Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **8** (2014), 111–122. doi:10.2298/AADM131128023D

DETECTING WHEELS

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A *wheel* is a graph made of a cycle of length at least 4 together with a vertex that has at least three neighbors in the cycle. We prove that the problem whose instance is a graph G and whose question is "does G contain a wheel as an induced subgraph" is NP-complete. We also settle the complexity of several similar problems.

1. INTRODUCTION

In this article, all graphs are finite and simple. If G and H are graphs, we say that G *contains* H when H is isomorphic to an induced subgraph of G.

A *prism* is a graph made of three vertex-disjoint chordless paths $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$, $P_3 = a_3 \dots b_3$ of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except those of the two triangles.

A *pyramid* is a graph made of three chordless paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least 1, two of which have length at least 2, vertex-disjoint except at a, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to a.

A *theta* is a graph made of three internally vertex-disjoint chordless paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least 2 and such that no edges exist between the paths except the three edges incident to a and the three edges incident to b.

A *hole* in a graph is a chordless cycle of length at least 4. Observe that the lengths of the paths in the three definitions above are designed so that the union of any two of the paths form a hole. A *wheel* is a graph formed by a hole H (called the *rim*) together with a vertex (called the *center*) that has at least three neighbors in the hole.

2010 Mathematics Subject Classification. 05C85, 05C75, 05C17.
Keywords and Phrases. Wheel, detection, Truemper configurations.

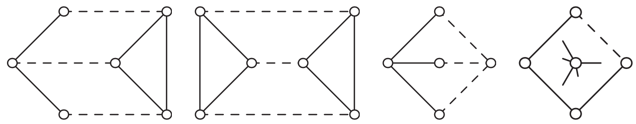

Figure 1. Pyramid, prism, theta and wheel (dashed lines represent paths)

A *Truemper configuration* is a graph isomorphic to a prism, a pyramid, a theta or a wheel (they were first considered by TRUEMPER [**13**]). Truemper configurations play an important role in the proof of several decomposition theorems as explained in a very complete survey of VUŠKOVIĆ [**14**]. Let us explain how with the example of perfect graphs.

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colours needed to assign a colour to each vertex of G in such a way that adjacent vertices receive different colours. The *clique number* of G, denoted by $\omega(G)$, is the maximum number of pairwise adjacent vertices in G. Every graph G clearly satisfies $\chi(G) \geq \omega(G)$, because the vertices of a clique must receive different colours. A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. A chordless cycle of length $2k+1$, $k \geq 2$, satisfies $3 = \chi > \omega = 2$, and its complement satisfies $k+1 = \chi > \omega = k$. An *antihole* is an induced subgraph H of G, such that $\overline{H}$ is a hole of $\overline{G}$ (where $\overline{G}$ denotes the complement of G). A hole (resp. an antihole) is *odd* or *even* according to the number of its vertices. A graph is *Berge* if it does not contain an odd hole nor an odd antihole. The following, known as the *strong perfect graph theorem* (*SPGT* for short), was conjectured by BERGE [**2**] in the 1960s and was the object of much research until it was finally proved in 2002 by CHUDNOVSKY, ROBERTSON, SEYMOUR and THOMAS [**6**].

Theorem 1 (CHUDNOVSKY, ROBERTSON, SEYMOUR, THOMAS 2002)**.** *A graph is perfect if and only if it is Berge.*

One direction is easy: Every perfect graph is Berge, since as we observed already odd holes and antiholes satisfy $\chi = \omega + 1$. The proof of the converse is very long and relies on structural graph theory. The main step is a *decomposition theorem*, not worth stating here, asserting that every Berge graph is either in a well-understood *basic* class of perfect graphs, or has some *decomposition*.

Let us now explain why Truemper configurations play a role in the proof. First, a Berge graph has no pyramid (because among the three paths of a pyramid, two have the same parity, and their union forms an odd hole). This little fact is used very often to provide a contradiction when describing the structure of a Berge graph. A long part of the proof of the SPGT is devoted to study the structure of a Berge graph that contains a prism, and another long part is devoted to a Berge graph that contains a wheel. And at the very end of the proof, it is proved that graphs not previously decomposed are bipartite, just as Berge thetas are. Note that prisms can be defined as line graphs of thetas. This use of Truemper configurations is seemingly something deep and general as suggested by the survey of VUŠKOVIĆ [**14**] about Truemper configurations and how they are used (sometimes implicitly) in the study of different graph classes.

Testing whether a graph contains different Truemper configurations is therefore a question of interest. In what follows, n stands for the number of vertices, and m for the number of edges of the input graph. Detecting a pyramid in an input graph can be done in time $O(n^9)$ (see Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [**4**]) and a theta in time $O(n^{11})$ (see Chudnovsky and Seymour [**7**]). Detecting a prism is NP-complete (see Maffray and Trotignon [**9**]). Detecting a prism or a pyramid can be done in time $O(n^5)$ (see Maffray and Trotignon [**9**]). Detecting a theta or a pyramid can be done in time $O(n^7)$ (see Maffray, Trotignon and Vušković [**10**]). Detecting a prism or a theta can be done in time $O(n^{35})$ (see Chudnovsky and Kapadia [**5**]).

The complexity of detecting a wheel was not known so far. We prove here that it is NP-complete, even when restricted to bipartite (and therefore perfect) graphs. Our proof relies on a variant of a classical construction of Bienstock [**3**] (that is the basis of all the above mentioned hardness results, but how to use it for wheels has not been discovered so far). An easy consequence is that detecting a wheel or a prism is NP-complete (because bipartite graphs contain no prisms, so for them detecting a wheel or a prism is equivalent to detecting a wheel). By the same argument, detecting a wheel or a pyramid is NP-complete. Also detecting a wheel, a pyramid or a prism is NP-complete.

k	theta	pyramid	prism	wheel	Complexity	Reference
0	yes	yes	yes	yes	$O(nm)$	[**8**][**12**]
1	yes	yes	yes	—	$O(n^7)$	[**9**][**10**]
2	yes	yes	—	yes	?	
3	yes	yes	—	—	$O(n^7)$	[**10**]
4	yes	—	yes	yes	?	
5	yes	—	yes	—	$O(n^{35})$	[**5**]
6	yes	—	—	yes	?	
7	yes	—	—	—	$O(n^{11})$	[**7**]
8	—	yes	yes	yes	NPC	
9	—	yes	yes	—	$O(n^5)$	[**9**]
10	—	yes	—	yes	NPC	
11	—	yes	—	—	$O(n^9)$	[**4**]
12	—	—	yes	yes	NPC	
13	—	—	yes	—	NPC	[**9**]
14	—	—	—	yes	NPC	
15	—	—	—	—	$O(1)$	

Table 1. Detecting Truemper configurations

In Table 1, we survey the complexity of detecting any combination of Truemper configurations. The structure to be detected is indicated with "yes". For instance line 5 of the table should be read as follows: the complexity of deciding whether a graph contains a theta or a prism is $O(n^{35})$. Observe that being able to detect a theta *or* a prism is equivalent to a recognition algorithm of the classes of graphs that do not contain thetas *and* prisms as induced subgraphs. Line 0 of the table follows from a result of Conforti, Cornuéjols, Kapoor and Vušković [**8**]. They call *universally signable graphs* the graphs that contain no Truemper configuration, and give a decomposition theorem for them. The complexity of recognizing universally signable graphs is obtained with an algorithm of Tarjan [**12**] that gives the decomposition tree of any graph with clique cutsets. For lines with a question mark, the complexity is not known. The complexities claimed in lines 8, 10, 12 and 14 of the table are proved in this article.

In Section 2 we give the basic reduction from 3-SAT that is used for all our hardness results. In Section 3, we adress the question of detecting a wheel, and several variants motivated by perfect graphs, such as detecting a wheel in a graph or its complements, and detecting variants of wheels (with different sets of constraints on the length of the rim, and the numbers of neighbors of the center). Some variants are polynomial, and some are NP-complete.

2. THE MAIN CONSTRUCTION

In this section, we give a variant of a classical construction due to Bienstock [**3**]. Let f be an instance of 3-SAT, consisting of m clauses $C_1, \dots, C_m$ on n variables $x_1, \dots, x_n$. Let us build a graph G_f with two specialized vertices a, b, such that there will be an induced cycle containing both a, b in G_f if and only if there exists a truth assignment for f. For later use, some edges of G_f will be labelled "black" and some will be labelled "red". Black edges should be thought of as "edges that can be subdivided", or as "edges that potentially belong to the hole". Red edges should be thought of as "edges that serve as chords" and as "non-subdivisible edges".

For each variable x_i $(i = 1, \dots, n)$, make a graph $G(x_i)$ with $8m + 8$ vertices $a_i, b_i, a'_i, b'_i, t_{i,0}, \dots, t_{i,2m}, f_{i,0}, \dots, f_{i,2m}, t'_{i,0}, \dots, t'_{i,2m}, f'_{i,0}, \dots, f'_{i,2m}$. Add black edges in such a way that $a_i t_{i,0} \dots t_{i,2m} b_i$, $a_i f_{i,0} \dots f_{i,2m} b_i$, $a'_i t'_{i,0} \dots t'_{i,2m} b'_i$ and $a'_i f'_{i,0} \dots f'_{i,2m} b'_i$ are chordless paths. Add the following red edges: $t_{i,2j} f_{i,2j}$, $f_{i,2j} t'_{i,2j}$, $t'_{i,2j} f'_{2j}$ and $f'_{i,2j} t_{i,2j}$ for $j = 0, \dots, m$. See Figure 2.

For each clause C_j $(j = 1, \dots, m)$, with $C_j = u_j^1 \vee u_j^2 \vee u_j^3$, where each u_j^p $(p = 1, 2, 3)$ is a literal from $\{x_1, \dots, x_n, \overline{x}_1, \dots, \overline{x}_n\}$, make a graph $G(C_j)$ with five vertices $c_j, d_j, v_j^1, v_j^2, v_j^3$ and six black edges so that each of c_j, d_j is adjacent to each of v_j^1, v_j^2, v_j^3. See Figure 3. For $p = 1, 2, 3$, if $u_j^p = x_i$ then add two red edges $v_j^p f_{i,2j-1}, v_j^p f'_{i,2j-1}$, while if $u_j^p = \overline{x}_i$ then add two red edges $v_j^p t_{i,2j-1}, v_j^p t'_{i,2j-1}$. See Figure 4.

The graph G_f is obtained from the disjoint union of the $G(x_i)$'s and the $G(C_j)$'s as follows. For $i = 1, \dots, n - 1$, add edges $b_i a_{i+1}$ and $b'_i a'_{i+1}$. Add an edge $b'_n c_1$. For $j = 1, \dots, m - 1$, add an edge $d_j c_{j+1}$. Introduce the two specialized vertices a, b and add edges aa_1, aa'_1 and bd_m, bb_n. See Figure 5. Clearly the size of G_f is polynomial (actually quadratic) in the size $n + m$ of f. An *f-graph* is any graph obtained from G_f by subdividing black edges of G_f (the subdivision is arbitrary: each edge is subdivided an arbitrary number of times, possibly zero).

Lemma 2. *Let f be an instance of 3-SAT and G an f-graph. Then, f admits a truth assignment if and only if G contains an induced cycle through a and b.*

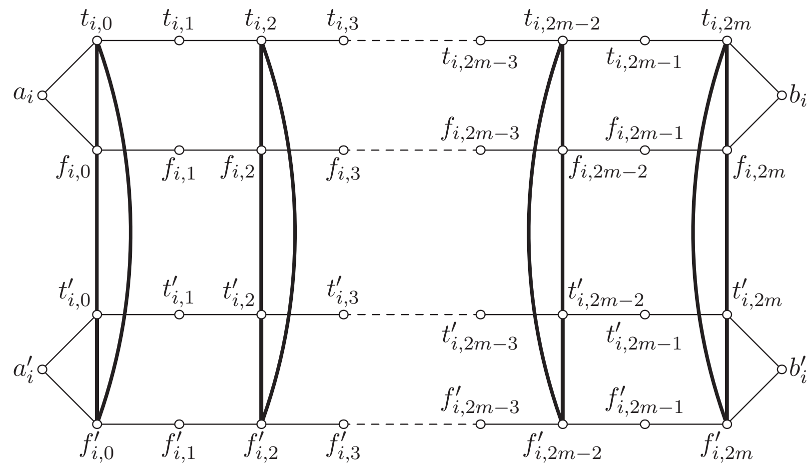

Figure 2. The graph $G(x_i)$

Proof. Suppose that f admits a truth assignment $\xi \in \{0,1\}^n$. We build an induced cycle H in G_f by selecting vertices as follows (we build an induced cycle in G later). Select a, b. For $i = 1, \dots, n$, select a_i, b_i, a'_i, b'_i; moreover, if $\xi_i = 1$ select $t_{i,0} \dots t_{i,2m}$ and $t'_{i,0} \dots t'_{i,2m}$, while if $\xi_i = 0$ select $f_{i,0} \dots f_{i,2m}$ and $f'_{i,0} \dots f'_{i,2m}$. For $j = 1, \dots, m$, since ξ is a truth assignment for f, at least one of the three literals of C_j is equal to 1, say $v_j^p = 1$ for some $p \in \{1,2,3\}$. Then select c_j, d_j and v_j^p. Now it is a routine matter to check that the selected vertices induce a cycle Z that contains a, b, and that Z is chordless, so it is an induced cycle.

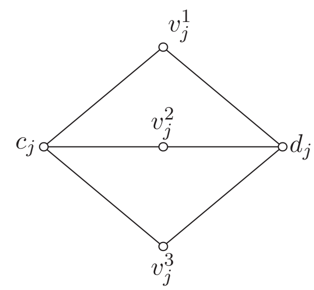

Figure 3. The graph $G(C_j)$

The main point is that there is no chord in Z between some subgraph $G(C_j)$ and some subgraph $G(x_i)$, for that would be either an edge $f_{i,2j-1}v_j^p$ (or $f'_{i,2j-1}v_j^p$) with $v_j^p = x_i$ and $\xi_i = 1$, or, symmetrically, an edge $t_{i,2j-1}v_j^p$ (or $t'_{i,2j-1}v_j^p$) with $v_j^p = \overline{x}_i$ and $\xi_i = 0$, in either case a contradiction to the way the vertices of Z were selected.

To build an induced cycle in G (instead of G_f), just subdivide the black edges of $E(G_f) \cap E(H)$ that were subdivided to obtain G.

For the converse statement, we write the proof for G_f, the proof is similar for G. Suppose that G_f admits an induced cycle Z that contains a, b. Clearly Z contains a_1, a'_1 since these are the only neighbours of a in G_f.

(1) *For $i = 1, \dots, n$, Z contains exactly $4m + 6$ vertices of $G(x_i)$: a_i, a'_i, b_i, b'_i, and either $t_{i,0} \dots t_{i,2m}$ and $t'_{i,0} \dots t'_{i,2m}$, or $f_{i,0} \dots f_{i,2m}$ and $f'_{i,0} \dots f'_{i,2m}$.*

First we prove the claim for $i = 1$. Since a, a_1 are in Z and a_1 has only three neighbours (namely $a, t_{1,0}, f_{1,0}$) exactly one of $t_{1,0}, f_{1,0}$ is in Z. Likewise exactly one of $t'_{1,0}, f'_{1,0}$ is in Z. If $t_{1,0}, f'_{1,0}$ are in Z, then the vertices $a, a_1, a'_1, t_{1,0}, f'_{1,0}$ are all in Z and because of the red edge, they induce a cycle that does not contain b, a contradiction. Likewise we do not have both $t'_{1,0}, f_{1,0}$ in Z. Therefore, up to symmetry we may assume that $t_{1,0}, t'_{1,0}$ are in Z. Thus $f_{1,0}$ is not in Z because of the red edge $t_{1,0}f_{1,0}$. Similarly, $f'_{1,0}$ is not in Z. It follows that $t_{1,1}$ and $t'_{1,1}$ are in Z.

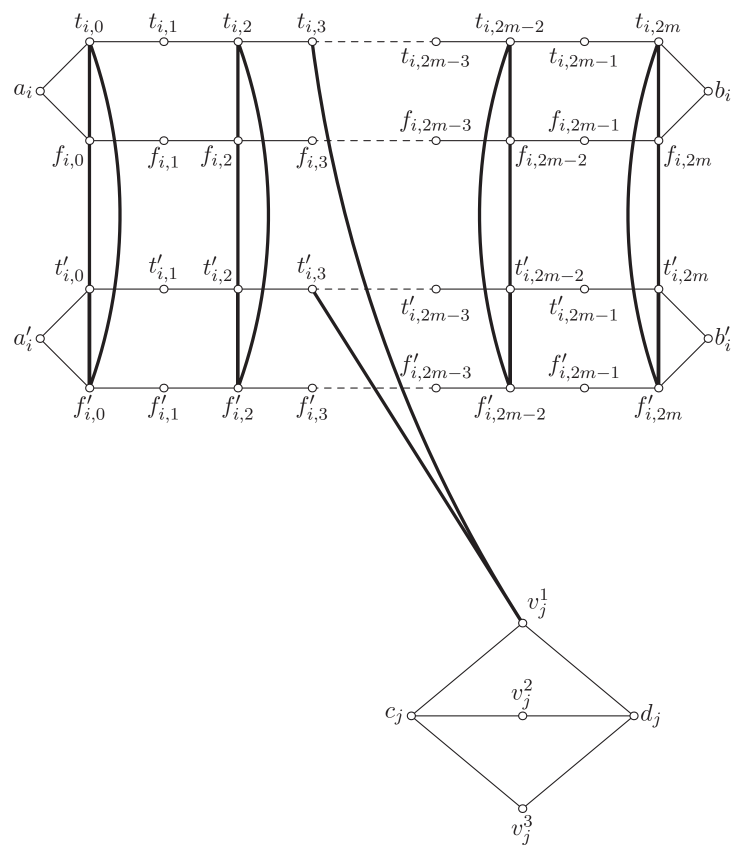

Figure 4. Red edges between $G(x_i)$ and $G(C_j)$ (here $j = 2$ and $u_j^1 = \overline{x_i}$)

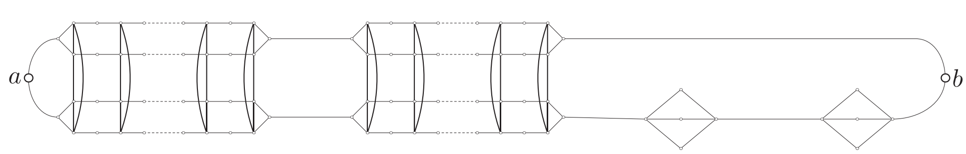

Figure 5. The graph G_f

If a vertex u_j^p of some $G(C_j)$ ($1 \leq j \leq m$, $1 \leq p \leq 3$) is in Z and is adjacent to $t_{1,1}$ then, since this u_j^p is also adjacent to $t'_{1,1}$, we see that the vertices $a, a_1, a'_1, t_{1,0}, t'_{1,0}, t_{1,1}, t'_{1,1}$ and u_j^p are all in Z and induce a hole that does not contain b, a contradiction. Thus the neighbour of $t_{1,1}$ in $Z \setminus t_{1,0}$ is not in any $G(C_j)$ ($1 \leq j \leq m$), so that neighbour is $t_{1,2}$. Likewise $t'_{1,2}$ is in Z. By the same argument, it can be proved that $t_{1,3}, \ldots, t_{1,2m}$ and $t'_{1,3}, \ldots, t'_{1,2m}$ are all in Z. Also, for $k = 1, \ldots, m$, $f_{1,2k}$ is not in Z because of the red edge $t_{1,2k}f_{1,2k}$ and similarly, $f'_{1,2k}$ is not in Z. Since $f_{1,2k-1}$ has degree at most 3, it cannot be in Z because one of its neighbor in Z would be $f_{1,2k-2}$ or $f_{1,2k}$. It follows that b_1 and b'_1 are in Z.

So the claim holds for $i = 1$. Since $f_{1,2m}$ is not in Z, we see that a_2 is in Z and similarly that a'_2 is in Z. Now the proof of the claim for $i = 2$ is essentially the same as for $i = 1$, and so on up to $i = n$. This proves (1).

(2) *Z contains no red edge of G_f.*

Follows directly from (1). This proves (2).

(3) *For $j = 1, \ldots, m$, Z contains c_j, d_j and exactly one of v_j^1, v_j^2, v_j^3.*

First we prove this claim for $j = 1$. By (1), b'_n is in Z and exactly one of $t'_{n,2m}, f'_{n,2m}$ is in Z, so (since b'_n has degree 3 in G_f) c_1 is in Z. So, by (2) Z contains exactly one of the paths $c_1v_1^1d_1$, $c_1v_1^2d_1$ or $c_1v_1^3d_1$. Thus, the neighbor of d_1 in $Z \setminus \{v_1^1, v_1^2, v_1^3\}$ must be c_2. Now the proof of the claim for $j = 2$ is the same as for $j = 1$, and similarly the claim holds up to $j = m$. This proves (3).

We can now make a Boolean vector ξ as follows. For $i = 1, \ldots, n$, if Z contains $t_{i,0}, t'_{i,0}$ set $\xi_i = 1$; if Z contains $f_{i,0}, f'_{i,0}$ set $\xi_i = 0$. By (1) this is consistent. Consider any clause C_j ($1 \leq j \leq m$). By (3) and up to symmetry we may assume that u_j^1 is in Z. If $u_j^1 = x_i$ for some $i \in \{1, .., n\}$, then the construction of G_f implies that $f_{i,2j-1}, f'_{i,2j-1}$ are not in Z, so $t_{i,0}, t'_{i,0}$ are in Z, so $\xi_i = 1$, so clause C_j is satisfied by x_i. If $u_j^1 = \overline{x}_i$ for some $i \in \{1, .., n\}$, then the construction of G_f implies that $t_{i,2j-1}, t'_{i,2j-1}$ are not in Z, so $f_{i,0}, f'_{i,0}$ are in Z, so $\xi_i = 0$, so clause C_j is satisfied by $\overline{x}_i$. Thus ξ is a truth assignment for f. This completes the proof of the lemma. □

A *hub* in a graph is a vertex that has at least three neighbors of degree at least 3. Note that the center of a wheel is a hub. This simple observation will be useful in the next section.

Theorem 3. *Let k be an integer. The problem of detecting an induced cycle of length at least k through two prescribed vertices a and b of degree 2 in an input graph is NP-complete, even when restricted to bipartite graphs with no hub.*

Proof. We consider an instance f of 3-SAT and we build an instance (G, a, b) of our problem such that f can be satisfied if an only if an induced cycle of length at least k of G goes through a and b.

To do so, we start with the graph G_f defined above. We now subdivide carefully chosen black edges. First, we subdivide the edge aa_1 k times so that any cycle through a has length at least k. Since every vertex of G is adjacent to at most 2 red edges, it is possible to eliminate all hubs by subdividing once each black edge. Now consider the vertices of degree at least 3 in G. They induce a graph $G_{\geq 3}$. The components of $G_{\geq 3}$ are cycles of length 4, paths on 3 vertices and isolated vertices.

Thus $G_{\geq 3}$ is bipartite, and we choose a bipartition into blue and green vertices (it is not unique since $G_{\geq 3}$ is not connected). Now the fact that the bipartition of $G_{\geq 3}$ can be extended to a bipartition of G depends only on the parity of the paths of black edges linking the components of $G_{\geq 3}$. It follows that by subviding one edge or no edges in each of these paths, the desired bipartite graph G can be obtained. The result follows from Lemma 2.

3. DETECTING WHEELS

Let $3 \leq k$ and $0 \leq \ell \leq k$ be integers. A (k, ℓ)-*wheel* is a graph made of a chordless cycle of length at least k together with a vertex that has at least l neighbors in the cycle. Thus a wheel is a $(4, 3)$-wheel. An article of Aboulker et al. [**1**] is devoted to the detection of $(3, 2)$-wheels (that are called *propellers*). The last fifty pages of the proof of the SPGT are devoted to studying Berge graphs that contain particular kinds of $(6, 3)$-wheels or their complement. These examples are our motivation for not restricting ourselves to the study of wheels only.

If $3 \leq k$ and $3 \leq \ell \leq k$ are integers, then the center of a (k, ℓ)-wheel is a hub. This simple observation is very useful for turning Lemma 2 into the next theorem.

We denote by $\Pi_{k,\ell}$ the problem whose instance is a graph G and whose question is "does G contain a (k, ℓ)-wheel (as an induced subgraph)?". We do not know the complexity of $\Pi_{k,\ell}$ when $\ell = 2$ and $k \geq 4$ (for large values of k, we believe that the question is quite challenging). The next theorem settles all the other cases.

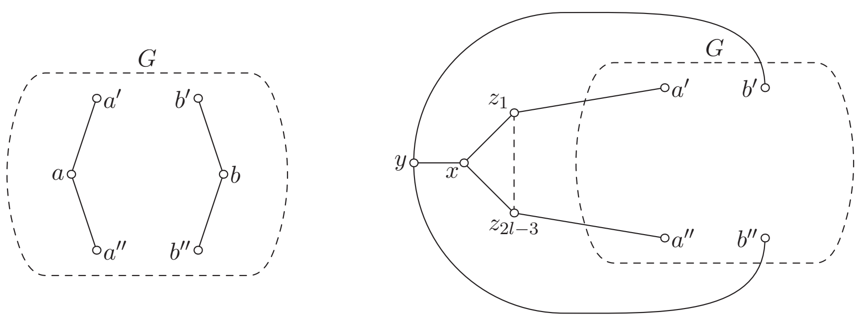

Figure 6. G and G'

Theorem 4. *Let $3 \leq k$ and $0 \leq \ell \leq k$ be integers. The problem $\Pi_{k,\ell}$ is polynomial if $\ell \leq 1$ and is NP-complete if $\ell \geq 3$ (and it remains NP-complete when restricted to bipartite instances). If $\ell = 2$ and $k = 3$ then $\Pi_{k,\ell}$ is polynomial.*

Proof. If $\ell = 0$, then $\Pi_{k,\ell}$ consists of detecting an induced cycle of length at least k, and if $\ell = 1$ it can be reduced easily to the detection of an induced cycle of length at least k through a prescribed vertex. These problems are clearly polynomial (here is a short sketch: enumerate each path P of length at most $k - 2$ through the vertex, delete the internal vertices of P and their neighbors, put back the ends of P and run a shortest path algorithm to link them). If $\ell = 2$ and $k = 3$, then the polynomiality of $\Pi_{k,\ell}$ is a result in [**1**] where it is refered to as the *detection of propellers*.

Let us now suppose $\ell \geq 3$. Consider an instance (G, a, b) of the problem from Theorem 3 (so G is bipartite). Call a', a'' the two neighbors of a and b', b'' the two neighbors of b. Build a graph G' as follows. Delete a and b from G. Add a path $z_1 \dots z_{2\ell-3}$ and the edges z_1a' and $z_{2\ell-3}a''$. Add a vertex x adjacent to $z_1, z_3, \dots, z_{2\ell-3}$. Add a vertex y adjacent to x, b' and b'' (see Fig. 6). After possibly subdividing once or twice z_1a', $z_{2\ell-3}a''$, yb' and yb'', G' is bipartite and the unique hub of G' is x.

A (k, ℓ)-wheel of G' must therefore be centered at x and must contain an induced path from a' to b' and an induced path from a'' to b'' (or an induced path from a' to b'' and an induced path from a'' to b'). In either case, G contains a hole that goes through a and b.

Conversely, if G contains a hole through a and b, then G' contains a (k, ℓ)-wheel (centered at x). □

The class of Berge graphs is self-complementary, and detecting a structure in a Berge graph or its complement is sometimes useful. This motivates the next problem. We denote by $\overline{\Pi}_{k,\ell}$ the problem whose instance is a bipartite graph G and whose question is "does one of G or $\overline{G}$ contain a (k, ℓ)-wheel as an induced subgraph?". The next theorem settles the complexity of $\overline{\Pi}_{k,\ell}$ in several cases (the other cases are open).

Theorem 5. *Let $3 \leq k$ and $0 \leq \ell \leq k$ be integers. The problem $\overline{\Pi}_{k,\ell}$ is polynomial if $k \leq 4$ and NP-complete if $k \geq 5$ and $\ell \geq 3$ (and it remains NP-complete when restricted to bipartite instances).*

Proof. Suppose that $k \leq 4$ (so $\ell \leq 4$). Consider a chordless path $v_1 \dots v_7$. In the complement, the vertices v_1, v_2, v_4, v_5, v_7 induce a $(4, 4)$-wheel (and therefore a (k, ℓ)-wheel for any $k \leq 4$ and $0 \leq \ell \leq k$). Since a (k, ℓ)-wheel on at least nine vertices contains an induced cycle of length at least 8, it also contains an induced path of length 7, and therefore a (k, ℓ)-wheel on 5 vertices in the complement. It follows that the answer to $\overline{\Pi}_{k,\ell}$ is yes if and only if the input graph or its complement contains a wheel on at most eight vertices. This can be tested by brute force enumeration in time $O(n^8)$.

When $k \geq 5$, the complement of a bipartite graph cannot contain a (k, ℓ)-wheel, because $\overline{C_5}$ is not bipartite and a cycle of length at least 6 contains a stable set of size 3 (that does not exist in the complement of a bipartite graph). Thus, the NP-completeness of $\overline{\Pi}_{k,\ell}$ directly follows from NP-completeness of $\Pi_{k,\ell}$. □

In fact, we can be faster for $\overline{\Pi}_{4,3}$, which is the problem of detecting a wheel in an input graph or in its complement. We need the following result which appears as Theorem 3.1 in Nikolopoulos and Palios [**11**].

Theorem 6. *There is an $O(n + m^2)$ time algorithm that determines whether an input graph G contains a hole of size at least five.*

We also need the next two little facts.

Lemma 7. *If a graph G contains a hole H of length at least 5, then either $G = H$ or one of G or $\overline{G}$ contains a wheel.*

Proof. Suppose that $G \neq H$ and let $w \notin V(H)$ be a vertex of G. If H is of length 5, then it is self-complementary, so in G or $\overline{G}$, w has at least three neighbors in H. If H is of length at least 6, then let $H = v_1 \dots v_6 \dots$. If G has no wheel, then w has at most two neighbors in H. Up to a relabelling, we may assume that w has at most one neighbor among $v_2, \dots, v_6$. It follows that $\{w, v_2, v_3, v_5, v_6\}$ induces a wheel in $\overline{G}$.

Lemma 8. *Let G be a graph. None of $G, \overline{G}$ contains a path on three vertices if and only if G is a complete graph or an independent graph.*

Proof. Suppose that G is neither independent nor complete. Since G is not independent, it has a connected component C with at least two vertices. So, C contain an edge vw. If G has another connected component C', then for some $u \in C'$, vuw is a path on three vertices in $\overline{G}$. So, we may assume that C is the only connected component of G. Since $C = G$ is not complete, it contains two non-adjacent vertices v', w'. A shortest path from v' to w' in G contains a path on three vertices. We proved that G contains a path on three vertices or its complement. The proof of the converse statement is clear.

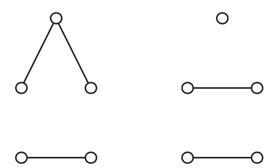

Figure 7. The two complements of wheels on five vertices

Theorem 9. *The problem $\overline{\Pi}_{4,3}$ can be solved in time $O(n^4)$.*

Proof. The first step of the algorithm is to check whether G or $\overline{G}$ contains a hole of length at least 5. This can be implemented in time $O(n^4)$ by Theorem 6, and if a hole is found, by Lemma 7 it can be decided whether G or $\overline{G}$ contains a wheel. So, we may assume from here on that none of G, $\overline{G}$ contains a hole of length at least 5. Hence, we just need to detect a wheel on five vertices. For convenience, we show how to detect the complement of a wheel (and we run this routine in the graph and in its complement).

Complement of wheels on five vertices are represented in Fig. 7. One is the disjoint union of an edge and a path on three vertices, the other one is the disjoint union of an edge and the complement of a path on three vertices. To decide whether a graph contains the complement of a wheel on five vertices, it is therefore enough to check all edges vw of G, and to decide for each of them whether the set S of vertices of G adjacent to none of v, w contains a path on three vertices or its complement. By Lemma 8, testing the desired property in S is easy to implement in time $O(n^2)$. Hence, the algorithm can be implemented to run in time $O(n^4)$. □

Theorem 9 and its proof suggest that graphs with no wheels and no complement of wheels form a restricted class that might have a simple structure. The class contains all split graphs, complete bipartite graphs, some (non-induced) subgraphs of them, and several particular graphs such as C_5, C_6 or P_5. We could not ellucidate its structure, and leave this as an open question.

Ackowledgements. Emilie Diot, Sébastien Tavenas and Nicolas Trotignon: CNRS, ENS de Lyon, LIP, Équipe MC2, INRIA, Université Lyon 1, Université de Lyon.

This work was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

Nicolas Trotignon is partially supported by the French Agence Nationale de la Recherche under reference ANR-10-JCJC-HEREDIA and ANR-14-BLAN-STINT.

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(Received July 20, 2013)
(Revised November 28. 2013)