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ON A FINITE SUM WITH POWERS OF COSINES

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A formula for the finite sum of powers of cosines with fractional multiples of $\pi/2$, viz.

$$S(n,m)=\sum_{k=1}^{m}(-1)^k\cos^{2n}\Big(\frac{k\pi}{2m+2}\Big),$$

where m and n are arbitrary positive integers, is derived. In the process new and interesting mathematical results are uncovered, particularly with regard to the Bernoulli and Euler polynomials, while other related series are discussed. It is found that the series always yields rational values, which can only be evaluated by using the integer arithmetic routines in a mathematical software package such as Mathematica.

1. INTRODUCTION

Finite sums with arbitrary powers of trigonometric functions have attracted a growing interest over the past few decades. To gain an appreciation of the activity and importance of this topic, the reader is urged to consult [**2, 3**] and examine the huge number of references cited therein. As stated in the first of these references, many finite trigonometric sums remain intractable or cannot be evaluated explicitly in closed form, even though they may possess beautiful reciprocity theorems. Recently, one of us [**14**] was able to derive elegant formulas for the case where the summand of such series was given in arbitrary inverse powers of cosines of fractional multiples of $\pi/2$. This problem, which arises in probability theory [**12**] and in the evaluation of lattice sums [**27**], was solved by using new results derived in [**15**] and the development of a fascinating empirical approach, which will in turn become the subject of a future work. On the other hand, a slightly different version of the

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series where the cosines were in fractional multiples of π had been solved previously in [**4, 6, 7**]. Yet, despite all this activity, the complementary problem, where the powers of the trigonometric functions in the summand are positive rather than negative, has proved not only to be equally difficult to solve, but is also, except for a few special cases described herein, still elusive.

Interest in those series where the summand is in positive powers of trigonometric functions with fractional multiples of π began with Greening et al. [**10**] when they solved a problem originally proposed by Quoniam. Their work was concerned with deriving a formula for the sum given by

$$S_p(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} 2^p \cos^p \left(\frac{k\pi}{n+1} \right),$$

for p, a positive even integer. Here $\lfloor x \rfloor$ represents the largest integer not exceeding x. Recently, Merca [**17, 18, 19, 20, 21**] studied variants of this sum, in particular the sums given by

$$S_p^+(n-1) = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \cos^p \left(\frac{k\pi}{n} \right),$$

where p is again a positive even integer, and

$$S_p^-(n-1) = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} (-1)^k \cos^p \left(\frac{k\pi}{n} \right),$$

where n and p are positive integers with the same parity. The main objective concerning these series is to derive elegant general formulas for them, even though in some cases asymptotic results have been either conjectured or determined [**18, 21**].

Prudnikov et al consider more general versions of these series involving powers of $\cos(kx)$ and $\sin(kx)$ in [**23**, Sec. 4.4.2], but these are expressed as awkward finite trigonometric sums. However, they do give special cases of the above series and those studied herein, which appear as Nos. 4.4.2.9 to 4.4.2.12 in [**23**]. Note that although these series possess different upper limits, these limits only affect the results slightly as can be seen in Section 3 of this paper.

It should also be mentioned that even more general trigonometric series, viz. in powers of $\sin(kx+y)$ and $\cos(kx+y)$, are presented respectively on pages 234 and 256 of [**11**]. Once again, they are expressed as awkward sums, while the special cases of these series are not only restricted to even integer powers of the trigonometric functions, but also possess different arguments and limits than those studied here. Since the awkward results in [**11, 23**] have been obtained via standard analytic techniques, it raises the question whether more general and elegant results for such series be obtained by using the new material in [**15**] as was done in [**14**] which, as stated previously, dealt with sums of inverse powers of trigonometric functions.

In [**18**] Merca conjectured that a $p \times p$ matrix A_p whose (n, m)-entry is

$$S(n,m) = \sum_{k=1}^{m} (-1)^k \cos^{2n} \left(\frac{k\pi}{2m+2} \right) \tag{1.1}$$

has a determinant given by

$$\det A_p = (-1)^p \, 2^{-p^2} \, \Gamma(p+1) \, .$$

Whilst this problem seems elementary, it turns out to be very difficult to solve. It has, however, been solved by one of us [**5**] as a result of evaluating (1.1) in the following cases: (1) where the entries above the main diagonal are constant, i.e., for $m > n$, (2) along the main diagonal, where $m = n$, and (3) along the first subdiagonal below the main diagonal, where $m = n - 1$. However, the solution based upon a general formula for $S(n, m)$ remains elusive. Therefore, in this paper we aim to draw upon recent developments in [**13, 14, 15, 16**] to uncover new and interesting mathematics in our quest to obtain a general formula for $S(n, m)$, but in the end, we shall see that standard analytic techniques will prevail.

Generally in classical analysis, one derives a formula and then programs it to obtain numerical values. Often, one has an idea of how these values will turn out. In this instance, however, we shall see that our result for $S(n, m)$ is arbitrary for each value of m and n and that one really has no idea of what the value of $S(n, m)$ is except when m is close to n. Therefore, it will be necessary to program our result for $S(n, m)$ by employing integer arithmetic routines in order to obtain the discrete values in rational form. That is, we shall see here that a computer program has become the solution to a problem in classical analysis rather than serving only as a means of facilitating number-crunching on a computer.

2. MAIN RESULT

The main result of this paper, whose first form is proved at the end of Section 4, but whose more intricate and fascinating second form becomes the subject of the following section, can be expressed succinctly by the following theorem:

Theorem 2.1. *The series* $S(n, m)$ *as defined by* (1.1) *is found to be given by*

$$S(n,m) = \begin{cases} -\dfrac{1}{2}, & m \geqslant n \, , \\ -\dfrac{1}{2} + \left(\dfrac{m+1}{2^{2n-1}}\right)\dbinom{2n}{n-m-1} + \dfrac{m}{2^{2n-1}}\Big((1-(-1)^{n-m})\, S_{\mathrm{even}} & \\ + \, (1+(-1)^{n-m})\, S_{\mathrm{odd}}\Big), & m < n \, , \end{cases} \tag{2.1}$$

where

$$S_{\mathrm{even}} = \sum_{j=0,2,4,\ldots}^{n-m-2} \delta_{\mathrm{mod}(n-m-j-1,2(m+1)),0} \binom{2n}{j}, \tag{2.2}$$

$$S_{\mathrm{odd}} = \sum_{j=1,3,\ldots}^{n-m-2} \delta_{\mathrm{mod}(n-m-j-1,2(m+1)),0} \binom{2n}{j}, \tag{2.3}$$

and $\delta_{j,i}$ represents the discrete or Kronecker delta function, which is equal to unity whenever the first index is equal to the second index, i.e. $i = j$, and zero otherwise ($i \neq j$). In the above $\mathrm{mod}(a, b)$ *denotes the modulo/modulus or remainder when the dividend a is divided by the divisor or base b.*

Remark 2.1. The Kronecker delta function appearing in the S_{even} and S_{odd} equals unity only when $(n-m-j-1)$ is exactly divisible by $2(m+1)$. E.g., for $n = 15$ and $m = 2$, the modulus function reduces to $\mathrm{mod}(12-j, 6)$, which yields zero only when $j = 0$ and $j = 6$. Therefore, the Kronecker delta function yields unity for these cases. For all other values of j, it yields zero since $\mathrm{mod}(12 - j, 6)$ is non-zero. As a consequence, we see that for $n = 15$ and $m = 2$, there are two contributions from S_{even}, viz. the $j = 0$ and $j = 6$ terms, and none from S_{odd} to $S(n, m)$. For $n = 14$ and $m = 2$, which results in $\mathrm{mod}(11 - j, 6)$ appearing in the index of the Kronecker delta function, we find that only the $j = 5$ term from S_{odd} contributes to $S(n, m)$.

We shall refer to the term of $-1/2$ on the rhs of both forms of (2.1) as the leading term. The reason for this will become apparent when we attempt to find an expansion in powers of $\pi/2(m+1)$ in Section 4. Although we shall express $S(n, m)$ in such a power series with the coefficients in terms of the Euler polynomials, we shall see that despite obtaining new results for the latter, it is a formidable challenge to obtain higher order terms in the series expansion. Ultimately, in Section 5 we shall use standard analytical techniques to prove the second or $m < n$ form of $S(n, m)$ in (2.1). We shall also observe that because of the modulus function appearing as an index in the Kronecker delta function, the above formula is, in reality, an algorithm. As a consequence, a special code is required to evaluate $S(n, m)$ for large values of n and m as has already been stated at the end of the introduction.

3. A FORMULA FOR $S(n, m)$ INVOLVING EULER POLYNOMIALS

We begin by expressing $S(n, m)$ in terms of terms of powers of sines as in [**14**]. Thus, $S(n, m)$ becomes

$$S(n, m) = \sum_{k=1}^{m} (-1)^{k+m+1} \sin^{2n}\left(\frac{k\pi}{2m+2}\right).$$

As a consequence, we can introduce a power series expansion for $\sin^{2n} z$ that is proved in [**15**, Section 13]. This is

$$\frac{z^\rho}{\sin^\rho z} \equiv \sum_{k=0}^{\infty} c_{\rho,k}\, z^{2k}, \tag{3.1}$$

where ρ can be any value including a complex number. The coefficients $c_{\rho,k}$, which are known as the generalised cosecant numbers, are given by

$$c_{\rho,k} = (-1)^k \sum_{\substack{n_1, n_2, n_3, \ldots, n_k \\ \sum_{i=1}^{k} i\, n_i = k}}^{k, \lfloor k/2 \rfloor, \lfloor k/3 \rfloor, \ldots, 1} (-1)^N (\rho)_N \prod_{i=1}^{k} \left(\frac{1}{(2i+1)!}\right)^{n_i} \frac{1}{n_i!}, \tag{3.2}$$

where $\sum_{i=1}^{k} n_i = N$ and $(\rho)_N$ is the Pochhammer notation for $\Gamma(\rho+N)/\Gamma(\rho)$. The equivalence symbol in (3.1) indicates that the series on the rhs of the above statement is divergent when $|\operatorname{Re} z| \geqslant \pi$, but elsewhere it is convergent. This is because of the zeros occurring at $z = \pm\pi$ in the denominator on the lhs of above equivalence. Since k ranges from 1 to m in $S(n,m)$, $k\pi/(2m+2)$ cannot exceed π. Hence, we can replace the equivalence symbol by an equals sign, which is a similar situation to that in [**14**, Eq.(7)]. It should also be mentioned that the coefficients $c_{\rho,k}$ satisfy other relations such as

$$\rho\, c_{\rho+1,k} = \sum_{j=0}^{k} (\rho - 2k + 2j)\, d_j\, c_{\rho,k-j}\ ,$$

and

$$c_{\mu+\nu,k} = \sum_{j=0}^{k} c_{\mu,j}\, c_{\nu,k-j}\ .$$

These results appear as Eqs. (274) and (275) in [**15**].

For the case of $\rho = -2n$, which is the case we are interested here, (3.2) simplifies drastically because $(-2n)_N$ vanishes when $N > 2n$. Furthermore, by introducing the equation form of (3.1) into No. I.1.9 of [**23**], we find that after some algebra the coefficients can also be written as

$$c_{-2n,k} = 2^{1-2n} \sum_{j=0}^{n-1} \frac{(-1)^{k-j}}{(2n+2k)!} \binom{2n}{j} \big(2(n-j)\big)^{2n+2k}\ .$$

For $n = 1$, this yields $c_{-2,k} = (-1)^k 2^{2k+1}/(2k+2)!$, which when introduced into (3.1) yields the power series expansion for $\sin^2 x$ given by No.1.412(1) in [**9**]. In a similar manner one can obtain the $n = 2$ result, which becomes

$$\sin^4 x = \sum_{j=2}^{\infty} \frac{(-1)^j\, 2^{2j-1}}{(2j)!} \left(2^{2j-2} - 1\right) x^{2j}.$$

By using the preceding results, we find that the sum $S(n,m)$ can be written as

$$S(n,m) = (-1)^{m+1} \sum_{j=0}^{\infty} c_{-2n,j} \Big(\frac{\pi}{2(m+1)}\Big)^{2n+2j} \sum_{k=1}^{m} (-1)^k k^{2n+2j}\ . \tag{3.3}$$

At this stage we introduce the Euler polynomials $E_n(x)$ into the analysis. With the aid of the identity, $E_n(x+1) + E_n(x) = 2x^n$, we can express the last sum in (3.3) as $(-1)^m E_{2n+2j}(m+1)/2$. Hence, we find that (3.3) reduces to

$$S(n,m) = -\frac{1}{2} \sum_{j=0}^{\infty} c_{-2n,j} \Big(\frac{\pi}{2(m+1)}\Big)^{2n+2j} E_{2n+2j}(m+1)\ . \tag{3.4}$$

As an aside, if there were no phase factor of $(-1)^k$ in the summand of the inner series in (3.3), then the sum over k would have yielded $B_{2n+2j+1}(m+1)/(2n+2j+1)$, which is known as Faulhaber's formula [**25**]. As a consequence, we obtain

$$\sum_{k=1}^{m} \sin^{2n}\Big(\frac{k\pi}{2m+2}\Big) = \sum_{j=0}^{\infty} c_{-2n,j}\Big(\frac{\pi}{2(m+1)}\Big)^{2n+2j}\, \frac{B_{2n+2j+1}(m+1)}{2n+2j+1}\,. \tag{3.5}$$

This result is virtually identical to the remainder obtained for the sum of inverse powers of cosines [**14**, Eq. (12)] with the only difference being that $2v$ has been replaced by $-2n$.

It should be noted that both (3.4) and (3.5) appear to be very compact, but actually they are complicated because series involving Euler and Bernoulli polynomials other than the generating functions are difficult to solve. This can be seen by the few results involving these special polynomials given in [**22**, Chapter 6], none of which can be related or applied to either (3.4) or (3.5). In addition, they are not really power series, but possess the behaviour associated with perturbation series. That is, as the order increases, extra terms are continually introduced, which means that summing over all orders from $j = 0$ to infinity represents a formidable problem. This will be seen more clearly in the next section, where we evaluate the leading terms for (3.4) and (3.5).

4. THE LEADING TERM

In this section we show as indicated in Section 2 that the leading term of $S(n, m)$ is a constant equal to $-1/2$ for any value of n and m. Furthermore, for $m \geqslant n$ we show that it represents the sole contribution to $S(n, m)$ as indicated in Theorem 2.1.

In order to determine the leading term and possible remainder, we need to evaluate the summation over j in (3.3). This is complicated by the fact that the summation index appears in the order of the Euler polynomials, i.e., $2n+2j$. Hence, the dominant term in (3.3) is due to the highest order term in $E_{2n+2j}(m+1)$ or $(m+1)^{2n+2j}$, but to evaluate the sum, we require a general formula in j for the coefficient. The same applies to the other orders if we wish to evaluate their contribution to the sum. Yet, the literature [**1, 9, 24**] only provides lists of the polynomials for specific orders. One could perhaps conjecture that the leading order term has a coefficient of unity, but we really need to derive general formulas for the coefficients of the powers in the Euler polynomials.

4.1. Coefficients of Euler Polynomials

From [**15**, Theorem 14] we know that the Euler polynomials can be expressed in terms of the secant polynomials $d_k(t)$, according to

$$E_{2k}(x) = (-1)^k 2^{-2k}\Gamma(2k+1)\, d'_k(2x-1)\,, \tag{4.1}$$

where the prime denotes the first derivative. In (4.1) the secant polynomials are given by

$$d_k(t) = \sum_{j=0}^{k} \frac{(-1)^j}{(2j+1)!}\, d_{k-j}\, t^{2j+1}\,, \tag{4.2}$$

where the coefficients d_k are referred to as the secant numbers with $d_0 = 1$, $d_1 = 1/2$, $d_2 = 5/24$, $d_3 = 61/720$, etc. These monotonically decreasing positive rationals can be obtained by putting $\rho = 1$ and replacing $1/(2i+1)!$ by $1/(2i)!$ in (3.2) or by using the following result:

$$d_k = \sum_{j=0}^{k-1} \frac{(-1)^{k-j-1}}{(2k-2j)!}\, d_j = \frac{2}{(2\pi)^{2k+1}} \Big(\zeta(2k+1, 1/4) - \zeta(2k+1, 3/4) \Big)\,.$$

In the above equation $\zeta(s,z)$ represents the Hurwitz zeta function.

If we introduce (4.2) into (4.1), then we find that

$$E_{2k}(x) = \frac{(-1)^k}{2^{2k}}\, \Gamma(2k+1) \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(2k-2j)!}\, d_j \sum_{\ell=0}^{2k-2j} \binom{2j}{j} (-1)^{\ell} (2x)^{2k-2j-\ell}\,.$$

Moreover, by denoting the coefficients of $E_{2k}(x)$ as $E_{2k,i}$, where i ranges from zero to $2k$, we arrive at

$$E_{2k,i} = \frac{(-1)^{k+i}}{2^{2k-i}}\, \Gamma(2k+1) \sum_{\substack{j=1 \\ j \geqslant i/2}}^{k} \frac{(-1)^j}{(2j)!} \binom{2j}{2j-i} d_{k-j}\,. \tag{4.3}$$

When $i = 2k$, j can only equal k. Consequently, we find that $E_{2k,2k} = 1$, which is the only even-powered coefficient that does not vanish. For i an odd number, (4.3) reduces to

$$E_{2k,2i+1} = \frac{(-1)^{k+1}}{2^{2k-2i-1}}\, \frac{(2k)!}{(2i+1)!} \sum_{j=0}^{k} \frac{(-1)^j}{(2j-2i-1)!}\, d_{k-j}\,. \tag{4.4}$$

The sum in (4.4) is equal to $d_k^{(2i+2)}(1)$, where the superscript denotes that the derivative must be taken $2i+2$ times.

From [**15**, Theorem 12], we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, d_{k+1}(t) = -d_k(t)\,, \tag{4.5}$$

and

$$d_k(1) = -\left(2^{2k+2} - 1\right) c_{k+1}(1), \tag{4.6}$$

while from Eq. (348) in the same reference, we see that

$$c_k(1) = -\frac{2}{\pi^{2k}}\,\zeta(2k). \tag{4.7}$$

Hence, repeated differentiation of (4.5) yields another result for $d_k^{(2i+2)}(1)$, which can be expressed in terms of $c_{k-i}(1)$ via (4.6). Then by introducing (4.7) we find after a little algebra that

$$E_{2k,2i+1} = 4(-1)^{k+i}\,\frac{(2k)!}{(2i+1)!}\,\left(1-2^{2i-2k}\right)\frac{\zeta(2k-2i)}{\pi^{2k-2i}}\,, \tag{4.8}$$

where $\zeta(k)$ is the Riemann zeta function.

For the highest orders this result yields

$$E_{2k,2k-1} = -k\,, \quad E_{2k,2k-3} = \frac{k}{3}\,(k-1/2)(k-1)\,,$$
$$E_{2k,2k-5} = -\frac{2k}{15}\,(k-1/2)(k-1)(k-3/2)(k-2)\,,$$

and so on. A similar result can be obtained for $E_{2k+1}(x)$ by following the same procedure except instead of (4.1) we use

$$E_{2k+1}(x) = (-1)^k 2^{-2k-1}\,\Gamma(2k+1)\,d_k(2x-1)\ . \tag{4.9}$$

This is also derived in [**15**, Theorem 14]. As a consequence, the coefficients of the Euler polynomials to odd order are found to be

$$E_{2k+1,2i} = (-1)^{k+i+1}\,\frac{\Gamma(2k+2)}{\Gamma(2i+1)}\left(4-2^{2i-2k}\right)\frac{\zeta(2k-2i+2)}{\pi^{2k-2i+2}}\,,$$

while $E_{2k+1,2k+1}$ is equal to unity.

4.2. Coefficients of Bernoulli Polynomials

The method in the previous subsection can be adapted to obtain the coefficients of the Bernoulli polynomials with the difference being that Eqs. (399) and (400) in [**15**] are used instead of (4.1) and (4.9). These are

$$B_{2k}(t) = (-1)^k 2^{-k}\,\Gamma(2k+1)\,c_k(2t-1), \tag{4.10}$$

and

$$B_{2k+1}(t) = (-1)^{k+1} 2^{-2k-1}\,\Gamma(2k+2)\,c'_{k+1}(2t-1). \tag{4.11}$$

The $c_k(t)$ are known as the cosecant polynomials. In [**15**, Sec. 15] they are defined as

$$c_k(t) = \sum_{j=0}^{k}\frac{(-1)^k}{(2j)!}\,c_{k-j}t^{2j}. \tag{4.12}$$

To obtain general formulas for the coefficients of the Bernoulli polynomials, we introduce the above form and its derivative into (4.10) and (4.11), respectively. Then one equates the coefficients of like powers on both sides of the resulting equations. We leave the details as an exercise for the reader, although the final results are found to be

$$B_{2k,2k-1} = -k\,, \quad B_{2k,2i} = \frac{(-1)^{k+i+1}}{2^{2k-2i-1}} \frac{\Gamma(2k+1)}{\Gamma(2i+1)} \frac{\zeta(2k-2i)}{\pi^{2k-2i}}\,,$$

and

$$B_{2k+1,2k} = -\Big(k+\frac{1}{2}\Big), \quad B_{2k+1,2i+1} = \frac{(-1)^{k+i+1}}{2^{2k-2i-1}} \frac{\Gamma(2k+2)}{\Gamma(2i+2)} \frac{\zeta(2k-2i)}{\pi^{2k-2i}}\,,$$

while $B_{2k,2k} = B_{2k+1,2k+1} = 1$.

4.3. Series Behaviour

From the preceding results, we find that

$$E_{2n+2j}(m+1) = (m+1)^{2n+2j} + \sum_{i=0}^{n+j-1} \frac{(-1)^{i+1}}{2^{2i+1}} C_i\,(2n+2j)(2n+2j-1)$$
$$\times\,(2n+2j-2)\cdots(2n+2j-2i)(m+1)^{2n+2j-2i-1}\,, \tag{4.13}$$

where $C_0 = 1$, $C_1 = 1/3$, $C_2 = 2/15$ and the other constants can be evaluated via (4.8). By introducing (4.13) into (3.4), we obtain a series in powers of $1/(m+1)$. The leading term in the resulting expansion is given by

$$S_0(n,m) = -\frac{1}{2}\sum_{j=0}^{\infty} c_{-2n,j}\Big(\frac{\pi}{2}\Big)^{2j+2n} = -\frac{1}{2}\,, \tag{4.14}$$

where (3.1) has been used to express the summation in (4.14) as $\sin^{2n+2j}(\pi/2)$ or unity. Consequently, (3.4) can be expressed as

$$S(n,m) = -\frac{1}{2} - \frac{1}{2}\sum_{j=0}^{\infty} c_{-2n,j}\,\Big(\frac{\pi}{2}\Big)^{2n+2j} \sum_{i=0}^{n+j-1} \frac{(-1)^{i+1}\,C_i}{(2m+2)^{2i+1}}\,(2n+2j)\cdots(2n+2j-2i)\,.$$

Expanding in powers of $1/(m+1)$ yields

$$S(n,m) = -\frac{1}{2} + \frac{C_0}{4(m+1)} \sum_{j=0}^{\infty} c_{-2n,j}\,(2n+2j)\Big(\frac{\pi}{2}\Big)^{2n+2j} - \frac{C_1}{16(m+1)^3}$$
$$\times \sum_{j=0}^{\infty} c_{-2n,j}\,(2n+2j)(2n+2j-1)(2n+2j-2)\Big(\frac{\pi}{2}\Big)^{2n+2j} + \cdots\,. \tag{4.15}$$

The series over j in the above result can be written as

$$\sum_{j=0}^{\infty} c_{-2n,j}\,(2n+2j)\cdots(2n+2j-2i)\Big(\frac{\pi}{2}\Big)^{2n+2j} = \frac{\mathrm{d}^{2i+1}}{\mathrm{d}x^{2i+1}} \sin^{2n} x \Big|_{x=\pi/2}\,, \tag{4.16}$$

where (3.4) has been used again. Moreover, by using No. I.1.9 from [**23**] again, we have

$$\frac{\mathrm{d}^{2i+1}}{\mathrm{d}x^{2i+1}} \sin^{2n} x = \frac{1}{2^{2n}-1} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2n}{k} \frac{\mathrm{d}^{2i+1}}{\mathrm{d}x^{2i+1}} \cos\bigl(2(n-k)\,x\bigr)\,. \tag{4.17}$$

Because an odd number number of derivatives is being taken, we always end up with $\sin(2(n-k)\,x)$ on the rhs, which means, in turn, that the lhs of (4.17) vanishes for $x = \pi/2$. Therefore, all the terms in (4.15) that are expressible in the form of (4.16) vanish. However, this does not mean that all terms in $S(n,m)$ vanish. As stated previously, (3.4) is a perturbation series involving both m and n. We should expect terms to appear that are dominated by n rather than only by m as implied by (4.15). We can see that a transition occurs in the expansion when i is equal to its upper limit in (4.13). For this value of i, a factor of $(-1)^{n+j}(2m+2)^{-(2n+2j)}$ now emerges, which has the effect of altering $x = \pi/2$ in (4.16) to $x = i\pi/4(m+1)$. In addition, the gamma function, viz. $\Gamma(2n+2j+2)$, appears in the coefficients, turning the series into an asymptotic series. As a consequence, the resulting series will not vanish in the same manner as (4.17). Such series, however, are formidable to evaluate. Since this series represents one of many correction terms and not the solution to $S(n,m)$, there seems to be little gain in making an effort to evaluate it, especially as we shall show how to evaluate $S(n,m)$ exactly in the following section.

We are also in a position to make a comparison with the situation when the Euler polynomials are replaced by the Bernoulli polynomials as in (3.5). The leading order term in this series, which we denote as $S^+(n,m)$, is given by

$$S_0^+(n,m) = (m+1) \sum_{j=0}^{\infty} \frac{c_{-2n,j}}{(2n+2j+1)} \Bigl(\frac{\pi}{2}\Bigr)^{2n+2j} .$$

The above can be expressed as an integral over the series expansion in (3.1). Consequently, it can be replaced by the trigonometric power on the lhs, thereby yielding

$$S_0^+(n,m) = \frac{2(m+1)}{\pi} \int_0^{\pi/2} \mathrm{d}t\, \sin^{2n} t = \frac{(m+1)}{\Gamma(n+1)}\,(1/2)_n\,. \tag{4.18}$$

In the above equation $(x)_n$ represents the Pochhammer notation of $\Gamma(x+n)/\Gamma(x)$. Unlike $S(n,m)$, however, there is now a contribution arising from the second highest order terms in the Bernoulli polynomials, viz. $B_{2k+1,2k}$, which yields a constant of $-1/2$ as we found for $S_0(n,m)$. Hence, we find that

$$S^+(n,m) = \frac{(m+1)}{\Gamma(n+1)}\,(1/2)_n - \frac{1}{2} + \frac{1}{12(m+1)} \sum_{j=0}^{\infty} c_{-2n,j} \Bigl(\frac{\pi}{2}\Bigr)^{2n+2j} + \cdots .$$

With the aid of the duplication formula for the gamma function, viz. 8.335(1) in [**9**], the first term on the rhs can also be written as $2^{-2n}(m+1)\binom{2n}{n}$. Once again, we see that the correction terms in descending powers of $m+1$ possess the same vanishing series over j as in (4.15). In fact, the correction terms for $S(n,m)$ and $S^+(n,m)$ only differ in the factors multiplying the series over j. Hence, the transitional behaviour observed for $S(n,m)$ will also apply to $S^+(n,m)$.

To complete this section, we now show that the leading term is always equal to $-1/2$, for $m \geqslant n$. That is, we are basically going to prove the first form of Theorem 2.1. The method we use will serve as the basis for the proof of the more intricate and fascinating form of the theorem appearing in the next section. If we express the cosine in (1.1) as the sum of two exponentials and introduce the binomial series expansion into the resulting equation, then after some algebra we obtain

$$S(n,m) = \frac{1}{2^{2n}} \sum_{\ell=0}^{2n} \binom{2n}{\ell} \sum_{k=1}^{m} (-1)^k e^{-ik(n-\ell)\pi/(m+1)} . \tag{4.19}$$

The summation over k is easily evaluated since it represents a finite geometric series. Separating the odd and even numbers in the summation over ℓ yields

$$S(n,m) = \frac{1}{2^{2n}} \left(\sum_{\ell=1,3,\ldots}^{2n-1} \binom{2n}{\ell} \left(\frac{(-1)^{m+\ell-n} - e^{i(\ell-n)\pi/(m+1)}}{1+e^{i(\ell-n)\pi/(m+1)}} \right) + \sum_{\ell=0,2,\ldots}^{2n} \binom{2n}{\ell} \left(\frac{(-1)^{m+\ell-n} - e^{i(\ell-n)\pi/(m+1)}}{1+e^{i(\ell-n)\pi/(m+1)}} \right) \right). \tag{4.20}$$

When $m-n$ is equal to an even number, the above reduces to

$$S(n,m) = -\frac{1}{2^{2n}} \sum_{\ell=1,3,\ldots}^{2n-1} \binom{2i}{\ell} + \frac{1}{2^{2n}} \sum_{\ell=0,2,\ldots}^{2n} \binom{2n}{\ell} \left(\frac{1-e^{i(\ell-n)\pi/(m+1)}}{1+e^{i(\ell-n)\pi/(m+1)}} \right). \tag{4.21}$$

The first sum on the rhs of (4.21) is again half the binomial series expansion of $(1+x)^{2n}$ with $x=1$ and thus, the entire term yields $-2^{2n-1}/2^{2n}$ or $-1/2$. The second term on the rhs vanishes because the summand for each value of l is cancelled by the summand corresponding to $2n-\ell$. It should also be noted that the denominator in (4.20) does not vanish at any stage since it has been assumed that $m \geqslant n$.

For the case where $m-n$ is odd, we adopt the same procedure, but in this instance we find that the first term on the rhs of (4.20) vanishes, while the second term equals $-1/2$. Therefore, for $m \geqslant n$, we can equate (3.4) to $-1/2$, which yields the following result:

$$\sum_{j=0}^{\infty} c_{-2n,j} \left(\frac{\pi}{2(m+1)} \right)^{2n+2j} E_{2n+2j}(m+1) = 1 .$$

The analysis from (4.19) to (4.21) can also be applied to $S^+(n,m)$. Then one finds that the odd values of ℓ yield the same value as the first term on the rhs of

(4.21), while the even values vanish except for the $\ell = n$ term, which needs to be isolated. Thus, for $n < 2m + 2$, we find that

$$S^{+}(n,m) = -\frac{1}{2} + \frac{(m+1)}{2^{2n}} \binom{2n}{n}.$$

which is not only identical to the leading term in (4.18), but is also another version of No. 4.4.2.12 in [**23**]. According to (3.5), this means for $n < 2m$ that

$$\sum_{j=0}^{\infty} c_{-2n,j} \left(\frac{\pi}{2m}\right)^{2j+2n} \frac{B_{2n+2j+1}(m)}{2n+2j+1} = -\frac{1}{2} + \frac{m}{2^{2n}} \binom{2n}{n}. \tag{4.22}$$

5. PROOF OF MAIN RESULT

In this section we prove the more intricate and fascinating case in Theorem 2.1, i.e., when $m < n$. We begin by putting $m = n - \ell$. This enables us to express (1.1) as

$$S(n, n-\ell) = \sum_{k=1}^{n-\ell} (-1)^k \cos^{2n}\left(\frac{k\pi}{2n-2\ell+2}\right), \tag{5.1}$$

where ℓ is a positive integer. We have already proved the $\ell \leqslant 0$ case or the first form in Theorem 2.1 at the end of the previous section by replacing the cosines with sums of exponentials and applying the binomial series expansion. Now, as a result of expressing $S(n,m)$ in the above form, we shall be able to apply the standard techniques of interchanging the order of the sums, introducing the geometric series formula and separating the resulting sum into odd and even numbers, which are often used in the evaluation of formulas for finite trigonometric sums, e.g. see Glasser and Cosgrove [**8**]. Thus, (5.1) becomes

$$S(n, n-\ell) = \frac{1}{2^{2n}} \sum_{k=1}^{n-\ell} (-1)^k \left(e^{ik\pi/(2n+2-2\ell)} + e^{-ik\pi/(2n+2-2\ell)}\right)^{2n}. \tag{5.2}$$

Expanding (5.2) via the binomial theorem, we obtain

$$S(n, n-\ell) = \frac{1}{2^{2n}} \sum_{k=1}^{n-\ell} (-1)^k \sum_{j=0}^{2n} \binom{2n}{j} e^{-i(n-j)k\pi/(n+1-\ell)}. \tag{5.3}$$

Remark 5.1. Interestingly, (5.3) can extended to situations involving non-integral powers, i.e., where n can be replaced by a complex number, ν. To handle this situation, the binomial series has been generalised in [**13**]. If we consider this extension, then we will no longer have a finite series as in (5.3), but an infinite series, resulting in a far more formidable problem. Moreover, the $\ell = n-1$ case reduces to $S(n,1) = -\cos^{2n}(\pi/4) = -1/2^n$, which can be used as a check on the results obtained in this section.

We now divide the series over j in (5.3) into three smaller series: (1) from $j=0$ to $\ell-1$, (2) from $j=\ell$ to $2n-\ell$, and (3) from $j=2n-\ell$ to $2n$. This step is crucial for observing an important property occurring in the second of these series. Therefore, (5.3) can be written as

$$(5.4)\qquad S(n,n-\ell)=\frac{1}{2^{2n}}\biggl(\sum_{k=1}^{n-\ell}\Bigl(\sum_{j=0}^{\ell-2}\binom{2n}{j}e^{-i(\ell-j-1)k\pi/(n+1-\ell)}+\sum_{j=2n-\ell+2}^{2n}\binom{2n}{j}\times e^{-i(\ell-j-1)k\pi/(n+1-\ell)}+\sum_{j=l}^{2n-\ell}\binom{2n}{j}e^{-i(\ell-j-1)k\pi/(n+1-\ell)}\biggr)+2^{1-2n}(n-\ell)\binom{2n}{\ell-1},$$

where the last term has been evaluated directly by putting $j=\ell-1$ and $j=2n-\ell+1$ into (5.4).

We shall denote the final series in (5.4) by $S_2(n,n-\ell)$, while the first two series are denoted by $S_1(n,n-\ell)$. The first two series will be referred to as the cosine terms because they can be combined as follows:

$$(5.5)\qquad S_1(n,n-\ell)=\frac{1}{2^{2n}}\biggl(\sum_{k=1}^{n-\ell}\Bigl(2\cos\Bigl(\frac{(\ell-1)k\pi}{n+1-\ell}\Bigr)+4n\cos\Bigl(\frac{(\ell-2)k\pi}{n+1-\ell}\Bigr)+\cdots+2\binom{2n}{\ell-2}\cos\Bigl(\frac{k\pi}{n+1-\ell}\Bigr)\Bigr)\biggr)+2^{1-2n}(n-\ell)\binom{2n}{\ell-1}.$$

Later, we express the cosines as exponentials in order to carry out the sums over k. The last term in (5.5) represents the dominant contribution to $S(n,n-\ell)$ apart from the leading term of $-1/2$, which will become clearer as we evaluate $S_2(n,n-\ell)$.

We now consider $S_2(n,n-\ell)$. After interchanging the j and k summations and evaluating the summation over k as a finite geometric series, we obtain

$$(5.6)\qquad S_2(n,n-\ell)=\frac{1}{2^{2n}}\sum_{j=l}^{2n-\ell}\binom{2n}{j}\biggl(\frac{e^{-i(\ell-j-1)\pi/(n+1-\ell)}+(-1)^{\ell+j}}{1-e^{-i(\ell-j-1)\pi/(n+1-\ell)}}\biggr).$$

For ℓ odd, (5.6) can be separated as follows:

$$(5.7)\qquad S_2(n,n-\ell)=\frac{1}{2^{2n}}\sum_{j=\ell,\,\ell+2,\,\ell+4,\ldots}^{2n-\ell}\binom{2n}{j}\biggl(\frac{e^{-i(\ell-j-1)\pi/(n+1-\ell)}+(-1)^{\ell+j}}{1-e^{-i(\ell-j-1)\pi/(n+1-\ell)}}\biggr)+\frac{1}{2^{2n}}\sum_{j=\ell+1,\,\ell+3,\,\ell+5,\ldots}^{2n-\ell-1}\binom{2n}{j}\biggl(\frac{e^{-i(\ell-j-1)\pi/(n+1-\ell)}+(-1)^{\ell+j}}{1-e^{-i(\ell-j-1)\pi/(n+1-\ell)}}\biggr).$$

Since ℓ has been assumed to be odd, the first series in the above result is over odd integers, while the second is over even integers. Therefore, for the first series

$(-1)^{\ell+j}$ is equal to unity, while for the second series it equals -1. Hence, (5.7) reduces to

$$S_2(n,n-\ell) = \frac{1}{2^{2n}} \sum_{j=\ell,\,\ell+2,\,\ell+4,\ldots}^{2n-\ell} \binom{2n}{j} \left(\frac{e^{-i(\ell-j-1)\pi/(n+1-\ell)}+1}{1-e^{-i(\ell-j-1)\pi/(n+1-\ell)}} \right) - \frac{1}{2^{2n}} \sum_{j=\ell+1,\,\ell+3,\,\ell+5,\ldots}^{2n-\ell-1} \binom{2n}{j}. \tag{5.8}$$

The final term in the above result is responsible for the leading term of $-1/2$ in $S(n,n-\ell)$.

We now show that for the most part the first series in (5.8) vanishes according to the lemma given below. However, before we can present the lemma, we require the following definition.

Definition 5.1. *An integer function f is said here to be* cyclic *over a range of integers ℓ if for a fixed integer n, $f(n-\ell) = f(\ell)$.*

Lemma 5.1. *If*

(1) *f is cyclic over the integers between ℓ and $n-\ell$, with n a fixed integer,*

(2) *$z(n-\ell) = \bar{z}(\ell)$, and*

(3) *$z\bar{z} = \alpha^2$, where α is arbitrary and real,*

then we find that for odd values of n,

$$\sum_{j=\ell}^{n-\ell} f(j) \left(\frac{\alpha \pm z}{\alpha \mp z} \right) = 0, \tag{5.9}$$

while for n even, we have

$$\sum_{j=\ell}^{n-\ell} f(j) \left(\frac{\alpha \pm z}{\alpha \mp z} \right) = f(n/2) \left(\frac{\alpha \pm z(n/2)}{\alpha \mp z(n/2)} \right). \tag{5.10}$$

Proof. Because of the cyclic nature of $f(j)$, we can split the series into two parts, the first part consisting of half the series in (5.9) and the second part with j replaced by $n-\ell$. For odd values of n, this means that

$$\sum_{j=\ell}^{n-\ell} f(j) \left(\frac{\alpha+z}{\alpha-z} \right) = \frac{1}{2} \left(\sum_{j=l}^{n-\ell} f(j) \left(\frac{\alpha+z}{\alpha-z} \right) + \sum_{j=\ell}^{n-\ell} f(j) \left(\frac{\alpha+\bar{z}}{\alpha-\bar{z}} \right) \right). \tag{5.11}$$

In obtaining this result we have used the second condition which states that $z(n-\ell) = \bar{z}(\ell)$. The third condition applies once the series in (5.11) are combined. After a little algebra one finds that the resultant series yields zero. For even values of n, we can split the sum as in (5.11) except for the $j = n/2$ term because in this case $z(n/2) \neq \bar{z}(n/2)$. Consequently, we need to evaluate the summand for $j = n/2$ separately, which is the result given on the rhs of (5.10). This completes the proof. □

It is also not necessary that consecutive integers only appear in the sums of (5.9) and (5.10). In fact, the above proof is valid provided the $j = n - \ell$ term appears with the $j = \ell$ term in the sums. Otherwise, it has to be evaluated separately. Thus, we could have considered $j = \ell + 1, \ell + 3, \ell + 5, \ldots$, which is the situation we have with the first series in (5.8). Furthermore, for our case n in Lemma 5.1 is replaced by $2n$. Then the $j = n$ term has to be isolated from the first sum in (5.8) as in the case of (5.10). This term, however, vanishes since it yields $\exp(i\pi) + 1$ in the numerator. For the other values in the first sum of (5.7), we note that $z = \exp(-i(\ell - j - 1)\pi/(n + 1 - \ell))$, which obeys the second condition in the lemma when j is replaced by $2n - j$. Therefore, (5.8) becomes

$$(5.12) \qquad S_2(n, n-\ell) = -\frac{1}{2^{2n}} \sum_{j=\ell+1,\,\ell+3\,,\ldots}^{2n-\ell-1} \binom{2n}{j} = -\frac{1}{2^{2n}} \left(\sum_{j=0,2,\ldots}^{2n} \binom{2n}{j} - 2 \sum_{j=0,2,\ldots}^{\ell-1} \binom{2n}{j} \right).$$

The first series in the last member is simply half the binomial series expansion of $(1 + x)^{2n}$ with $x = 1$. Hence, (5.12) simplifies to

$$(5.13) \qquad S_2(n, n-\ell) = -\frac{1}{2} + \frac{1}{2^{2n-1}} \sum_{j=0,2,4,\ldots}^{\ell-1} \binom{2n}{j}.$$

Let us turn our attention to the case when ℓ is even in (5.7). Again, $(-1)^{\ell+j}$ is unity for the first series in (5.7). Therefore, according to Lemma 5.1, it vanishes except for the $j = n$ term. The latter, however, only contributes when n is even, but then the numerator equals zero again. As a result, we are left with the same series appearing in the intermediate line of (5.12), except now $\ell + 1, \ell + 3, \ldots, 2n - \ell - 1$, are odd. Thus, we find that

$$(5.14) \qquad S_2(n, n-\ell) = -\frac{1}{2^{2n}} \sum_{j=\ell+1,\,\ell+3\,,\ldots}^{2n-\ell-1} \binom{2n}{j} = \frac{1}{2^{2n}} \left(- \sum_{j=1,3,\ldots}^{2n-1} \binom{2n}{j} + 2 \sum_{j=1,3,\ldots}^{\ell-1} \binom{2n}{j} \right).$$

The first series in the final member of (5.14) is once again equal to half the binomial theorem expansion of $(1 + x)^{2n}$ with $x = 1$. By combining (5.13) and (5.14), we arrive at

$$S_2(n,n-\ell) = -\frac{1}{2} + \frac{(1-(-1)^\ell)}{2^{2n}} \sum_{j=0,2,4,\ldots}^{\ell-1} \binom{2n}{j} + \frac{(1+(-1)^\ell)}{2^{2n}} \sum_{j=1,3,5,\ldots}^{\ell-1} \binom{2n}{j}. \tag{5.15}$$

As indicated earlier, there are cosine terms in (5.5) that also need to be evaluated before $S(n,n-\ell)$ can be evaluated. More formally, these are given by

$$S_c(n,n-\ell) = 2^{1-2n} \sum_{k=1}^{n-\ell} \sum_{j=0}^{\ell-2} \binom{2n}{j} \cos\Big(\frac{(\ell-j-1)k\pi}{n+1-\ell}\Big). \tag{5.16}$$

To evaluate these terms, we write them as

$$\begin{aligned} S_c(n,n-\ell) = \frac{1}{2^{2n}} \sum_{k=1}^{n-\ell} \Big(& e^{i(\ell-1)k\pi/(n+1-\ell)} + e^{-i(\ell-1)k\pi/(n+1-\ell)} \\ & + 2ne^{i(\ell-2)k\pi/(n+1-\ell)} + 2n\, e^{-i(\ell-2)k\pi/(n+1-\ell)} + \cdots \\ & + \binom{2n}{\ell-2} e^{ik\pi/(n+1-\ell)} + \binom{2n}{\ell-2} e^{-ik\pi/(n+1-\ell)} \Big). \end{aligned}$$

Each series represents a finite geometric series and thus, can be summed. It turns out that as a result of the Lemma 5.1, when ℓ is odd, the terms involving odd numbers of π cancel, while those with even numbers of π yield -2. The opposite applies when ℓ is even. This means that the cosine terms reduce to

$$\begin{aligned} S_c(n,n-\ell) = \frac{1}{2^{2n}} \Big(& \frac{(1-(-1)^\ell)}{2}\,(-2) \sum_{j=0,2,\ldots}^{\ell-2} \binom{2n}{j} \\ & + \frac{(1+(-1)^\ell)}{2}\,(-2) \sum_{j=1,3,\ldots}^{\ell-2} \binom{2n}{j} \Big). \end{aligned} \tag{5.17}$$

There is, however, a problem with the above result. It occurs whenever $k(\ell-j-1)/(n+1-\ell)$ is an even integer resulting in the appearance of simple poles when the summation over k is carried out. Hence, we need to isolate these terms in the cosine series so that they can be corrected. Since k ranges from 1 to $n-\ell$, there are at most $n-\ell$ values where $(\ell-j-1)k/(n+1-\ell)$ can yield an even integer. E.g., for $\ell = n-1$, $(n-j)/2-1$ is only even when $n-j$ is twice an odd number. For $n=6$ and $\ell=5$, though $k=1$, it means that the cosine terms decrement unitarily in multiples of $\pi/2$ from $\cos(4\pi/2)$ to $\cos(\pi/2)$. Therefore, the only term yielding unity in this sequence of cosines is the first term. This term, which yields a value of $2^{1-12} = 1/2048$, needs to be treated on its own.

To obtain $S_1(n,n-\ell)$, we must add the first term on the rhs of (5.5) to the above result. This yields

$$S_1(n,n-\ell) = \frac{1}{2^{2n}} \Big(2(n-\ell)\binom{2n}{\ell-1} - \big(1-(-1)^\ell\big) \sum_{j=0,2,\ldots}^{\ell-2} \binom{2n}{j} \tag{5.18}$$

$$- \big(1 + (-1)^{\ell}\big) \sum_{j=1,3,\ldots}^{\ell-2} \binom{2n}{j}\Bigg) + S_{\rm cor}(n-\ell, n)\,,$$

where $S_{\rm cor}(n, n-\ell)$ represents all the correction terms arising from the simple poles mentioned in the previous paragraph. Furthermore, by combining (5.15) with (5.18), we obtain $S(n, n-\ell)$, which is given by

$$S(n, n-\ell) = -\frac{1}{2} + \frac{(n-\ell+1)}{2^{2n-1}} \binom{2n}{\ell-1} + S_{\rm cor}(n-\ell, n)\,. \tag{5.19}$$

Table 5.1 presents values of $S(n, n-\ell)$ as given by (5.19) without the correction terms for the first eight values of ℓ. As we shall now see, the correction terms are dependent upon the explicit values of n and ℓ.

To obtain a result for the correction terms, let us consider $\ell = 5$ and $n = 6$ in $S(n, n-\ell)$. Then (5.16) reduces to

$$S_c(6,1) = 2^{-11} \sum_{k=1}^{1} \sum_{j=0}^{4} \binom{12}{j} \cos\Big(\frac{(4-j)k\pi}{2}\Big)\,. \tag{5.20}$$

Because the double series is finite, it can be evaluated quite easily. In fact, there are only two terms that make a contribution in the summation over j, viz. the $j = 0$ and $j = 2$ terms. Hence, we obtain

$$S_c(6,1) = 2^{-11}\Big(1 - \binom{12}{2}\Big)\,. \tag{5.21}$$

Now let us determine the result via (5.17), which is

$$S_c(1,6) = 2^{-12}(-2) \sum_{j=0}^{3} \binom{12}{j} = 2^{-11}\Big(-1 - \binom{12}{2}\Big)\,. \tag{5.22}$$

Comparing (5.22) with (5.21), we see that there is a discrepancy in the sign of the $j = 0$ term. This is due to interchanging the j and k summations in the process of obtaining (5.17), which yields -1, whereas the other way around it is equal to unity only.

As remarked earlier, this discrepancy only occurs when $(\ell-j-1)\pi/(n+1-\ell)$ is equal to an even number of multiples of π. As a consequence, the correction terms in (5.19) can be expressed as

$$S_{\rm cor}(n, n-\ell) = 2^{1-2n}(n-\ell)\Bigg(\big(1 - (-1)^{\ell}\big) \sum_{j=0,2,4,\ldots}^{\ell-2} \delta_{{\rm mod}(\ell-j-1),2(n+1-\ell),0} \binom{2n}{j} + \big(1 + (-1)^{\ell}\big) \sum_{j=1,3,\ldots}^{\ell-2} \delta_{{\rm mod}(\ell-j-1,2(n+1-\ell)),0} \binom{2n}{j}\Bigg), \tag{5.23}$$

ℓ	$S(n,n-\ell)$
1	$-\frac{1}{2}+2^{1-2n}\,n$
2	$-\frac{1}{2}+2^{2-2n}\,n(n-1)$
3	$-\frac{1}{2}+2^{1-2n}\,n(n-2)(2n-1)$
4	$-\frac{1}{2}+\left(\frac{1}{3}\right)2^{2-2n}\,n(n-1)(n-3)(2n-1)$
5	$-\frac{1}{2}+\left(\frac{1}{3}\right)2^{-2n}\,n(n-1)(n-4)(2n-1)(2n-3)$
6	$-\frac{1}{2}+\left(\frac{1}{15}\right)2^{1-2n}\,n(n-1)(n-2)(n-5)(2n-1)(2n-3)$
7	$-\frac{1}{2}+\left(\frac{1}{45}\right)2^{-2n}\,n(n-1)(n-2)(n-6)(2n-1)(2n-3)(2n-5)$
8	$-\frac{1}{2}+\left(\frac{1}{315}\right)2^{1-2n}\,n(n-1)(n-2)(n-3)(n-7)(2n-1)(2n-3)(2n-5)$

Table 5.1. The first eight values of $S(n,n-\ell)$ with the correction term $S_{\rm cor}(n,n-\ell)$ omitted

where, as previously mentioned, $\mathrm{mod}(a,b)$ is the modulus function yielding the remainder when a is divided by b. Finally, we arrive at

$$S(n,n-\ell)=-\frac{1}{2}+\frac{(n-\ell+1)}{2^{2n-1}}\binom{2n}{\ell-1} + \frac{(n-\ell)}{2^{2n-1}}\Bigg((1-(-1)^\ell)\sum_{j=0,2,4,\ldots}^{\ell-2}\delta_{\mathrm{mod}(\ell-j-1,2(n+1-\ell)),0}\binom{2n}{j} + (1+(-1)^\ell)\sum_{j=1,3,\ldots}^{\ell-2}\delta_{\mathrm{mod}(\ell-j-1,2(n+1-\ell)),0}\binom{2n}{j}\Bigg). \tag{5.24}$$

If we replace ℓ by $n-m$ in the above result, then we obtain the second form of (2.1). This completes the proof.

To conclude this section, we note that the correction term will only make a significant contribution as ℓ increases with respect to n. That is, for small values of ℓ, the correction term is negligible. In fact, for $\ell=1$ and 2, the correction term vanishes. Hence, the results in Table 5.1 represent the exact values of $S(n,n-1)$ and $S(n,n-2)$. For the other values of ℓ the correction term often vanishes, but even when it does not, the results in the table dominate or yield very accurate approximations. As a consequence, from (3.4) we find that

$$\sum_{j=0}^{\infty} c_{-2n,j}\Big(\frac{\pi}{2n}\Big)^{2j} E_{2n+2j}(n) = \Big(\frac{\pi}{2n}\Big)^{-2n}\big(1-2^{2-2n}n\big)\,, \tag{5.25}$$

and

$$\sum_{j=0}^{\infty} c_{-2n,j}\Big(\frac{\pi}{2(n-1)}\Big)^{2j} E_{2n+2j}(n-1) = \Big(\frac{\pi}{2(n-1)}\Big)^{-2n}\big(1-2^{3-2n}n(n-1)\big)\,. \tag{5.26}$$

From (5.24) we can also see why we were unable to obtain higher order terms other than the leading term for $S(n,m)$ in Section 4.3. The form for the series given by (3.3) yields inverse or descending powers of $m+1$. Since $m = n-\ell$, we see that according to (5.24), $S(n,m)$ is linear in $m+1$ multiplied by combinatorial factors. For example, the term just after the leading term possesses the binomial factor of $\binom{2n}{\ell-1}$ or $(2n)!/(n-m-1)!\,(n+m+1)!$. That is, m appears inside the factorials in the denominator. To expand these terms, we would have to consider two distinct cases: (1) $m+1 \approx n$, and (2) $n \gg m+1$. Both cases produce different expansions, but according to Section 4.3, there was no indication that the expansion was affected by the size of $m+1$ in relation to n. In addition, the remaining terms or $S_{\rm cor}$ in (5.24) are also linear in m multiplied by combinatorial factors. When the correction term does not vanish, it is different for each value of ℓ and n. That is, because of the arbitrariness in $S_{\rm cor}$, one must specify the actual values of ℓ and n before the correction term can be evaluated, which means that there is no need to expand. As a consequence of these points, it is a impractical to derive a power series expansion for $S(n,m)$, which explains why in Section 4.3 the symbol $+\ldots$ was ultimately introduced after the leading term instead of the Landau gauge symbols, $O()$ or $o()$.

6. EVALUATION

In summary, we have seen that the series $S(n,m)$ given by (1.1) or alternatively by (5.1) can be expressed as the sum of: (1) $S_1(n,n-\ell)$, which in turn can be expressed as a finite sum of cosine terms plus a combinatorial term as in (5.5), (2) $S_2(n,n-\ell)$ given by (5.15) and (3) the last term on the rhs of (5.4). The cosine terms in (5.5) were later denoted by $S_c(n,n-\ell)$ as in (5.16). Determining a closed form solution for them proved to be quite tricky as it involved the isolation of singularities in the sum, which vary according to the values of n and ℓ. We denoted all of them by $S_{\rm cor}(n,n-\ell)$ and were able eventually to derive a general result for them given by (5.23). With the aid of this result we were placed in a position where we could derive a general result for $S(n,m)$. This is given here as either (2.1) or (5.24) with m replaced by $n-\ell$.

This, however, is by no means the end of the story. Although the correction term given by (5.23) appears to be a formula, it is in reality an algorithm because of the indices in the Kronecker delta function. Since the correction term is different for each value of n and ℓ and cannot be expressed in higher order terms involving the Landau gauge symbols, in order to determine the values of $S(n, n-\ell)$, we need to create a computer program, but not an ordinary one at that. This is because from (5.24) it can be seen that the values of $S(n, n-\ell)$, like their counterparts for inverse powers of cosines [**14**], are rational. However, only floating point numbers will be generated if the result is programmed in a standard third generation language such as C/C++. Therefore, to obtain the rational values for $S(n, n-\ell)$, we require the integer arithmetic routines in a software package such as Mathematica [**26**].

It should be noted that the series $S(n, m)$ can be easily programmed in Mathematica via the following statement:

$$\mathrm{S[n\_, m\_] :=(-1)^\wedge(m + 1)\ Sum[(-1)^\wedge k\ Sin[Pi\ k/(2m + 2)]^\wedge(2n), \{k, 1, m\}]}\ ,$$

where Sum is the routine for evaluating sums in Mathematica. For relatively small values of n, the results are printed out in rational form. E.g., with the aid of the Simplify routine, the above statement prints out $S(6, 4) = -241/512$, but for large values such as $S(31, 22)$, one can only obtain a decimal number, viz. $S(31, 22) = -0.499\,999\,966\,274\,691\,631\,535\ldots$. Therefore, if we use the above statement to evaluate $S(n, m)$ for large values of n, then we are unable to tell whether we are dealing with rational numbers or not.

One could also program the cosine term $S_c(n, n-\ell)$ or (5.16) in a Mathematica module with $S_1(n, n-\ell)$ and $S_2(n, n-\ell)$, but once again, as n becomes large, only decimal values will be generated. Thus, our only means of obtaining rational or integer arithmetic values for $S(n, m)$ is to create a Mathematica module based on (5.24) or the result in Theorem 2.1. In order to accomplish this, the Kronecker delta function in S_{even} and S_{odd} must be replaced by a combination of the If and Mod (modulus) routines in Mathematica, while the sums become nested Do loops. The final version of the module, which aims to evaluate rational values of $S(n, m)$ and its component series, is presented in the appendix.

In this compact module there are two distinct calls invoking the Mod routine. The first, Mod[l,2], appears in two If statements in order to separate the even and odd parts of $S_c(n, n-\ell)$, $S_1(n, n-\ell)$ and $S_2(n, n-\ell)$. The second distinct call, Mod[(l-j-1),2*(n+1-l)], is designed to separate the terms in the correction term from the other non-singular terms in $S_1(n, n-\ell)$. In this instance there is no need to double the correction term as in Section 5 since it is no longer included in $S_1(n, n-\ell)$. The correction term denoted by ecor in the module is initialised to zero before it is calculated within the nested Do loops. When the module is run for $n = 100$ and $\ell = 87$ ($m = 13$), we obtain

$$S(100, 13) = -\frac{278201444165047686149439523134249722521781906841532727739503}{1004336277661868922213726307713226626576376871114245522063336}\ ,$$

while for $n = 120$ and now an even value of ℓ, say 104 ($m = 16$), the module gives

$$S(120, 16) = -\frac{472510442765908971202603436647285028330874830901470161946762434025979 1}{138034926935811275748695117245540509049022179443407731103250484475985 92}.$$

Both calculations took less than 0.01 sec on a Sony VAIO laptop with 2 GB RAM. The module also prints out the values of $S_c(n, n-\ell)$, $S_1(n, n-\ell)$, and $S_2(n, n-\ell)$. For the computationally inclined, a more extensive description of the code is presented in the appendix. As expected, both decimal forms of these results agree with the corresponding decimal values obtained via the first Mathematica statement given earlier for S[n_, m_]. Therefore, we have seen that a computer solution, not standard techniques in classical analysis, can only yield the rational values of $S(n, m)$, which is an indication of the mystery and beauty of this problem.

APPENDIX

Below we present the Mathematica module called Sf, which calculates the rational values of the series $S(n, m)$ as given by (1.1). The rational values arise from the fact that the program invokes the inherent or intrinsic integer arithmetic routines in Mathematica since floating point or decimal numbers are not introduced at any stage. To execute this program, one must type in the values of n and ℓ or $m - n$, which are represented by n and l in the module. The module begins by initialising the cosine term ecos to zero before encountering an If statement. Then it determines whether ℓ is even or not in the first Mod (modulus) statement. This is necessary so that only the odd or even part of $S_{\rm cor}$ in (5.23) is evaluated. The odd part is represented by the sum over even values of j in (5.23) or S_{even} in Theorem 2.1, while the even part is represented by the sum over odd values of j or S_{odd} in Theorem 2.1. If ℓ is odd, then the first Do loop is executed. Otherwise the second Do loop is executed. In both Do loops there is another If statement involving Mod[l-j-1,2*(n+1-l)]. The combination of the If and Mod statements replaces the discrete or Kronecker delta function in (5.23). Whenever the second If statement is true, the correction term ecor in the appropriate series over j in (5.23) is calculated and added to the current value of the cosine term or ecos. Once the Do loop is completed, the final value of the cosine term or $S_c(n, m)$ as given by (5.16) is printed out. The program proceeds to calculate the rational value of $S_1(n, m)$ by adding the final term on the rhs of (5.5) denoted by e1 to the final value of ecos, thereby yielding a value for the variable s1. This, too, is printed out by the module. Next the value of $S_2(n, n-\ell)$ as given by (5.15) is evaluated. Because this quantity is composed of even and odd parts represented by the two series on the rhs of (5.15), another If statement appears. This statement also invokes Do loops according to whether l is even or not. The Do loops sum the various terms in the series, denoted by e3, to the variable e2, which has been initialised to equal the leading term of -1/2. The final value of e2 represents the value of $S_2(n, m)$ and is also printed out. The program concludes by printing out the value of $S(n, m)$, which is calculated simply by adding e2 to s1.

```
Sf[n_,l_] := Module[{},
   ecos = 0;
   If[Mod[l,2]} != 0,
     Do[
       If[Mod[(l-j-1),2*(n+1-l)]!=0,ecor=-2^(1-2n)Binomial[2n,j],
```

```
            ecor = 2^(1-2n)(n-l)Binomial[2n,j]];
      ecos = ecor + ecos,{j,0,l-2,2}],
    Do[
      If[Mod[(l-j-1),2*(n+1-l)]!=0,ecor=-2^(1-2n)Binomial[2n,j],
            ecor = 2^(1-2n)(n-l)Binomial[2n,j]];
      ecos = ecor + ecos,{j,1,l-2,2}]
    ];
  Print["The cosine terms equal ", ecos];
  e1 = 2^(1-2n) (n-l) Binomial[2n,l-1];
  s1 = ecos+e1;
  Print["The value of S_1[", n, ",",n-l, "] is ",s1];
  e2=-1/2;
  If[Mod[l,2] == 0,
    Do[
       e3 = 2^(1-2n)Binomial[2n,j];
       e2 = e3 + e2,{j,1,l-1,2}],
    Do[
       e3 = 2^(1-2n)Binomial[2n,j];
       e2 = e3 + e2,{j,0,l-1,2}]
    ];
  Print["The value of S_2[", n,",", n-l, "] is ",e2];
  Print["The value of S[", n, ",", n-l,"] is ",s1+e2]
]
```

If we set n and ℓ equal to 105 and 90 respectively, and enter the command Timing[Sf[105,90]], then Mathematica produces the following output:

The cosine terms equal

$$-\frac{474809601546190988190234236565175704861400930424752729492543}{514220174162876888173427869549172032807104958010493707296440 32}$$

The value of $S_1[105,15]$ is

$$\frac{694993284530530564121409412366123992977138384670662682421870 7}{5142201741628768881734278695491720328071049580104937072964403 2}$$

The value of $S_2[105,15]$ is

$$-\frac{2474121626379286915475213315687887697393673068180556171964219 5}{5142201741628768881734278695491720328071049580104937072964403 2}$$

The value of S[105,15] is

$$-\frac{4343575053341690310922372810844149668985680379662825902203}{1255420347077336152767157884641533283220471088892806902579 2}$$

{0.00400, Null}

From this output we see that the code takes only 0.004 seconds to compute the various series in the module on a Sony VAIO laptop with 2GB RAM. By introducing the last value into the Mathematica statement N[ ], we obtain the decimal value of $S(105, 15)$, which equals -0.345985714.... On the other hand, if we use the more compact Mathematica statement for $S(n, m)$ given in Section 6, then we obtain the following output:

$$1/4056481920730334084789450257203\!2 - \mathrm{Cos}[\pi/32]^{210} + \mathrm{Cos}[\pi/16]^{210} - \mathrm{Cos}[3\pi/32]^{210} + \mathrm{Cos}[\pi/8]^{210} - \mathrm{Cos}[5\pi/32]^{210} + \mathrm{Cos}[3\pi/16]^{210} - \mathrm{Cos}[7\pi/32]^{210} - \mathrm{Sin}[\pi/32]^{210} + \mathrm{Sin}[\pi/16]^{210} - \mathrm{Sin}[3\pi/32]^{210} + \mathrm{Sin}[\pi/8]^{210} - \mathrm{Sin}[5\pi/32]^{210} + \mathrm{Sin}[3\pi/16]^{210} - \mathrm{Sin}[7\pi/32]^{210}.$$

This result is of little benefit since we cannot even tell if the value is rational or not. Nevertheless, we can obtain a decimal value by enclosing it again in N[ ]. Then we find that the resulting value is identical to the decimal value of $S(105, 15)$ given above.

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