > 0$, $\alpha - th$ order fractional difference of the product fg is given in this form*

$$\overset{t}{\nabla_0^{\alpha}}(fg)(t) = \sum_{n=0}^{t} \binom{\alpha}{n}\left[\overset{t-n}{\nabla_0}{}^{\alpha-n}f(t - n)\right]\left[\nabla^n g(t)\right],$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha - n + 1)},$$

and f, g are defined on $\mathbb{N}_0$, and t is a positive integer.

Definition 2.1. The exponential function of discrete fractional calculus with ∇-operator is defined by

$$\widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = (1 - a)\sum_{n=0}^{\infty} \frac{a^n(t + 1)^{\overline{(n+1)\alpha - 1}}}{\Gamma((n + 1)\alpha)},$$

where $|a| < 1$ and $t \geq 0$.

We note that the above infinite series is absolutely convergent, for its proof see [**5**].

Theorem 2.2 ([**4**]). *The exponential function $\widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}})$ is a solution of the IVP (1.1)–(1.2).*

Definition 2.3 ([**4**])**.** The discrete Mittag-Leffler function is defined as

$$F_{\alpha,\beta}(a, t^{\overline{\mu}}) = \sum_{k=0}^{\infty} \frac{a^k t^{\overline{k\mu}}}{\Gamma(k\alpha + \beta)},$$

where μ is any real number.

Next we prove some properties of the exponential functions.

Lemma 2.4. *The following are valid.*
(i) $\widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = (1 - a)(t + 1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(a, (t + \alpha)^{\overline{\alpha}})$.
(ii) $\nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = (1 - a) F_{\alpha,1}(a, (t + 1)^{\overline{\alpha}})$.

Proof. The proof of (i) can be found in the paper [**4**]. To prove (ii), we use power rule (Lemma 2.1) and the fact that the nabla exponential function is convergent.

$$\nabla_0^{-(1-\alpha)} \widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = \nabla_0^{-(1-\alpha)}(1 - a) \sum_{n=0}^{\infty} \frac{a^n (t + 1)^{\overline{(n+1)\alpha-1}}}{\Gamma((n + 1)\alpha)}$$

$$= (1 - a) \sum_{n=0}^{\infty} \frac{a^n \nabla_0^{-(1-\alpha)}(t + 1)^{\overline{(n+1)\alpha-1}}}{\Gamma((n + 1)\alpha)}$$

$$= (1 - a) \sum_{n=0}^{\infty} \frac{a^n (t + 1)^{\overline{n\alpha}}}{\Gamma(n\alpha + 1)} = (1 - a) F_{\alpha,1}(a, (t + 1)^{\overline{\alpha}}).$$

Lemma 2.5. *The following are valid.*
(i) $\overset{t-1}{\nabla_0^{\alpha}} \widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = a\widehat{e}_{\alpha,\alpha}(a, (t - 1)^{\overline{\alpha}}), \quad t = 1, 2, \ldots$
(ii) $\overset{t}{\nabla_0^{\alpha}} \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = a \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}})$.

Proof. To prove (i), we first shift the equation (1.1) one unit left. Hence we have

$$\overset{t-1}{\nabla_0^{\alpha}} y(t) = ay(t - 1), \quad t = 2, 3, \ldots$$

Then the result follows

$$\overset{t-1}{\nabla_0^{\alpha}} \widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = a\widehat{e}_{\alpha,\alpha}(a, (t - 1)^{\overline{\alpha}}).$$

To prove (ii), we apply the definition of discrete fractional difference on the left side of the equation. Hence we have

$$(2.3) \qquad \overset{t}{\nabla_0^{\alpha}} \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = \nabla \overset{t}{\nabla_0^{\alpha-1}} \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}).$$

Then, we use Lemma 2.2 by calling $\overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = f(t)$. Since $f(0) = 0$ we obtain

$$\overset{t}{\nabla_1^{\alpha-1}} \nabla f(t) = \nabla \overset{t}{\nabla_0^{\alpha-1}} f(t).$$

Thus we have

$$\overset{t}{\nabla}{}_{0}^{\alpha}\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}}) = \nabla\overset{t}{\nabla}{}_{0}^{\alpha-1}\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})$$

$$= \overset{t}{\nabla}{}_{1}^{\alpha-1}\nabla\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}}) = \overset{t}{\nabla}{}_{1}^{\alpha-1}\overset{t-1}{\nabla}{}_{0}^{\alpha}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}}).$$

Using the equality in (i), we have

$$\overset{t}{\nabla}{}_{1}^{\alpha-1}\overset{t-1}{\nabla}{}_{0}^{\alpha}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}}) = \overset{t}{\nabla}{}_{1}^{\alpha-1}a\widehat{e}_{\alpha,\alpha}(a,(t-1)^{\overline{\alpha}}) = a\sum_{s=1}^{t}\frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\,\widehat{e}_{\alpha,\alpha}(a,(s-1)^{\overline{\alpha}})$$

$$= a\sum_{u=0}^{t-1}\frac{(t-1-\rho(u))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\,\widehat{e}_{\alpha,\alpha}(a,u^{\overline{\alpha}}) = a\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}}).$$

Thus, we conclude that

$$\overset{t}{\nabla}{}_{0}^{\alpha}\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}}) \;=\; a\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}}).$$

Lemma 2.6. *The following equality holds:*

$$\overset{t}{\nabla}{}_{0}^{\alpha}\,(t\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})) = a\,t\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}}) + \alpha\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}}).$$

Proof. We use Leibniz rule stated in Lemma 2.3 to obtain the equality.

$$\overset{t}{\nabla}{}_{0}^{\alpha}(t\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})) = \sum_{n=0}^{t}\binom{\alpha}{n}\left[\overset{t-n}{\nabla}{}_{0}^{\alpha-n}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})\right][\,\nabla^{n}t]$$

$$= \binom{\alpha}{0}\left[\overset{t}{\nabla}{}_{0}^{\alpha}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})\right]t + \binom{\alpha}{1}\left[\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})\right]\nabla t$$

$$= \left[\overset{t}{\nabla}{}_{0}^{\alpha}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})\right]t + \alpha\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\,\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})$$

$$= a\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})\,t + \alpha\overset{t-1}{\nabla}{}_{0}^{\alpha-1}\,\widehat{e}_{\alpha,\alpha}(\lambda,t^{\overline{\alpha}}),$$

since $\widehat{e}_{\alpha,\alpha}(a,t^{\overline{\alpha}})$ solves the IVP (1.1)–(1.2).

3. THE CASORATIAN AND LINEAR INDEPENDENCE

We consider the following linear fractional difference equation

$$(3.1) \qquad p_{n}\nabla_{0}^{(n\alpha)}y(t) + p_{n-1}\nabla_{0}^{((n-1)\alpha)}y(t) + \cdots + p_{1}\nabla_{0}^{\alpha}y(t) + p_{0}y(t) = 0$$

where $\nabla_{0}^{(n\alpha)} = \underbrace{\nabla_{0}^{\alpha}\nabla_{0}^{\alpha}...\nabla_{0}^{\alpha}}_{n-\text{times}}$ and p_{i} are constants for $0 \le i \le n$ with $p_{n} \ne 0$. The corresponding fractional differential equation is known as the sequential fractional

differential equation in the literature [**3, 19**]. The sequential fractional difference equations have been first introduced by C. GOODRICH [**15**].

Let $\{y_1, y_2, ..., y_n\}$ be a set of functions which are defined on the discrete interval I. Then the following determinant, denoted by, $C[y_1, y_2, \ldots, y_n]$ is called the Casoratian

$$\begin{vmatrix} \nabla_0^{-(1-\alpha)} y_1(t) & \nabla_0^{-(1-\alpha)} y_2(t) & \cdots & \nabla_0^{-(1-\alpha)} y_n(t) \\ \nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_1(t) & \nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_2(t) & \cdots & \nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_n(t) \\ \nabla_0^{-(1-\alpha)} \nabla_0^{(2\alpha)} y_1(t) & \nabla_0^{-(1-\alpha)} \nabla_0^{(2\alpha)} y_2(t) & \cdots & \nabla_0^{-(1-\alpha)} \nabla_0^{(2\alpha)} y_n(t) \\ \vdots & & & \vdots \\ \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y_1(t) & \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y_2(t) & \cdots & \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y_n(t) \end{vmatrix}.$$

Theorem 3.1. *Let $\{y_1, y_2, \ldots, y_n\}$ be a set of n solutions of the equation (3.1). Then the set is linearly independent if and only if the Casoratian is not identically equal to zero for all $t \geq 0$.*

Proof. We prove for the case $n = 2$. The proof can be easily done for any n. Let $y_1(t)$ and $y_2(t)$ be two solutions of the following problem

$$(3.2) \qquad \left\{ \; p\nabla_0^\alpha \nabla_0^\alpha y(t) + q\nabla_0^\alpha y(t) + ry(t) = 0, \; t = 1, 2, \ldots \right.$$

for all $t \geq 0$, $0 < \alpha \leq 1$, and where $p \neq 0$ and q, r are constants.

Let y_1 and y_2 be linearly dependent. Then there exist a nonzero constant k such that

$$(3.3) \qquad\qquad\qquad y_2(t) = ky_1(t),$$

for all $t \geq 0$.

This implies that

$$C[y_1, y_2] = (\nabla_0^{-(1-\alpha)} y_1(t))(\nabla_0^{-(1-\alpha)} \nabla_0^\alpha ky_1(t)) - (\nabla_0^{-(1-\alpha)} ky_1(t))(\nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_1(t))$$

$$= k(\nabla_0^{-(1-\alpha)} y_1(t))(\nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_1(t)) - k(\nabla_0^{-(1-\alpha)} y_1(t))(\nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_1(t)) = 0$$

for all $t \geq 0$.

Conversely, we assume that $y_1 \neq 0$ and $y_2 \neq 0$ for all $t \geq 0$ and $C[y_1, y_2](t) = 0$ for all $t \geq 0$.

Consider the following two equations in two unknowns

$$(3.4) \qquad\qquad \left\{ \begin{array}{l} k_1 \nabla_0^{-(1-\alpha)} y_1(0) + k_2 \nabla_0^{-(1-\alpha)} y_2(0) = 0 \\ k_1 \nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_1(0) + k_2 \nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_2(0) = 0. \end{array} \right.$$

This system of equations can be represented by the matrix equation

$$\begin{bmatrix} \nabla_0^{-(1-\alpha)} y_1(0) & \nabla_0^{-(1-\alpha)} y_2(0) \\ \nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_1(0) & \nabla_0^{-(1-\alpha)} \nabla_0^\alpha y_2(0) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0.$$

Since the determinant of the coefficient matrix is zero, there is a nontrivial solution k_1, k_2 of the above equation. We note that the case $k_1 = 0$ and $k_2 \neq 0$ is not possible since the equation

$$k_1 \nabla_0^{-(1-\alpha)} y_1(0) + k_2 \nabla_0^{-(1-\alpha)} y_2(0) = 0$$

becomes $k_2 y_2(0) = 0$ which is not possible.

Now we consider

$$y(t) = k_1 y_1(t) + k_2 y_2(t).$$

We can easily verify that $y(t)$ is a solution of the initial value problem

$$\begin{cases} p \nabla_0^\alpha \nabla_0^\alpha y(t) + q \nabla_0^\alpha y(t) + r y(t) = 0, \ t=1, \ 2, \dots \\ \nabla_0^{-(1-\alpha)} y(t)\,|_{t=0} = y(0) = 0 \ \text{ and } \ \nabla_0^{-(1-\alpha)} \nabla_0^\alpha y(0) = 0. \end{cases}$$

First we note that the initial value problem can be transformed into a linear system of nabla fractional difference equations using the method of change of variables such that

$$y_1(t) = y(t) \implies \nabla_0^\alpha y_1(t) = \nabla_0^\alpha y(t) = y_2(t)$$

$$y_2(t) = \nabla_0^\alpha y(t) \implies \nabla_0^\alpha y_2(t) = \nabla_0^\alpha \nabla_0^\alpha y(t) = -\frac{q}{p} \nabla_0^\alpha y(t) - \frac{r}{p} y(t) = -\frac{q}{p} y_2(t) - \frac{r}{p} y_1(t).$$

Thus, we have the following dynamic system

$$\nabla_0^\alpha Y(t) = \begin{bmatrix} 0 & 1 \\ -\dfrac{r}{p} & -\dfrac{q}{p} \end{bmatrix} Y(t),$$

with an initial condition

$$(3.5) \quad \nabla_0^{-(1-\alpha)} Y(t)|_{t=0} = \begin{bmatrix} \nabla_0^{-(1-\alpha)} y_1(t) \\ \nabla_0^{-(1-\alpha)} y_2(t) \end{bmatrix}_{t=0} = \begin{bmatrix} \nabla_0^{-(1-\alpha)} y(t) \\ \nabla_0^{-(1-\alpha)} \nabla_0^\alpha y(t) \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.

By the existence and uniqueness theorem proven in [4], we conclude that

$$k_1 y_1(t) + k_2 y_2(t) = 0$$

for all $t \geq 0$. Since k_1 and k_2 are both nonzero, y_1 and y_2 are linearly dependent.

Theorem 3.2. *Suppose that $y_1(t), y_2(t), \dots, y_n(t)$ are linearly independent solutions of* (3.1). *Then every solution $y(t)$ of* (3.1) *can be written as*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t), \quad \text{for some constants } c_1, c_2, \dots, c_n.$$

Proof. Let $y(t)$ be an arbitrary but fixed solution of (3.1). Consider the system of fractional nabla difference and sum equations.

$$c_1 \nabla_0^{-(1-\alpha)} y_1(0) + c_2 \nabla_0^{-(1-\alpha)} y_2(0) + \cdots + c_n \nabla_0^{-(1-\alpha)} y_n(0) = \nabla_0^{-(1-\alpha)} y(0)$$
$$c_1 \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} y_1(0) + c_2 \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} y_2(0) + \cdots + c_n \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} y_n(0) = \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} y(0)$$
$$\vdots$$
$$c_1 \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y_1(0) + \cdots + c_n \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y_n(0) = \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y(0)$$

The system above can be represented as matrix form such that

$$[Y] = \begin{bmatrix} \nabla_0^{-(1-\alpha)} y(0) \\ \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} y(0) \\ \vdots \\ \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y(0) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \nabla_0^{-(1-\alpha)} y_1(0) & \nabla_0^{-(1-\alpha)} y_2(0) & \cdots & \nabla_0^{-(1-\alpha)} y_n(0) \\ \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} y_1(0) & \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} y_2(0) & \cdots & \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} y_n(0) \\ \vdots & & & \vdots \\ \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y_1(0) & \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y_2(0) & \cdots & \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y_n(0) \end{bmatrix}}_{[A]} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{[C]}.$$

The Casoratian $C[y_1, y_2, \ldots, y_n](0) \neq 0$, since the set of solutions is linearly independent. Therefore, $\det A \neq 0$ which means the matrix A is invertible. Apply $[A]^{-1}$ to each side of the system above from the left, we have $[C] = [A]^{-1} [Y]$.

Let $c_1, c_2, \ldots, c_n$ be the unique solution of the above system. Set

$$v(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t).$$

Note that $v(t)$ is a solution and

$$\nabla_0^{-(1-\alpha)} v(0) = \nabla_0^{-(1-\alpha)} y(0)$$
$$\nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} v(0) = \nabla_0^{-(1-\alpha)} \nabla_0^{\alpha} y(0)$$
$$\vdots$$
$$\nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} v(0) = \nabla_0^{-(1-\alpha)} \nabla_0^{((n-1)\alpha)} y(0).$$

By the uniqueness of the solution, we have $y(t) = v(t)$. Hence we have

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t).$$

4. SOLVING SEQUENTIAL LINEAR DIFFERENCE EQUATIONS OF FRACTIONAL ORDER

The sequential discrete fractional equation is given by

(4.1) $p \nabla_0^{\alpha} \nabla_0^{\alpha} y(t) + q \nabla_0^{\alpha} y(t) + r y(t) = 0$ for $t = 1, 2, \ldots$

where $0 < \alpha < 1$ and where $p \neq 0$ and q, r are constant coefficients. The characteristic equation of (4.1) is given as

$$p\lambda^2 + q\lambda + r = 0.$$

Assume that λ_1 and λ_2 are positive real roots of the characteristic equation such that they are less than 1. By using the fact that any given equation can be represented by its characteristic roots, we have

$$(4.2) \qquad \nabla_0^\alpha \nabla_0^\alpha y(t) - (\lambda_1 + \lambda_2)\nabla_0^\alpha y(t) + (\lambda_1\lambda_2)y(t) = 0.$$

Theorem 4.1. *If $\lambda_1 \neq \lambda_2$, then the general solution of (4.1) is*

$$y(t) = c_1 \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) + c_2 \widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}),$$

where c_1 and c_2 are constant parameters.

Proof. One can easily verify that $\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})$ and $\widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}})$ are solutions of the equation (4.2). Next we show that these two are linearly independent solutions. The Casoratian of these two functions is

$$C\,[y_1, y_2] = \begin{vmatrix} \nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) & \nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}) \\ \nabla_0^{-(1-\alpha)}\nabla_0^\alpha \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) & \nabla_0^{-(1-\alpha)}\nabla_0^\alpha \widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}) \end{vmatrix}$$

$$= \begin{vmatrix} \nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) & \nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}) \\ \lambda_1\nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) & \lambda_2\nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}) \end{vmatrix}$$

$$= (\lambda_2 - \lambda_1)\nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}})\nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}).$$

The above last expression is not equal to zero since $\lambda_1 \neq \lambda_2$ and by Lemma 2.4. Hence the set of solutions $\{\widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}),\ \widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}})\}$ is linearly independent and by Theorem 3.2, we conclude that the general solution of (4.1) is

$$y(t) = c_1 \widehat{e}_{\alpha,\alpha}(\lambda_1, t^{\overline{\alpha}}) + c_2 \widehat{e}_{\alpha,\alpha}(\lambda_2, t^{\overline{\alpha}}),$$

where c_1 and c_2 are constant parameters.

Theorem 4.2. *If $\lambda_1 = \lambda_2(= \lambda)$, then the general solution of (4.1) is*

$$y(t) = c_1 \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + c_2 t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$$

where c_1, c_2 are constant parameters.

Proof. We first show that $t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$ satisfies the equation (4.2). Applying the formula in Lemma 2.6 we have

$$\nabla_0^\alpha \nabla_0^\alpha (t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})) = \lambda^2 t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha\lambda \overset{t-1}{\nabla_0^{\alpha-1}}\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha\overset{t}{\nabla_0^\alpha}\overset{t-1}{\nabla_0^{\alpha-1}}\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}).$$

Then it follows that

$$\nabla_0^\alpha \nabla_0^\alpha(t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})) - 2\lambda \nabla_0^\alpha(t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})) + \lambda^2(t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}))$$

$$= \lambda^2 t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha\lambda \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha \overset{t}{\nabla_0^\alpha} \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$$

$$- 2\lambda \left[\lambda t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \right] + \lambda^2(t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}))$$

$$= \lambda^2 t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha\lambda \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha\lambda \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})$$

$$- 2\lambda^2 t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) - 2\alpha\lambda \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \lambda^2 t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) = 0,$$

where we used Lemma 2.5 and Lemma 2.6 repeatedly.

Next we show that the set $\left\{ t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}),\ \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \right\}$ is linearly independent. In fact, we calculate the Casoratian of this set and we obtain

$$\begin{vmatrix} \nabla_0^{-(1-\alpha)}(t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})) & \nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \\ \nabla_0^{-(1-\alpha)}\nabla_0^\alpha(t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})) & \nabla_0^{-(1-\alpha)}\nabla_0^\alpha\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \end{vmatrix}$$

$$= \begin{vmatrix} \nabla_0^{-(1-\alpha)}(t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})) & \nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \\ \nabla_0^{-(1-\alpha)}(\lambda t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + \alpha \overset{t-1}{\nabla_0^{\alpha-1}} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}})) & \lambda\nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \end{vmatrix}$$

$$= -\left[\alpha \overset{t-1}{\nabla_0^{-(1-\alpha)}} \widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) \right] \nabla_0^{-(1-\alpha)}\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}).$$

The Casoratian is not identically equal to zero. Finally, by Theorem 3.2, the general solution of (4.2) is

$$y(t) = c_1 t\,\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}) + c_2\widehat{e}_{\alpha,\alpha}(\lambda, t^{\overline{\alpha}}),$$

where c_1, c_2 are constants.

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