

COINCIDENCE PROBLEMS FOR GENERALIZED CONTRACTIONS

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In this paper, we establish some new existence, uniqueness and Ulam-Hyers stability theorems for coincidence problems for two single-valued mappings. The main results of this paper extend the results presented in O. MLEŞNIŞTE: *Existence and Ulam-Hyers stability results for coincidence problems*, J. Non-linear Sci. Appl., **6** (2013), 108–116. In the last section two examples of application of these results are also given.

1. INTRODUCTION

Fixed point theory is one of the traditional theories in Mathematics and has a broad set of applications. In the existing literature on this theory, contractive conditions on the mappings play a vital role in proving the existence and uniqueness of a fixed point. Banach's contraction principle is one of the most widely used fixed point theorem in all of Analysis. Due to its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis and it has many applications in solving nonlinear equations. Generalizations of this principle have been obtained in several directions. The theory of generalized contractions has been, in the last years, subject to an intensive development (see [**3**, **8**, **12**, **13**]).

On one hand, let us consider the following coincidence problem: let X, Y be two nonempty sets and $s, t : X \rightarrow Y$ be two mappings.

$$(1) \quad \text{find } x \in X \text{ such that } s(x) = t(x).$$

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It is well known that the coincidence problem (1) is, under appropriate conditions, equivalent to a fixed point problem. In this sense, in 1968 K. GOEBEL [6], using the Banach contraction principle, proved the following nice result: Let X be an arbitrary nonempty set and let (Y, ρ) be a metric space. Suppose that $s, t : X \rightarrow Y$ are two mappings such that $t(X)$ is complete, $s(X) \subseteq t(X)$ and $\rho(s(x), s(y)) \leq k\rho(t(x), t(y))$, for some constant $k \in [0, 1)$ and for all $x, y \in X$, then problem (1) has a solution. This result allowed the author to give conditions for existence of solutions of the differential equation $x'(t) = f(t, x(t))$. For existence results on the coincidence problem, for instance, see [16].

On the other hand, the stability problem of functional equations originated from a question of ULAM [17], in 1940, concerning the stability of group homomorphisms. In the following year, HYERS [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Thereafter, this type of stability is called the Ulam-Hyers stability. For Ulam-Hyers stability results in the case of fixed point problems and coincidence point problems see [1, 9, 12, 14, 15, 16].

Recently, O. MLEŞNIȚE in [11] obtained several results on the existence of solutions for the coincidence problem (1) using the fixed point technique and she obtained as well as the Ulam-Hyers stability for such coincidence problem. In this note, we study the existence, uniqueness and Ulam-Hyers stability for the coincidence problem and thus, we may extend most of the results given in [11], our techniques also allow us to give a generalization of [18, Theorem 2.1]. Finally, we apply such results to solve a nonlinear integral equation and a differential equation.

2. PRELIMINARIES

We start this paper with some notations and introductory notions.

Let (X, d) , (Y, ρ) be two metric spaces and let $f : X \rightarrow Y$ be a mapping.

- (a) f is called a contraction if there exists a constant $k \in [0, 1)$ such that

$$\rho(f(x), f(y)) \leq k \cdot d(x, y), \text{ for each } x, y \in X.$$

If $k = 1$, then f is called nonexpansive.

- (b) f is a dilatation if there exists a constant $h > 1$ such that

$$\rho(f(x), f(y)) \geq h \cdot d(x, y), \text{ for each } x, y \in X.$$

If $h = 1$, then f is said to be expansive.

Let Z be a nonempty set and $f : Z \rightarrow Z$ a mapping. We denote by

$$\text{Fix}(f) := \{z \in Z \mid f(z) = z\}$$

the fixed point set of the mapping f .

Let (X, d) , (Y, ρ) be two metric spaces and $s, t : X \rightarrow Y$ be two mappings. Let us consider the coincidence problem (1).

Definition 2.1 ([15]). *The coincidence problem (1) is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each solution $w^* \in X$ of the approximative coincidence problem*

$$(2) \quad \rho(s(w^*), t(w^*)) \leq \varepsilon$$

there exists a solution z^ of (1) such that*

$$(3) \quad d(w^*, z^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) = ct$ for each $t \in \mathbb{R}_+$ then the coincidence problem (1) is said to be Ulam-Hyers stable.

Definition 2.2 ([13]). *A mapping $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be a comparison function if it is increasing and the iterate $\phi^k(t) \rightarrow 0$, as $k \rightarrow +\infty$.*

REMARK 2.3. On one hand, if $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a comparison function, then ϕ is increasing, $\phi(t) < t$, for each $t > 0$, $\phi(0) = 0$ and ϕ is continuous at 0. On the other hand, if $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous increasing function such that $\phi(t) < t$, for each $t > 0$, $\phi(0) = 0$, then ϕ is a comparison function, see [8, Chapter 1].

Definition 2.4. *Let (X, d) , (Y, ρ) be two metric spaces. A mapping $f : X \rightarrow Y$ is a ϕ -contraction if $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a comparison function and*

$$\rho(f(x), f(y)) \leq \phi(d(x, y)), \text{ for all } x, y \in X.$$

Definition 2.5. *Let (X, d) , (Y, ρ) be two metric spaces. A mapping $f : X \rightarrow Y$ is said to be a separate contraction if there exist two functions $\varphi, \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying*

- (i) $\psi(0) = 0$, ψ is strictly increasing;
- (ii) $\rho(f(x), f(y)) \leq \varphi(d(x, y))$;
- (iii) $\psi(r) \leq r - \varphi(r)$ for $r > 0$.

The following results will be the key in the proof of our results. The first one is known in the literature as MATKOWSKI-RUS's theorem [13]. The second one can be found in [10]. At this point, we refer to [8, Chapter 1], where KIRK gives an overview of the sharpening of the Banach contraction principle.

Theorem 2.6. *Let (X, d) be a complete metric space and $f : X \rightarrow X$ a ϕ -contraction. Then f is a Picard operator, i.e., f has a unique fixed point $x^* \in X$ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$, for all $x \in X$.*

Theorem 2.7. *Let (X, d) be a complete metric space. Suppose $f : X \rightarrow X$ is a separate contraction. Then f has a unique fixed point in X .*

3. COINCIDENCE PROBLEM FOR TWO SINGLE-VALUED MAPPINGS

3.1. PREPARATORY RESULTS

First, let us give an example of a mapping which is ϕ -contraction, but is not a contraction. In [2, 10] the reader will find more sophisticated examples of separate contractions.

EXAMPLE 3.8. Let us consider $f : [0, \frac{1}{\sqrt{2}}] \rightarrow [0, \frac{1}{\sqrt{2}}]$ defined by $f(x) = x - x^3$ and we consider a metric d ($d(x, y) \in \mathbb{R}_+$, $d(x, y) = |x - y|$). In [4] it was proved that f is not a contraction but it is a separated contraction, taking $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by

$$\varphi(r) = \begin{cases} r \left(1 - \frac{r^2}{4}\right), & \text{if } r \leq 1, \\ \frac{3}{4}r, & \text{if } r > 1, \end{cases}$$

and $\psi(r) = r - \varphi(r)$. We can observe that φ previously defined satisfies the following properties: φ is continuous on $[0, +\infty)$, $\varphi(0) = 0$ and $\varphi(r) < r$, for all $r \in [0, +\infty)$. Remark 2.3 shows that φ is a comparison function. Therefore, f is a ϕ -contraction.

Following the arguments given in [5, Proposition 3.1] we obtain:

Lemma 3.9. *Let (X, d) and (Y, ρ) be two complete metric spaces. Let $f : X \rightarrow Y$ be an injective and continuous mapping such that $f^{-1} : f(X) \rightarrow X$ is uniformly continuous. Then $f(X)$ is a closed subset of Y .*

Proof. Let (x_n) be a sequence of elements of $f(X)$ such that (x_n) converges to x_0 . We have to prove that $x_0 \in f(X)$. Indeed, since (x_n) is a Cauchy sequence and f^{-1} is uniformly continuous it is easy to see that $(f^{-1}(x_n))$ is also a Cauchy sequence. Hence we may assume that $(f^{-1}(x_n))$ converges to $y_0 \in X$.

Finally, by using that f is a continuous mapping we conclude that (x_n) converges to $f(y_0)$ which means that $x_0 = f(y_0) \in f(X)$. \square

Next, we are going to show that if in Lemma 3.9 the hypothesis f is continuous is removed, then the result does not hold.

EXAMPLE 3.10. Define the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as follows:

$$f(x) = \begin{cases} 2x, & \text{if } x < 1, \\ e^x, & \text{if } x \geq 1. \end{cases}$$

It is clear that f is not continuous at $x = 1$. Moreover, $|f(x) - f(y)| \geq |x - y|$ for every $x, y \in \mathbb{R}^+$ which yields that $f^{-1} : f(\mathbb{R}^+) \rightarrow \mathbb{R}^+$ is uniformly continuous. Nevertheless, $2 \in f(\mathbb{R}^+) \setminus f(\mathbb{R}^+)$ and this implies that $f(\mathbb{R}^+)$ is not a closed subset.

The following lemma is a sharpening of the main result in [6].

Lemma 3.11. *Let $X \neq \emptyset$ be an arbitrary set and let (Y, ρ) be a metric space. Suppose that:*

1. (Y, ρ) is complete,
2. $t : X \rightarrow Y$ is an injection with $t(X)$ closed in Y ,
3. $s : X \rightarrow Y$ is a mapping such that $s(X) \subseteq t(X)$,
4. $s \circ t^{-1} : t(X) \rightarrow t(X)$ is a ϕ -contraction.

Then the coincidence problem (1) has a unique solution.

Proof. Since $t(X)$ is a closed subset of Y , then $(t(X), \rho)$ is complete metric space.

Moreover, by assumption (4), the mapping $h := s \circ t^{-1} : t(X) \rightarrow t(X)$ is a ϕ -contraction. Thus, Theorem 2.6 guarantees that there exists a unique $z^* \in t(X)$ such that $z^* = h(z^*)$, i.e. there exists a unique $x_0 \in X$ such that $t(x_0) = z^*$ satisfying

$$t(x_0) = s(t^{-1}(t(x_0))) = s(x_0),$$

therefore x_0 is the unique solution of the coincidence problem.

3.2. THE MAIN RESULTS

In this section we present the main results of this paper which extend the previous ones (for instance see [11, Theorems 1.6, 1.8, 1.11, 1.13]). In this sense, our first main result is the following theorem which guarantees the existence of a unique solution of the coincidence problem (1).

Theorem 3.12. *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose that:*

- (i) $t : X \rightarrow Y$ is an expansive mapping,
- (ii) the mapping $s : X \rightarrow Y$ is a ϕ -contraction,
- (iii) $s(X) \subseteq t(X)$.

Then the coincidence problem (1) has a unique solution.

Proof. From (i) we deduce that $t : X \rightarrow Y$ is an injection. Thus, t admits an inverse $t^{-1} : t(X) \rightarrow X$. It is not difficult to show that t^{-1} is a nonexpansive mapping, i.e.,

$$d(t^{-1}(y_1), t^{-1}(y_2)) \leq \rho(y_1, y_2), \text{ for each } y_1, y_2 \in t(X).$$

Therefore t^{-1} is uniformly continuous.

Now, we are going to distinguish two cases:

Case 1. Suppose that $t : X \rightarrow Y$ is continuous. In this case, by Lemma 3.9 we infer that $t(X)$ is a closed subset of Y . Hence $(t(X), \rho)$ is a complete metric space.

Since $s(X) \subseteq t(X)$, we may introduce the mapping $h : t(X) \rightarrow t(X)$ defined by

$$h(y) := (s \circ t^{-1})(y).$$

Let us see that h is a ϕ -contraction. Indeed, since s is a ϕ -contraction, we have

$$(4) \quad \rho(h(y), h(z)) = \rho(s(t^{-1}(y)), s(t^{-1}(z))) \leq \phi(d(t^{-1}(y), t^{-1}(z))).$$

Now, since ϕ is increasing and t^{-1} is nonexpansive, from equation (4) we obtain:

$$\rho(h(y), h(z)) \leq \phi(\rho(y, z)),$$

which means that h is a ϕ -contraction. Then, by Lemma 3.11 we obtain the conclusion.

Case 2. Suppose that t is not a continuous mapping. In this case, we consider the complete metric space $(\overline{t(X)}, \rho)$. Since $t^{-1} : t(X) \rightarrow X$ is a uniformly continuous mapping, we may define:

$$(5) \quad \tilde{t}^{-1}(y) := \begin{cases} t^{-1}(y), & \text{if } y \in t(X); \\ \lim_{n \rightarrow \infty} t^{-1}(x_n) \text{ where } (x_n) \subseteq t(X) \text{ such that } x_n \rightarrow y, & \text{if } y \in \overline{t(X)}. \end{cases}$$

It is easy to see that $\tilde{t}^{-1} : \overline{t(X)} \rightarrow X$ is also a nonexpansive mapping.

As in the case 1, we can prove that the mapping $h : \overline{t(X)} \rightarrow \overline{t(X)}$ defined by $h(y) := (s \circ \tilde{t}^{-1})(y)$ is a ϕ -contraction and by Theorem 2.6 h has a unique fixed point, say $x_0 \in \overline{t(X)}$. Let us see that $x_0 \in t(X)$. Indeed, since $x_0 = h(x_0)$, by assumption (iii) we infer that $x_0 = s(\tilde{t}^{-1}(x_0)) \in s(X) \subseteq t(X)$, which means that the coincidence problem has a unique solution. \square

Regarding the Ulam-Hyers stability problem the ideas given in [12, Theorem 2.3] allow us to obtain the second main result.

Theorem 3.13. *Let (X, d) , (Y, ρ) be two complete metric spaces. Suppose that all the hypotheses of Theorem 3.12 hold and additionally that the function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\beta(r) := r - \phi(r)$ is strictly increasing and onto. Then the coincidence problem (1) is generalized Ulam-Hyers stable.*

Proof. Let h be the mapping defined in the proof of Theorem 3.12.

On one hand, let $\varepsilon > 0$ and $w^* \in X$ be a solution of (2), i.e., $\rho(s(w^*), t(w^*)) \leq \varepsilon$. Taking $u^* := t(w^*)$, we infer that $h(u^*) = s(w^*)$ and moreover, $\rho(u^*, h(u^*)) \leq \varepsilon$.

On the other hand, let $w \in X$ be the unique solution of the problem (1), then if we call $u = t(w)$ we obtain that $Fix(h) = \{u\}$, because h is a Picard operator. Consequently,

$$\rho(u, u^*) \leq \rho(h(u), h(u^*)) + \rho(u^*, h(u^*)) \leq \phi(\rho(u, u^*)) + \rho(u^*, h(u^*)).$$

Therefore,

$$(6) \quad \beta(\rho(u, u^*)) := \rho(u, u^*) - \phi(\rho(u, u^*)) \leq \rho(u^*, h(u^*)).$$

Since t is an expansive mapping, equation (6) yields

$$d(w^*, w) \leq \rho(t(w^*), t(w)) = \rho(u^*, u) \leq \beta^{-1}(\varepsilon).$$

Consequently, the coincidence problem (1) is β^{-1} -generalized Ulam-Hyers stable. \square

Bearing in mind that if $t : X \rightarrow Y$ is a dilatation, then t is an expansive mapping. As a consequence of Theorem 3.12 and Theorem 3.13, we obtain the following result.

Corollary 3.14. *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose both that the mapping $t : X \rightarrow Y$ is a dilatation and $s : X \rightarrow Y$ is a ϕ -contraction with $s(X) \subseteq t(X)$. Then*

1. *the coincidence problem (1) has a unique solution.*
2. *If in addition, the function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\beta(r) := r - \phi(r)$ is strictly increasing and onto, then the coincidence problem (1) is generalized Ulam-Hyers stable.*

Another consequence of Theorem 3.12 and Theorem 3.13 is the following result that generalizes to Goebel's Theorem, see [6].

Corollary 3.15. *Let X be a nonempty set and let (Y, ρ) be a metric space. Suppose that the mapping $t : X \rightarrow Y$ is an injection, $s(X) \subseteq t(X)$, $t(X)$ is a complete subspace of Y and there exists $k \in [0, 1)$ such that $\rho(s(x), s(y)) \leq k\rho(t(x), t(y))$. Then*

1. *the coincidence problem (1) has a unique solution.*
2. *the coincidence problem (1) is generalized Ulam-Hyers stable.*

Proof. Since $t : X \rightarrow Y$ is an injection, it is clear that $d(x, y) := \rho(t(x), t(y))$ is a metric on X . Moreover, since $(t(X), \rho)$ is complete, it is easy to see that (X, d) is also complete.

By definition of d we infer that t is an isometry (and thus, it is an expansive mapping). Moreover, since $\rho(s(x), s(y)) \leq k\rho(t(x), t(y)) = kd(x, y)$, we obtain that s is a contraction.

Therefore we have:

- (a) (X, d) and $(t(X), \rho)$ are complete metric spaces,
- (b) $t : X \rightarrow t(X)$, is an expansive mapping,
- (c) $s : X \rightarrow t(X)$ is a contraction.

Thus, we achieve the result applying Theorem 3.12 and Theorem 3.13.

Theorem 3.16. *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose that:*

(i) *$t : X \rightarrow Y$ is a mapping such that there exists a function $\phi_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, continuous, increasing, $\phi_1(r) > r$ and $\phi_1(0) = 0$ and satisfying the following relation:*

$$\rho(t(y), t(z)) \geq \phi_1(d(y, z)), \quad \text{for all } y, z \in X,$$

- (ii) the mapping $s : X \rightarrow Y$ is lipschizian with constant $k_s > 0$,
- (iii) $s(X) \subseteq t(X)$,
- (iv) $r < \phi_1\left(\frac{r}{k_s}\right)$.

Then the coincidence problem (1) has a unique solution.

Proof. Following the arguments developed in the proof of Theorem 3.12, it is clear that $\tilde{t}^{-1} : \overline{t(X)} \rightarrow X$ defined like in equation (5) is a ϕ_1^{-1} -contraction.

If now, we define the mapping $h : \overline{t(X)} \rightarrow \overline{t(X)}$ defined by $h(y) := (s \circ \tilde{t}^{-1})(y)$, we obtain,

$$\rho(h(y), h(z)) = \rho\left(s(\tilde{t}^{-1}(y)), s(\tilde{t}^{-1}(z))\right) \leq k_s \left(d(\tilde{t}^{-1}(y), \tilde{t}^{-1}(z))\right) \leq k_s \phi_1^{-1}(\rho(y, z)).$$

Taking $\psi(r) = k_s \phi_1^{-1}(r)$. Clearly, we have that ψ is continuous, increasing, $\psi(0) = 0$ and by assumption (iv), $\psi(r) < r$. This yields that h is ψ -contraction. Consequently, by Theorem 2.6 h has a unique fixed point, say $x_0 \in \overline{t(X)}$. Let us see that $x_0 \in t(X)$. Indeed, since $x_0 = h(x_0)$ by assumption (iii) we infer that $x_0 = s(\tilde{t}^{-1}(x_0)) \in t(X)$, which means that the coincidence problem has a unique solution. \square

Next result is a generalization of [18, Theorem 2.1].

Corollary 3.17. *Let (X, d) be a complete metric space and let $t : X \rightarrow X$ be an onto mapping satisfying condition (i) of Theorem 3.16. Then t has a unique fixed point.*

Proof. Define $s : X \rightarrow X$ by $s(x) = x$. Then s and t fulfill the hypotheses of Theorem 3.16. Thus there exists a unique $x_0 \in X$ such that $t(x_0) = s(x_0)$, which means that x_0 is the unique fixed point of t . \square

Using the same tools that in the proof of Theorem 3.13 we obtain the following result regarding to the Ulam-Hyers stability.

Theorem 3.18. *Let $(X, d), (Y, \rho)$ be two complete metric spaces. Suppose that all the hypotheses of Theorem 3.16 hold and additionally that the function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\beta(r) := r - k_s \phi_1^{-1}(r)$ is strictly increasing and onto. Then the coincidence problem (1) is generalized Ulam-Hyers stable.*

As a consequence of Theorem 3.16 and Theorem 3.18 we get the following result.

Corollary 3.19. *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose both that the mapping $t : X \rightarrow Y$ is a dilatation with constant $k_t > 1$ and $s : X \rightarrow Y$ is a Lipschitz mapping with constant $k_s > 0$ such that $s(X) \subseteq t(X)$ and $k_s < k_t$. Then*

1. the coincidence problem (1) has a unique solution,
2. the coincidence problem (1) is Ulam-Hyers stable.

Proof. In this case, $\phi_1(r) = k_t \cdot r$. Therefore $\phi_1^{-1}(r) = \frac{1}{k_t} \cdot r$, hence $\psi(r) =: k_s \phi_1^{-1}(r) = \frac{k_s}{k_t} \cdot r$. Which yields that the function $\beta(r) = r - \psi(r) = \left(1 - \frac{k_s}{k_t}\right)r$ is increasing and onto. \square

Next, we give an example of a non dilatation function which satisfies condition (i) of Theorem 3.16.

EXAMPLE 3.20. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the function defined by $f(x) = e^x - 1$ and we consider in \mathbb{R}_+ the usual metric d , that is, $d(x, y) := |x - y|$ for any $x, y \in \mathbb{R}_+$.

Let us see that f is a non dilatation function. Otherwise, there exists $h > 1$ such that $|f(x) - f(y)| \geq h|x - y|$ for every $x, y \in \mathbb{R}_+$. However, since $h > 1$ there exists $y_0 > 0$ such that $e^{y_0} < h$. Therefore, given $x, y \in [0, y_0]$ there is $\xi \in]x, y[$ such that

$$|f(x) - f(y)| = |e^x - 1 - e^y + 1| = |e^x - e^y| = |e^\xi(x - y)| < h|x - y|.$$

This is a contradiction.

Finally, let us show that f satisfies condition (i) in Theorem 3.12. Let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the function defined by $\phi(r) = e^r - 1$, for all $r \in \mathbb{R}_+$.

- $\phi(r) > r$ for all $r \geq 0$ (because $e^r = \sum_{k=0}^{\infty} \frac{r^k}{k!}$),
- $\phi(0) = 0$,
- $|f(x) - f(y)| = |e^x - 1 - e^y + 1| = |e^x - e^y| = e^x |1 - e^{y-x}| = e^x (e^{|y-x|} - 1) \geq e^{|x-y|} - 1 \implies d(f(x), f(y)) \geq \phi(d(x, y))$.

Theorem 3.21. *Let (X, d) and (Y, ρ) be two complete metric spaces. Suppose that the mapping $t : X \rightarrow Y$ is expansive and the mapping $s : X \rightarrow Y$ is a separate contraction (i.e. $\exists \varphi, \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies the conditions of Definition 2.5) and $s(X) \subseteq t(X)$. Then*

- (i) *If φ is nondecreasing, the coincidence problem (1) has a unique solution.*
- (ii) *If ψ is onto, the coincidence problem (1) is generalized Ulam-Hyers stable.*

Proof. Consider the complete metric space: $(\overline{t(X)}, \rho)$. Since $t^{-1} : t(X) \rightarrow X$ is a nonexpansive mapping, it is a uniformly continuous mapping. Therefore, it is easy to see that $\tilde{t}^{-1} : \overline{t(X)} \rightarrow X$ defined in (5) is also a nonexpansive mapping.

Let $h : \overline{t(X)} \rightarrow \overline{t(X)}$ be the mapping defined by $h(u) = s(\tilde{t}^{-1}(u))$. In order to show the existence result, as in the proof of Theorem 3.12, it will be enough to see that h has a fixed point. In this sense, it is easy to see that

$$(7) \quad \rho(h(u), h(v)) \leq \varphi\left(d(\tilde{t}^{-1}(u), \tilde{t}^{-1}(v))\right),$$

since t^{-1} is nonexpansive and by (i), φ is nondecreasing, from inequality (7) we obtain:

$$\rho(h(u), h(v)) \leq \varphi(\rho(u, v)).$$

The above inequality shows that h is a separate contraction and then by Theorem 2.7, h has a unique fixed point.

Finally, in order to prove that the coincidence problem is generalized Ulam-Hyers stable, we argue as in the proof of Theorem 3.13. If u, u^*, w and w^* are as in such proof, we have:

$$\rho(u, u^*) \leq \rho(h(u), h(u^*)) + \rho(u^*, h(u^*)) \leq \varphi(\rho(u, u^*)) + \rho(u^*, h(u^*)).$$

Therefore,

$$\psi(d(u, u^*)) \leq \rho(u, u^*) - \varphi(\rho(u, u^*)) \leq \rho(u^*, h(u^*)),$$

Consequently,

$$d(w^*, w) \leq \rho(t(w^*), t(w)) = \rho(u, u^*) \leq \psi^{-1}(\rho(u^*, h(u^*))).$$

4. APPLICATIONS

In this section we will show two examples of applications of our results to an integral equation and a differential equation of second order with a non homogeneous Dirichlet condition.

4.1. AN APPLICATION TO INTEGRAL EQUATIONS

We will present now an application of Theorem 3.16 and Theorem 3.18.

Let us consider the following equation:

$$(8) \quad \int_0^t f(s, u(s)) ds + 2 - e^{u(t)} = 0,$$

where $f : [0, 1] \times \mathbb{R} \rightarrow [-1, +\infty)$ satisfies the following conditions:

- (i) $f(\cdot, \cdot)$ is a continuous function on $[0, 1] \times \mathbb{R}$,
- (ii) f is nonexpansive in the second variable, that is,

$$|f(t, x) - f(t, y)| \leq |x - y|, \text{ for all } t \in [0, 1] \text{ and } x, y \in \mathbb{R}.$$

Then the equation (8) has a unique solution in $C^+([0, 1])$, where

$$C^+([0, 1]) := \{u : [0, 1] \rightarrow \mathbb{R} \text{ continuous and } u(t) \geq 0 \text{ for all } t \in [0, 1]\},$$

and moreover it is Ulam-Hyers stable.

We can write equation (8) in the following form:

$$e^{u(t)} - 1 = \int_0^t f(s, u(s)) ds + 1.$$

We define the mappings $T, S : C^+([0, 1]) \rightarrow C^+([0, 1])$ by

$$Tu(t) = e^{u(t)} - 1, \text{ and } Su(t) = \int_0^t f(s, u(s)) ds + 1$$

and we denote by $\|\cdot\|_\infty$ a sup-norm in $C^+([0, 1])$, given by $\|x\|_\infty := \max_{t \in [0, 1]} |x(t)|$. Equation (8) can be rewritten as a coincidence problem in the following form:

$$(9) \quad \text{find } u \in C^+([0, 1]) \text{ such that } Tu = Su.$$

Next, we will show that the mappings T and S , defined above, satisfy the hypotheses of Theorem 3.16. For each $t \in [0, 1]$, we have:

$$\begin{aligned} \|Tu - Tv\|_\infty &\geq |Tu(t) - Tv(t)| = |e^{u(t)} - 1 - e^{v(t)} + 1| = e^{u(t)} |1 - e^{v(t)-u(t)}| \\ &= e^{u(t)} (e^{|v(t)-u(t)|} - 1) \geq e^{|u(t)-v(t)|} - 1. \end{aligned}$$

The above argument yields $\|Tu - Tv\|_\infty \geq e^{\|u-v\|_\infty} - 1$. If now we define $\phi_1(r) = e^r - 1$, clearly:

$$\|Tu - Tv\|_\infty \geq \phi_1(\|u - v\|_\infty),$$

where $\phi_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, increasing, $\phi_1(r) > r$ and $\phi_1(0) = 0$, for all $r \geq 0$. So, the hypothesis (i) of Theorem 3.16 holds.

For the mapping S we have:

$$\begin{aligned} |Su(t) - Sv(t)| &= \left| \int_0^t f(s, u(s)) ds - \int_0^t f(s, v(s)) ds \right| \leq \int_0^t |f(s, u(s)) - f(s, v(s))| ds \\ &\leq \int_0^t |u(s) - v(s)| ds. \end{aligned}$$

Then, we obtain

$$\|Su - Sv\|_\infty \leq \int_0^t \|u - v\|_\infty ds \leq \|u - v\|_\infty,$$

this means that S is a nonexpansive mapping. Thus, the hypotheses (ii) and (iv) of Theorem 3.16 hold.

Now let us see that $T(C^+([0, 1])) = C^+([0, 1])$.

Given $u \in C^+([0, 1])$. Define $v : [0, 1] \rightarrow \mathbb{R}^+$ by $v(t) := \ln(u(t) + 1)$. Clearly, $v \in C^+([0, 1])$ and moreover, for each $t \in [0, 1]$,

$$Tv(t) = e^{v(t)} - 1 = u(t).$$

In particular, this means that $S(C^+([0, 1])) \subseteq T(C^+([0, 1]))$. Finally, notice that hypothesis (iv) holds, since $k_s < 1$. Therefore, T and S fulfill the hypotheses

of Theorem 3.16 and then the coincidence problem (9) has a unique solution. This means that equation (8) has a unique solution.

For the second conclusion, since $\phi_1^{-1}(r) = \ln(r+1)$, if we define $\beta(r) := r - \ln(r+1)$, since β is a continuous strictly increasing function, $\lim_{r \rightarrow 0^+} \beta(r) = 0$ and $\lim_{r \rightarrow +\infty} \beta(r) = +\infty$, then β is strictly increasing and onto.

All the hypotheses of Theorem 3.18 hold, then the coincidence problem (9) is β^{-1} -generalized Ulam-Hyers stable. Thus, equation (8) is generalized Ulam-Hyers stable.

4.2. EXISTENCE OF CLASSICAL SOLUTIONS TO A DIFFERENTIAL EQUATION OF SECOND ORDER

We will present now an application of Theorems 3.12 and 3.13, which allows us to study the existence and uniqueness of a classic solution for a differential equation of second order.

Let $Y := (C([0, 1]), \|\cdot\|_0)$ be the Banach space of the continuous functions $u : [0, 1] \rightarrow \mathbb{R}$, where $\|u\|_0 := \sup\{|u(t)| : t \in [0, 1]\}$.

Denote $C^2([0, 1]) := \{u : [0, 1] \rightarrow \mathbb{R} : u'' \in C([0, 1])\}$. This allows us to introduce the following linear space

$$X := \{u \in C^2([0, 1]) : u(0) = u(1) = 0\},$$

if on this linear space we define the norm $\|u\|_2 := \max\{\|u\|_0, \|u'\|_0, \|u''\|_0\}$, then $(X, \|\cdot\|_2)$ is a Banach space.

Notice that, if $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then g defines a mapping $N_g : Y \rightarrow X$ by $N_g(u)(t) = g(t, u(t), u'(t))$. This mapping is called superposition (or Nemytskii) operator generated by g . The following three lemmas are of foremost importance for our subsequent analysis.

Lemma 4.22. *Let u be an element in X . Then $\|u\|_2 = \|u''\|_0$.*

Proof. Let u be a fixed element in X . Since $u(0) = 0$, the Mean value theorem says that given $t \in (0, 1]$ there exists $z_t \in (0, t)$ such that $u(t) = u(t) - u(0) = u'(z_t)t$, hence $|u(t)| \leq |u'(z_t)|t$, which implies that

$$(10) \quad \|u\|_0 \leq \|u'\|_0.$$

Since $u \in X$ we have that $u(0) = u(1) = 0$, applying again the Mean value theorem, there will be $z \in (0, 1)$ such that $u'(z) = 0$. Now, if we argue as above we infer that

$$(11) \quad \|u'\|_0 \leq \|u''\|_0.$$

From (10) and (11) we conclude that $\|u\|_2 = \|u''\|_0$.

Lemma 4.23. *The mapping $T : X \rightarrow Y$ defined by $T(u)(t) = u''(t)$ is an expansive surjection.*

Proof. First, let us prove that T is an expansive mapping. Indeed, by Lemma 4.22, we know that if $u, v \in X$, then $\|u - v\|_2 = \|u'' - v''\|_0$, therefore,

$$\|Tu - Tv\|_0 = \|u'' - v''\|_0 = \|u - v\|_2.$$

Second, let us see that T is onto. Indeed, given $u \in Y$ it is enough to consider

$$w(t) := \int_0^t v(\tau) d\tau - t \int_0^1 v(\tau) d\tau, \quad \text{where } v(\tau) := \int_0^\tau u(s) ds,$$

since in this case, $w \in X$ and $T(w)(t) = u(t)$.

Lemma 4.24. *Let $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and there exists $k \in [0, 1)$ such that*

$$|g(t, x, y, z) - g(t, u, v, w)| \leq k \max\{|x - u|, |y - v|, |z - w|\},$$

for all $(t, x, y, z), (t, u, v, w) \in [0, 1] \times \mathbb{R}^3$. Then the superposition operator $N_g : X \rightarrow Y$ is k -contractive.

Proof. Since $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then it is clear that $N_g : X \rightarrow Y$ is well defined. Bearing in mind the definition of the norms given in both Banach spaces X and Y we have

$$\begin{aligned} |N_g(u)(t) - N_g(v)(t)| &= |g(t, u(t), u'(t), u''(t)) - g(t, v(t), v'(t), v''(t))| \\ &\leq k \max\{|u(t) - v(t)|, |u'(t) - v'(t)|, |u''(t) - v''(t)|\}, \end{aligned}$$

Since $u, v \in X$, Lemma 4.22 yields that

$$|N_g(u)(t) - N_g(v)(t)| \leq k \|u'' - v''\|_0 \leq k \|u - v\|_2,$$

and therefore,

$$\|N_g(u) - N_g(v)\|_0 \leq k \|u'' - v''\|_0 \leq k \|u - v\|_2. \quad \square$$

We want to study the existence of classical solutions for the differential equation with a non homogeneous Dirichlet condition

$$(12) \quad \begin{cases} u''(t) - g(t, u(t), u'(t)) = f(t), & t \in [0, 1] \\ u(0) = \xi, \quad u(1) = \nu, \end{cases}$$

where $f \in Y$ is a fixed function.

First, let us notice that Eq.(12) is equivalent to the differential equation with the Dirichlet condition

$$(13) \quad \begin{cases} u''(t) - g(t, u(t) + (\nu - \xi)t + \xi, u'(t) + (\nu - \xi)) = f(t), & t \in [0, 1] \\ u(0) = 0, \quad u(1) = 0 \end{cases}$$

Thus, our goal will be to study the existence of classical solutions to Eq.(13).

For this purpose we define

$$T : X \rightarrow Y \text{ by } T(u)(t) = u''(t)$$

and

$$S : X \rightarrow Y \text{ by } S(u)(t) = N_{\tilde{g}}(u)(t) + f(t),$$

where $\tilde{g}(t, x, y) = g(t, x + (\nu - \xi)t + \xi, y + (\nu - \xi))$.

In order to show that Eq.(13) has a unique classical solution is enough to find a unique element u_0 in X such that $T(u_0) = S(u_0)$. That is, we need to see that the coincidence problem has a unique solution.

In this sense, as a consequence of the above three lemmas and Theorem 3.12 we obtain the following positive result.

Theorem 4.25. *If $g : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is under the hypotheses of Lemma 4.24, then Problem (13) has a unique solution.*

Proof. By Lemma 4.23, we know that T is an expansive mapping and $S(X) \subseteq T(X)$. Moreover, by Lemma 4.24 it is clear that S is a k -contractive mapping, thus we achieve the result applying Theorem 3.12.

EXAMPLE 4.26. Among these, we mention the problem of the forced oscillations of finite amplitude of a pendulum. The amplitude of oscillation $u(t)$ is a solution of the problem

$$(14) \quad \begin{cases} u''(t) - a^2 \sin(u(t)) = z(t), & t \in [0, 1] \\ u(0) = 0, & u(1) = 0, \end{cases}$$

where the driving force $z(t)$ is periodical and odd. The constant $a \neq 0$ depends on the length of the pendulum and on gravity. If $|a| < 1$, then problem (14) has a unique solution.

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