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MORE ON NON-REGULAR BIPARTITE INTEGRAL GRAPHS WITH MAXIMUM DEGREE 4 NOT HAVING ±1 AS EIGENVALUES

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A graph is integral if the spectrum (of its adjacency matrix) consists entirely of integers. The problem of determining all non-regular bipartite integral graphs with maximum degree four which do not have ± 1 as eigenvalues was posed in K.T. Balińska, S.K. Simić, K.T. Zwierzyński: *Which non-regular bipartite integral graphs with maximum degree four do not have* ± 1 *as eigenvalues*? Discrete Math., **286** (2004), 15–25. Here we revisit this problem, and provide its complete solution using mostly the theoretical arguments.

The paper is dedicated to professor Dobrilo Đ. Tošić on occasion of his 81st birthday.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple graph of order n $(= |G|)$ and size m $(= ||G||)$. $A(G)$ denotes the $(0, 1)$-adjacency matrix of G. Its spectrum is also called the *spectrum* of G, and denoted by $Sp(G)$ – note, it is real since $A(G)$ is symmetric. We assume that the eigenvalues of G are given in non-increasing order: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. Recall, $\lambda_1(G)$ is a simple eigenvalue if G is connected. Moreover, if not told otherwise, all graphs to be considered (but their subgraphs) will be connected. In particular, $\lambda_1(G)$ is called the *index* of G. For a given $\lambda \in Sp(G)$, $m(\lambda; G)$ denotes its *multiplicity* – note, since $A(G)$ is symmetric, the algebraic and geometric multiplicities of λ are equal. Let

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$\mu_1(G) > \mu_2(G) > \cdots > \mu_r(G)$ and $m_1, m_2, \ldots, m_r$, be the distinct eigenvalues of G along with their multiplicities. Then, assuming that $Sp(G)$ is a multiset, we write

$$Sp(G) = [\mu_1(G)^{m_1}, \mu_2(G)^{m_2}, \ldots, \mu_r(G)^{m_r}].$$

If G is bipartite, then its spectrum is symmetric with respect to the origin (see [**4, 6**]). So $\pm\mu$ are the eigenvalues of the same multiplicity. The equation $A\mathbf{x} = \mu\mathbf{x}$ is called the *eigenvalue equation* for $\mu \in Sp(G)$. Any non-zero vector $\mathbf{x}$ satisfying it is an *eigenvector* of the (labelled) graph G. $M_k(G) = \sum_{i=1}^{n} \lambda_i(G)^k$ $(k \geq 0)$ denotes the k-th *spectral moment* of G. It counts the total number of closed walks of length k starting and terminating at vertices of G (see [**4, 6**]).

P_n and C_n denote the path and cycle of order n, respectively; $K_{m,n}$ is the bi-complete graph on $m+n$ vertices; in particular, $S_n = K_{1,n-1}$ denotes the star of order n. Let $\Gamma(v; G) = \{w : w \sim v\}$; as usual, $d_v = \deg(v) = |\Gamma(v; G)|$, $\Delta(G) = \max_{v \in V(G)} d_v$ and $\delta(G) = \min_{v \in V(G)} d_v$. A vertex of degree 1 is called a *pendant vertex*. In particular, for trees, any other vertex is called an *interior vertex*. $G - u$ $(G - U)$ denotes the subgraph of G obtained by deleting a vertex u (resp. a vertex set U) from G. If $U \subseteq V(G)$ then $\langle U \rangle$ denotes the subgraph of G induced by U. $H \subseteq G$ denotes that H is an induced subgraph of G ($\subset$ stands for a proper induced subgraph). If $H \subset G$ and $U \subset V(G) \setminus V(H)$ then $H + U = \langle V(H) \cup U \rangle$. $G \cup H$ stands for the (disjoint) union of two graphs. Further on, if the graph name is clear from the context, it will be omitted.

A graph is *integral* if its eigenvalues are integers. For all other facts from the spectral graph theory (including integral graphs) the reader is referred to one of the books [**4, 6**]. In this paper we solve the problem posed in [**3**]. The main result of this paper reads:

Theorem 1.1. *Apart from S_5 $(= K_{1,4})$, there are just three non-regular bipartite (connected) integral graphs with maximum degree four which do not have ± 1 as eigenvalues (see Fig. 1.1).*

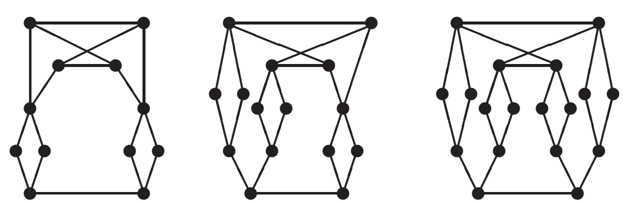


Figure 1.1. Three integral graphs

The rest of the paper is organized as follows: In Section 2, to make it more self-contained, we include basic observations from [**3**] (and [**1**]). In Section 3 we develop new ideas to be used in Section 4 for proving Theorem 1.1.

2. PRELIMINARIES

Let $\mathscr{S}$ be the set of all connected integral graphs with maximum degree four which are non-regular and bipartite. Note, if $G \in \mathscr{S} \setminus \{S_5\}$ then $Sp(G) = [3,\ 2^a,\ 1^b,\ 0^c,\ (-1)^b\ (-2)^a,\ -3]$. The case $a = 0$ was considered in [**2**] and settled in [**7**]. Here we concentrate on the case $b = 0$ and $a > 0$.[1] The quest for these graphs (or the corresponding set $\mathscr{S}'$) was initiated in [**3**]. If $G \in \mathscr{S}'$ then

$$Sp(G) \;=\; [3,\ 2^a,\ 0^c,\ (-2)^a,\ -3] \quad (a > 0).$$

Here we want to study our problem just theoretically, and so to subsume the search done in [**7**] by an exact algorithm which was very time consuming. However, this can put forward further ideas for studying the class $\mathscr{S}$ (needless to say, the intermediate goal can refer to the class with $c = 0$). Observe also that the graphs from $\mathscr{S}'$ are of diameter at most 4 (since $\mathrm{diam}(G) \leq |Sp(G)| - 1$ for any connected graph – see [**6**], p. 59).

The graphs from $\mathscr{S}'$ with at most 16 vertices (see Fig. 1.1) were found by a computer search in [**1**] (and later in [**7**]). Here we pursue only those graphs with at least 17 vertices, in order to show (almost theoretically) that, except the third graph of Fig. 1.1, there are no others. Besides many observations found in [**3**] (to be mentioned later), the most striking ones concern their order ($n \leq 29$), size ($m \leq 41$, since $a \leq 8$), their tentative degree sequences (see Table 1 therein). They are determined by the triplets (ν_4, ν_3, ν_2), where $\nu_i = |\{v : d_v = i\}|$ $(i = 4, 3, 2)$ – so $\delta(G) > 1$, as proved in [**3**]. There are 83 in total such triplets. One of them $(3, 14, 10)$ was not discarded in [**3**] by using Proposition 3.5(3^o), as was confirmed later by a simple program written in Oz, the language for constraint programming. So just 82 triplets remained unresolved.

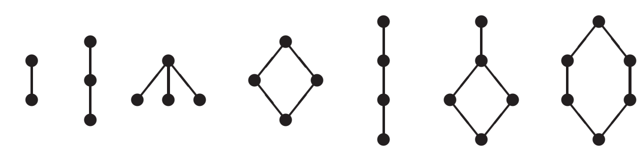
Let m, f, g, q, p, e, h denote the number of subgraphs of a graph G which are depicted in Fig. 2.1 (identified by labels below them). Following [**1**] we have:

Lemma 2.1. *Under the notation above, if G is bipartite then*:

1^o $M_2(G) = 2m$;

2^o $M_4(G) = 2m + 4f + 8q$;

3^o $M_6(G) = 2m + 12f + 12g + 48q + 6p + 12e + 12h$.



(*m*) (*f*) (*g*) (*q*) (*p*) (*e*) (*h*)

Figure 2.1. The graphs relevant to Lemma 2.1

[1] It is worth mentioning, that these two special cases are also relevant to some considerations from [**1**] – see Proposition 2.4 therein.

Note first that $m = 4a + 9$ (see 1^0 from above – recall $b = 0$). Therefore, since G is bipartite and m odd, G contains, in each colour class, an odd number of vertices of degree 3.

Lemma 2.2. *Under the notation above, if $G \in \mathscr{S}'$ then*

$$q = 5a + \frac{1}{4}(90 - 16\nu_4 - 9\nu_3 - 4\nu_2) \ \text{ and } \ \nu_3 \equiv 2 \ (mod\ 4).$$

Proof. By Lemma 2.1 (2^o), we can express q in terms of $M_4(G)$, m and f. Next we have: $M_4(G) = 2(3^4 + a2^4)$, $m = 4a+9$ and $f = \sum_{i=1}^{n} \binom{d_i}{2} = \frac{1}{2}(16\nu_4 + 9\nu_3 + 4\nu_2) - m$. So we easily get q; the rest follows from the integrality of q. □

The above lemma turns to be very powerful in discarding triplets (from Table 1 in [**3**]). Namely, since q is a non-negative integer, 39 triplets are eliminated at once, which results in its reduced form, here addressed as Table 2.1.

n	$a = 3$, $m = 21$	$a = 4$, $m = 25$	$a = 5$, $m = 29$	n	$a = 6$, $m = 33$	$a = 7$, $m = 37$	$a = 8$, $m = 41$
17	1:(1,6,10) 2:(3,2,12)	5:(1,14,2)		23	24:(1,18,4) 25:(3,14,6) 26:(5,10,8) 27:(7,6,10)		
18	3:(2,2,14)	6:(2,10,6) 7:(4,6,8) 8:(6,2,10)		24	28:(4,10,10) 29:(6,6,12) 30:(8,2,14)		
19	4:(1,2,16)	9:(1,10,8) 10:(3,6,10) 11:(5,2,12)		25	31:(5,6,14) 32:(7,2,16)		
20		12:(2,6,12) 13:(4,2,14)	14:(2,14,4) 15:(4,10,6)	26		33:(4,14,8) 34:(6,10,10) 35:(8,6,12)	
21			16:(1,14,6) 17:(3,10,8) 18:(5,6,10) 19:(7,2,12)	27		36:(5,10,12) 37:(7,6,14) 38:(9,2,16)	
22			20:(2,10,10) 21:(4,6,12) 22:(6,2,14)	28		39:(6,6,16) 40:(8,2,18)	41:(4,18,6)
23			23:(1,10,12)	29			42:(5,14,10) 43:(7,10,12)

Table 2.1. The reduced form of Table 1 from [**3**]. The entry format id:(ν_4,ν_3,ν_2)

Lemma 2.3. *Under the notation above, if $G \in \mathscr{S}'$ then*

(1) $$p + 2(e + h) = 114 + 4a + \nu_3.$$

Proof. Expressing $M_6(G)$ in two ways, by definition, and by Lemma 2.1 (3^o), we get $2(3^6+2^6a) = 2m+12f+12g+48q+12e+12h+6p$. Since $f = \sum_{i=1}^{n}\binom{d_i}{2}$ and $g = \sum_{i=1}^{n}\binom{d_i}{3}$, we obtain that $f+g = \frac{1}{6}\sum_{i=1}^{n} {d_i}^3 - \frac{1}{3}m$. Next $\frac{1}{6}\sum_{i=1}^{n} d_i^3 = 4^3\nu_4+3^3\nu_3+2^3\nu_2$. Since q is given by Lemma 2.2, the proof easily follows. □

For convenience, let $\hat{E} = p + 2(e + h)$, while $\hat{F} = 114 + 4a + \nu_3$. So (1) is equivalent to $\hat{E} = \hat{F}$. Note, the right hand side of (1), i.e. $\hat{F}$ is determined by the parameters of each instance (i.e. of the corresponding triplet from Table 2.1). On the other hand, its right hand side, i.e. $\hat{E}$, cannot be determined in advance (since p, e and h depend on the structure of G).

A graph G is *reflexive* if $\lambda_2(G) \le 2$. The next result is taken from [**8**] (cf. Theorem 3.2).

Lemma 2.4. *Let G be a connected graph having v as a cut-vertex, and let $G - v = \bigcup_{i=1}^{k} G_i$, where each G_i $(i = 1, 2, \dots, k)$ is connected. If $\lambda_1(G_1) \ge \lambda_1(G_2) \ge \cdots \ge \lambda_1(G_k)$ then:*

1^o $\lambda_2(G) > 2$, *if* $\lambda_1(G_1) > 2$ *and* $\lambda_1(G_2) \ge 2$;

2^o $\lambda_2(G) = 2$, *if* $\lambda_1(G_1) = \lambda_1(G_2) = 2$;

3^o $\lambda_2(G) < 2$, *if* $\lambda_1(G_1) \le 2$ *and* $\lambda_1(G_2) < 2$.

In remaining cases there are no definite rules.

Recall the *Smith graphs* (C_n – E_9, see Fig. 2.2) are connected graphs whose index is equal to 2. Connected graphs whose index is less than 2, also called the *reduced Smith graphs* (P_n – Z_8, see Fig. 2.2) are the proper subgraphs of the Smith graphs.

For short, let $\eta(G) = m(0; G)$ denote the *nullity* of G. Then we have:

Remark 2.5. Recall $\eta(C_{2i})$ is equal to 2 if i is even, or 0 otherwise. If T is a tree of order n whose maximal matching has cardinality k, then $\eta(T) = n - 2k$ (see [**4**], the Sachs Theorem). So we have:

- $\eta(G) = 3$, if $G = W_{2i+1}$ $(i \ge 2)$;
- $\eta(G) = 2$, if $G = W_{2i}$ $(i \ge 3)$, C_{4i} $(i \ge 1)$, E_8 and Y_{2i} $(i \ge 3)$;
- $\eta(G) = 1$, if $G = E_7$, E_9, Y_{2i+1} $(i \ge 2)$, Z_{2i+1} $(i \ge 2)$ and P_{2i+1} $(i \ge 0)$;
- $\eta(G) = 0$, if $G = C_{4i+2}$ $(i \ge 1)$, Z_{2i} $(i \ge 2)$ and P_{2i} $(i \ge 1)$.

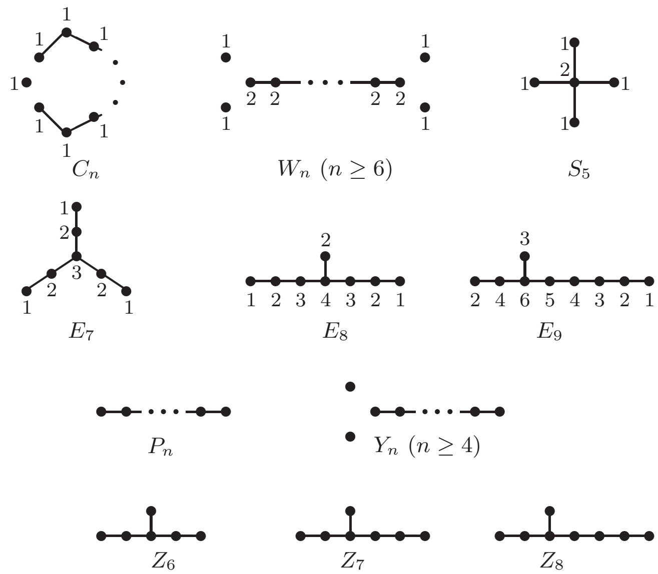


Figure 2.2. The Smith graphs[2] and the reduced Smith graphs

3. FURTHER TOOLS

Here we study various spectral and structural properties of graphs from $\mathscr{S}'$ (or even $\mathscr{S}$). In view of computational results already mentioned, we will assume further on that their order is at least 17.

3.1. Structural considerations

For any connected graph G, we introduce three types of partitions of its vertex set (the first two of them were considered in [**3**] but with less generality).

R-partition. Let R be a proper subgraph of G. Define $V_i(R) = \{u : \operatorname{dist}(u, V(R)) = i\}$ $(i \geq 0)$; here $\operatorname{dist}(u, V(R))$ denotes the distance between u and the closest vertex in R. Clearly, $V_0(R) = V(R)$. Then

$$V_0(R) \cup V_1(R) \cup \cdots \cup V_d(R) \quad (d \geq 1)$$

is an *R-partition* of G with respect to R. Note, if $V(R) = \{r\}$, then the corresponding partition (see [**1**]) is called the *distance partition* of G with respect to r (its root). We also write $V_{\leq k}(R) = \{u : \operatorname{dist}(u, R) \leq k\}$, $V_{\geq k}(R) = \{u : \operatorname{dist}(u, R) \geq k\}$ and $R^* = \langle V_{\geq 2}(R) \rangle$.

U-partition. Let U be a proper subset of $V(G)$, and let $G - U = H_0 \cup H_1 \cup \cdots \cup H_t$ for some $t \geq 0$ (here each H_i is connected). Then

$$U \cup V(H_0) \cup V(H_1) \cup \cdots \cup V(H_t)$$

[2]Numbers attached to vertices of the Smith graphs are the entries of the eigenvector corresponding to the index.

is a *U-partition* of G with respect to U.

If $G \in \mathscr{S}$, to choose U, we first observe $H_0 \subset G$ (usually H_0 is a Smith graph), and then take $U = V_1(H_0)$, i.e. the set of the first neighbours of H_0. Therefore $R = H_0$ and $R^* = H_1 \cup \cdots \cup H_t$, while $G - U = R \cup R^*$.

Using U-partitions we first prove:

Lemma 3.1. *Given $G \in \mathscr{S}$, let $H_0 \subset G$ be a Smith graph. Then*

1^o *Each H_i $(i \geq 1)$ is a Smith graph or a reduced Smith graph*;

2^o $|U| \geq a + 1 - s$, *where $s = s(U)$ is a number of the Smith graphs among H_i's $(i \geq 1)$*;

3^o *if $u \in U$ and H_i $(i \geq 0)$ is a Smith graph then u has a neighbour in H_i, and $s \leq 3$.*

Proof. To prove 1^o, it suffices to prove that $\lambda_1(H_i) \leq 2$ for each $i \geq 1$. Suppose to the contrary, i.e. that $\lambda_1(H_i) > 2$ for some i. Let $u \in U$ be a vertex adjacent to some vertex from H_i, and let $G' = G - U'$, where $U' = U \setminus \{u\}$. Then G' contains a component in which u is a cut-vertex, adjacent also to some vertex from H_0. But then (by Lemma 2.4(1^o) applied on that component), and by the Interlacing theorem (see, for example, [**6**], p. 17) we obtain $\lambda_2(G) > 2$, a contradiction.

To prove 2^o, observe first that the multiplicity of any eigenvalue of a graph changes at most by 1 if any vertex is deleted (an immediate consequence of the Interlacing theorem). Also, for any graph and each eigenvalue (say μ of multiplicity k), we can find k vertices to be deleted to arrive at a subgraph in which μ is not an eigenvalue anymore (see, for example, [**6**], p. 136). So if we delete from U all vertices but one, and from each H_i $(i \geq 1)$ if it is a Smith graph one vertex, i.e. $|U| - 1 + s$ in total, then (by Lemma 2.4(3^o)) we obtain a subgraph of G in which $\mu = \pm 2$ is not an eigenvalue at all. Therefore $a \leq |U| - 1 + s$.

To prove 3^o, suppose to the contrary, i.e. that $\Gamma(u) \cap V(H_i) = \emptyset$ for some $u \in U$ and H_i, where H_i is a Smith graph for some $i > 0$ (otherwise, if $i = 0$, we are done, by definition of U). If so, since G is connected, there exists a vertex $u' \in U$ such that $\Gamma(u') \cap V(H_i) \neq \emptyset$. Let $G_{uu'} = \langle \{u, u'\} \cup V(H_0) \cup V(H_i) \rangle$. Then u' is a cut-vertex in $G_{uu'}$, and $G_{uu'} - u'$ contains two components $H_0 + u$ and H_i. By Lemma 2.4(1^o) $\lambda_2(G_{uu'}) > 2$, and by the Interlacing theorem $\lambda_2(G) > 2$, a contradiction. In addition, $s \leq 3$ since $\Delta(G) = 4$.

Lemma 3.2. *If $G \in \mathscr{S}'$ and if $H_0 \in \{C_4, S_5\}$ then*

1^o *Each pendant vertex of H_i $(i \geq 0)$ has a neighbour in U*;

2^o $s \leq 2$, *where $s = s(U)$ is the number of the Smith graphs among H_i's $(i \geq 1)$*;

3^o $|U| \geq a - 1$, *with equality only if $s = 2$ (and then $\deg(u) \geq 3$ for each $u \in U$).*

Proof. First, since $\delta(G) = 2$, 1^o immediately follows.

To prove 2^o, in view of Lemma 3.1(3^o), we only need to prove that $s \neq 3$. On contrary, assume that $s = 3$. Then $a \leq |U| + 2$ (by Lemma 3.1(2^o)). On the other hand, $||G|| = \sum_{i=0}^{3} ||H_i|| + 4|U| + ||\langle U \rangle||$. So $4a + 9 = ||G|| \geq 16 + 4(a-2) + 1$ (1 is added since $||G||$ is odd). Therefore, $||H_i|| = 4$ for each $i < 3$, $|U| = a - 2$, and $||\langle U \rangle|| = 0$ (otherwise $\Delta(G) > 4$). Consequently, $||H_3|| = 5$. So $H_i \in \{C_4, S_5\}$ for $i < 3$, while $H_3 = W_6$ (see Fig. 2.2). If $H_i = S_5$ for some $i < 3$ then all vertices in U are equi-coloured (otherwise, by Lemma 3.1(3^o), $\Delta(G) > 4$). But then $\delta(G) < 2$ (since W_6 in the role of H_3 has pendant vertices in both colours). So $H_0 = H_1 = H_2 = C_4$. If $a \in \{3, 4, 5\}$ then $|U| \leq 3$, and $\delta(G) < 2$ (at least one pendant vertex in $H_3 = W_6$ is pendant in G). Otherwise, if $a \in \{6, 7, 8\}$ then $|G| = \sum_{i=0}^{3} |H_i| + |U| \leq a + 16$, a contradiction (see Table 2.1).

The proof of 3^o follows from 2^o and Lemma 3.1(2^o). □

If $H_0 \in \{C_4, S_5\}$ then $G - U$ consists of $t + 1$ components (together with H_0), which are the Smith graphs or reduced Smith graphs (by Lemma 3.1(1^o)). Without loss of generality, let $H_1, \ldots, H_s$ be the Smith graphs; so $H_{s+1}, \ldots, H_t$ are the reduced Smith graphs. Let $s^* = s^*(U)$ be the number of the reduced Smith graphs among H_i's (so $s^* = t - s$). We also write $H_1^* = H_{s+1}, \ldots, H_{s^*}^* = H_{s+s^*}$. Therefore, the following "graphical" representation of graphs from $\mathscr{S}'$ arises (see Fig. 3.1).

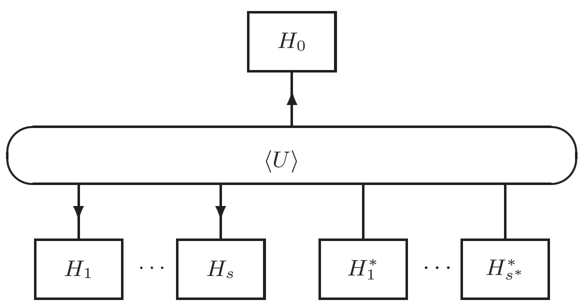


Figure 3.1. The structure of graphs $G \in \mathscr{S}'$ (or $\mathscr{S}$)

In Fig. 3.1 an oriented line stands if each vertex from U has at least one neighbour in the "terminal" subgraph (see Lemma 3.1(3^o)); an unoriented line stands if there are at least two edges between the subgraphs in question – note G is 2-connected (see [**3**] Proposition 2.1). Some important subgraphs of $G \in \mathscr{S}$, in view of the above representation, are:

- $R = H_0$, $R^* = \langle V_{\geq 2}(R) \rangle = \Big[\bigcup_{i=1}^{s} H_i \Big] \cup \Big[\bigcup_{j=1}^{s^*} H_j^* \Big]$, and $G - U = R \cup R^*$;

- $\langle U\rangle$ the *core* and $H_U = H_0 + U$ (in particular $H_u = H_0 + u$, where $u \in U$);
- $H_{i/j} = \langle V(H_i) \cup U \cup V(H_j)\rangle \quad (0 \le i < j \le t$; observe $H_{i/i} = H_i + U)$.

Clearly, for a fixed $H_0 \subset G$, U and H_i's ($i \ge 1$) are uniquely determined, and therefore we can define the following numeric quantities:

- $\ell(H_0) = |U|$;
- $\epsilon_c(U) = \sum_{i=1}^{s}(||H_i|| - |H_i| + 1)$ – the number of cycles among H_i's ($1 \le i \le s$);
- $\epsilon_u(U) = ||\langle U\rangle||$ – the number of *core edges*;
- $\mathrm{cross}(U; H_i) = |\{xy | x \in U, y \in V(H_i)|$ – the number of *cross edges* between corresponding subgraphs;
- $\epsilon_v(U) = \Big[\sum_{i=0}^{s}(\mathrm{cross}(U; H_i) - |U|)\Big] + \Big[\sum_{j=1}^{s^*}(\mathrm{cross}(U; H_j^*) - 2)\Big]$ – the total number of "extra" cross edges;
- $\epsilon(U) = \epsilon_c(U) + \epsilon_u(U) + \epsilon_v(U)$.

Note $\mathrm{cross}(U; H_i) \ge |U|$ (by Lemma 3.1(3^o)); also $\mathrm{cross}(U; H_j^*) \ge 2$ (since G is 2-connected). Further on we will omit H_0 and U from our notation if it is understood from the context.

Lemma 3.3. *Under the notation above, if $G \in \mathscr{S}$ and if H_0 is a Smith graph then*

1^o $s^* + \epsilon = ||G|| - |G| - (||H_0|| - |H_0|) - s(|U| - 1)$;

2^o $\sum_{u\in U} \deg(u) = (s + 1)|U| + 2s^* + 2\epsilon_u + \epsilon_v = (s + 1)|U| + 2(s^* + \epsilon) - 2\epsilon_c - \epsilon_v$.

Proof. Using Lemma 3.1, by a simple counting (based on Fig. 4.1) we obtain:

$$||G|| = ||H_0|| + [(s + 1)|U| + 2s^* + \epsilon_u + \epsilon_v] + \Bigg[\sum_{i=1}^{s}(|H_i| - 1) + \epsilon_c\Bigg] + \Bigg[\sum_{i=1}^{s^*}(|H_i^*| - 1)\Bigg],$$
$$|G| = |H_0| + |U| + \sum_{i=1}^{s}|H_i| + \sum_{i=1}^{s^*}|H_i^*|.$$

So 1^o follows, and also 2^o by counting, at each vertex of U, the edges incident to it. □

If $v \in V(H_i)$ $(0 \le i \le t)$, let $\mathrm{jump}(v) = \deg(v; G) - \deg(v; H_i)$. We say that v is a *jumper* (*non-jumper*) if $\mathrm{jump}(v) > 0$ (resp. $\mathrm{jump}(v) = 0$). In addition, v is a *simple jumper* if $\mathrm{jump}(v) = 1$, and v is a *k-jumper* if $\mathrm{jump}(v) \ge k$.

Given $G \in \mathscr{S}'$, observe that U is not only a separating set for G, but also for each $H_{i/j}$. Recall next that $\mathrm{diam}(G) \le 4$. So any two vertices of G are at distance

2 or 4 (or, 1 or 3), depending on colour classes they belong. So, if $v \in V(H_i)$ and $v' \in V(H_j)$ $(0 \le i < j \le t)$ belong to opposite colour classes then at least one of them is a jumper (otherwise $\mathrm{diam}(G) > 4$). So we have:

Lemma 3.4. *If* $G \in \mathscr{S}'$ *then*:

1^o *if* $v \in V(H_i)$ $(i \ge 1)$ *then* $\mathrm{dist}(v, U) \le 2$;

2^o *if* $\mathrm{diam}(H_i) \ge 5$ *then* $\epsilon_u + \epsilon_v \ge 1$;

3^o *if* $v \in V(H_i)$ *is a non-jumper then each vertex* $v' \in V(H_j)$ $(j \ne i)$ *in opposite colour is a jumper;*

4^o *if* $v, v' \in V(H_i)$ *are non-jumpers in opposite colours then each vertex in* $V(H_j)$ $(j \ne i)$ *is a jumper.*

Sketch proof. To prove 1^o, assume that there exists $v \in V(H_i)$ $(i \ge 1)$ such that $\mathrm{dist}(v, U) \ge 3$. Let $u \in U$ be a vertex such that $\mathrm{dist}(v, u) = \mathrm{dist}(v, U)$. Then u cannot be adjacent to all vertices of H_0 (otherwise $\Delta(G) > 4$). So $\mathrm{diam}(G) > 4$, a contradiction.

To prove 2^o consider two vertices, say v and w, of H_i $(i \ge 1)$ at distance 5 (in H_i). Clearly, they are in opposite colours. So their distance in G is 3 (since being non-adjacent). Now the shortest path among them in G should contain one or two (adjacent) vertices in U, and the rest easily follows.

Finally, if 3^o or 4^o do not hold then $\mathrm{diam}(G) > 4$, a contradiction. □

We now introduce the third partition to be used only in Section 4.

D-partition. Let $V(i) = \{v : \deg(v) = i\}$. Then

$$V(G) = V(\delta(G)) \cup V(\delta(G) + 1) \cup \cdots \cup V(\Delta(G))$$

is a *D-partition* of G. We also write $D(i) = \langle V(i) \rangle$.

Finally, needless to add, since the graphs in question are bipartite (so bicolourable), we can also observe yet another partition induced by the corresponding colouring. In view of it, U can be partitioned as $U = U_b \cup U_w$ (so vertices in U are either black or white). Let $\min(U) = \min\{|U_b|, |U_w|\}$. Further on, if not told otherwise, we will assume that $|U_b| \le |U_w|$.

3.2. Bounds on p, e and h (see Fig. 2.1)

Recall $\hat{E} = p + 2e + 2h$ and $\hat{F} = 114 + 4a + \nu_3$ (see Lemma 2.3). For any instance from Table 2.1 $\hat{F}$ can be immediately computed, but not $\hat{E}$ (the whole structure of a graph $G \in \mathscr{S}'$ is needed). So the best we can do then is to estimate $\hat{E}$ by estimating the above three quantities.

(P) – lower bounds on p. Let $p(e)$ be the number of paths in $G \in \mathscr{S}'$ of length 3 having e as the middle edge. If $e = uv$ then $p(e) = (d_u - 1)(d_v - 1)$, while $p = \sum_{uv \in E(G)} (d_u - 1)(d_v - 1)$. So p is of the form $x_1 y_1 + x_2 y_2 + \cdots + x_m y_m$, where $m = ||G||$ and the following conditions hold:

(a) $x_i, y_i \in \{1, 2, 3\}$, since $2 \leq d_w \leq 4$ for any $w \in V(G)$;

(b) if $k \in \{1, 2, 3\}$, then k is assigned in total to x_i $(1 \leq i \leq m)$ and/or to y_j $(1 \leq j \leq m)$, $f_k = \nu_{k+1}(k+1)$ times.

Therefore, we are in fact taking from the multi-set $\mathcal{F}(G) = [3^{f_3}, 2^{f_2}, 1^{f_1}]$, in the i-th draw (corresponding to the i-th edge of G) two elements at time to be assigned to variables x_i and y_i $(1 \leq i \leq m)$. Since the structure of G is not known (i.e. its edges), as a natural relaxation in estimating p, we will minimize just the expression of the form

$$(2) \qquad Z = x_1y_1 + x_2y_2 + \cdots + x_my_m, \quad \text{subject to (a) and (b).}$$

The following algorithm, more generally, minimizes Z by assigning the values to x_i's and y_i's from a multi-set $\mathcal{F}$ of $2m$ (non-necessarily distinct) numbers.

The minMAX-algorithm (minimizes Z, given $\mathcal{F}$ containing $2m$ non-necessarily distinct reals):

- *Step* 1: Set $i \leftarrow 1$ and $Z \leftarrow 0$;
- *Step* 2: Assign to x_i the smallest and to y_i the largest element from $\mathcal{F}$ (say $\hat{m}$ and $\hat{M}$, respectively); then set $Z \leftarrow Z + x_iy_i$ and update $\mathcal{F}$ by removing $\hat{m}$ and $\hat{M}$ from it;
- *Step* 3: If $i < m$ then set $i \leftarrow i + 1$ and go to Step 2, or else stop.

To prove the optimality, observe first that any algorithm for valuating Z can be structured as one above. The key decision appears in Step 2 (selection strategy). So let us examine whether this strategy can be changed to decrease Z. For this aim it suffices to consider Step 2 only in the first passing (then the rest follows by recursion). Let $\hat{m}$ and $\hat{M}$ be the smallest and the largest elements from $\mathcal{F}$, respectively. Then for some i and j we have that $(x_i, y_i) = (a_i, \hat{M})$ and $(x_j, y_j) = (\hat{m}, b_j)$. But then $(a_i - \hat{m})(b_j - \hat{M}) \leq 0$, or equivalently $\hat{m}\hat{M} + a_ib_j \leq a_i\hat{M} + \hat{m}b_j$. So it turns that in the first passing we can take $(x_1, y_1) = (\hat{m}, \hat{M})$, and this guaranties the optimality of our algorithm.

Remark 3.5. (i) It is easy to see that minimum of Z (i.e. $Z_{\min}$) can be obtained, provided $\mathcal{F}$ is an ordered list, by summing up the products of two members of the list in symmetric positions (note, the length of the list is even). So minMAX-algorithm and MAXmin-algorithm do the same.[3]

[3]It can be proved that maximum of Z (i.e. $Z_{\max}$) can by obtained by adopting the MAXMAX, or equivalently, the minmin strategy.

(ii) Usually, we can have some pre-requests, say, first to fix the values for certain pairs (x_i, y_i), and then to put focus on the rest. For example, if $\mathcal{F} = \{3^9, 2^5, 1^4\}$ then $Z_{\min} = \text{minMAX}(\mathcal{F}) = 42$. On the other hand, if the product "$3 \cdot 3$" has to appear $\alpha = 2$ times, while "$3 \cdot 2$" $\beta = 3$ times, then we first reduce $\mathcal{F}$ to $\mathcal{F}' = \{3^2, 2^2, 1^4\}$, and obtain $Z'_{\min} = 9\alpha + 6\beta + \text{minMAX}(\mathcal{F}') = 46$. Next, if the products "$1 \cdot 1$" have to appear $\alpha^* = 1$ times, while "$1 \cdot 2$" $\beta^* = 2$ times, then we first reduce $\mathcal{F}$ to $\mathcal{F}'' = \{3^9, 2^3, 1^0\}$, and obtain $Z''_{\min} = 1\alpha^* + 2\beta^* + \text{minMAX}(\mathcal{F}'') = 50$. It is noteworthy that for both constraints imposed that $Z'''_{\min} = 50$ (again!).

Assume now that for a graph $G \in \mathscr{S}'$ we know that there are α edges of degree $(4,4)$ (*red* edges), β edges of degree $(4,3)$ (*blue* edges), and also α^* edges of degree $(2,2)$ (*yellow* edges), β^* edges of degree $(2,3)$ (*green* edges). Edges which are coloured in red or blue will be addressed as *RB-edges*, while those which are coloured in yellow or green will be addressed as *YG-edges*. In view of these colourings, and Remark 3.5(ii), the input multi-set $\mathcal{F} = \{3^{4\nu_4}, 2^{3\nu_3}, 12\nu_2\}$ is reduced to

$$\mathcal{F}' = \{3^{4\nu_4-2\alpha-\beta}, 2^{3\nu_3-\beta-\beta^*}, 1^{2\nu_2-2\alpha^*-\beta^*}\},$$

and then the minMAX-algorithm is applied on it. Let $F(\alpha,\beta,\alpha^*,\beta^*)$ be the minimum of the target function for a given values of its arguments. Then

$$F(\alpha,\beta,\alpha^*,\beta^*) = 9\alpha + 6\beta + 2\beta^* + 1\alpha^* + \mathrm{minMAX}(\mathcal{F}').$$

We also write $f(\alpha,\beta) = F(\alpha,\beta,0,0)$ and $f^*(\alpha^*,\beta^*) = F(0,0,\alpha^*,\beta^*)$. From the above considerations we can immediately deduce the following constraints:

$$4\nu_4 - 2\alpha - \beta \geq 0,\ 3\nu_3 - \beta - \beta^* \geq 0,\ 2\nu_2 - 2\alpha^* - \beta^* \geq 0.$$

Observe also that $f(\alpha+1,\beta) \geq f(\alpha,\beta+1)$, and $f^*(\alpha^*+1,\beta^*) \geq f^*(\alpha^*,\beta^*+1)$ if the latter constraints do hold. Note, by introducing the parameters α^*, β^*, we have, in fact, plugged in the features of minmin or MAXMAX strategies in our modified algorithm.

From the discussion above, if not told otherwise, we will assume that

$$p \geq \max\{f(\alpha,\beta), f^*(\alpha^*,\beta^*)\}, \tag{3}$$

over some constraints guaranteed by the structure of a tentative graph. In this respect, observe that $\lambda_1(G) < 3$ if $\alpha = \beta = 0$ (namely, $\lambda_1(G) \leq \max_{uv\in E(G)} \sqrt{d_u d_v}$ – see, for example, [**6**], p. 241). Since $f(\alpha+1,\beta) \geq f(\alpha,\beta+1)$, if not told otherwise, we will assume that $\alpha \geq 0$ and $\beta \geq 1$.

(E) – lower bounds on *e*. Let $Q_1, Q_2, \ldots, Q_q$ be the quadrangles of G. Denote by $e(Q_i)$ the number of subgraphs (of G) obtained by adding to Q_i a hanging edge. Then $e(Q_i) = \ell(Q_i) + \kappa_i$, where $\ell(Q_i) = |U_i|$ and κ_i is the number of vertices in U_i having just two neighbours in Q_i (note, since G is bipartite, any vertex in U_i can have at most two neighbours in Q_i). Note, if $\kappa_i > 0$ then $K_{2,3} \subset G$. Next we obtain

$$e = \sum_{i=1}^{q} e(Q_i) = \sum_{i=1}^{q} \ell(Q_i) + \sum_{i=1}^{q} \kappa_i \geq q\ell_{\min} + \kappa, \tag{4}$$

where $\ell_{\min} = \min_{1\leq i\leq q}\{\ell(Q_i)\}$ and $\kappa = \sum_{i=1}^{q} \kappa_i$. Note, if $\kappa_i \geq 1$ for some i then $\kappa \geq 3$ because then at least 3 quadrangles have the same property.

(H) – lower bounds on *h*. There are many ways in which hexagons can arise in $G \in \mathscr{S}'$. To identify some of them, assume that $H_0 = C_4$. Let $h(v)$ $(h(e))$ be the number of hexagons passing through vertex v (resp. edge e). More generally, let $h(H)$ be the number of hexagons in H ($\subseteq G$); clearly, $h(H) \leq h(G)$.

(a) If $vv' \in E(H_i)$ ($i \geq 1$) then $h(vv') \geq \text{jump}(v) \cdot \text{jump}(v')$. Indeed, if v is a jumper (and also v') then v (resp. v') is adjacent to at least one vertex in U, which in turn is adjacent to at least one vertex in $V(H_0)$. Since neither of these six vertices coincide (at each "level" they belong to opposite colour classes), and since the encountered vertices in H_0 are adjacent, we are done. In addition, if two vertices in U are adjacent (so $\epsilon_u > 0$) we then encounter an additional hexagon. Therefore

$$h \geq \sum_{vv' \in \cup_{i=1}^{t} E(H_i)} \text{jump}(v) \cdot \text{jump}(v') + \epsilon_u. \tag{5}$$

It is also worth mentioning that some additional hexagons can arise by "aggregating" two quadrangles with just one common edge (as is the case if the number of quadrangles in G is too large with respect to the order, or size G; more details will be given in Section 4).

(b) Consider the distance partition of a vertex v. If we can find a vertex, say w, in $V_3(v)$ so that two paths starting at v meet at w but not before, then a hexagon arises. In the context of jumpers, if $v \in V(H_i)$ ($i \geq 1$) is a 2-jumper but not a 3-jumper, then either $h(v) \geq 2$ (two vertices of H_0 are in role of w), or otherwise $q(v) \geq 1$ (i.e. we encounter a quadrangle passing through v). Moreover if v is 3-jumper but not a 4-jumper then $h(v) \geq 4$ and $q(v) \geq 1$. Finally, if v is a 4-jumper then $h(v) \geq 8$ and $q(v) \geq 2$.

(c) Let $H \subset G$. In many situations $h(H)$ can be found just by a computer search. In some discussions in Section 4 the following two graphs are of interest: A_7 (it is obtained from C_6 by adding a vertex adjacent to three mutually non-adjacent vertices of C_6), and B_8 (it consists of three copies of C_4 having a common edge – a *book* graph). They are small but $h(A_7) = 4$, while $h(B_8) = 3$ (each has three pairs of quadrangles sharing a common edge). On the other hand, if two quadrangles share two common edges (as in $K_{2,3}$) then $h(K_{2,3}) = 0$.

3.3. Further spectral tools

As already noted, the multiplicity of any eigenvalue of some graph changes at most by one if any vertex is deleted. Consequently, if $v \in V(G)$ and $\mu \in Sp(G)$ then

$$m(\mu; G) - 1 \leq m(\mu; G - v) \leq m(\mu; G) + 1.$$

Recall, v is the *downer* (*neutral*, *Parter*) vertex in G with respect to μ if $m(\mu; G-v)$ is equal to $m(\mu; G) - 1$ (resp. $m(\mu; G)$, $m(\mu; G) + 1$). The following result is taken from [**10**] (see Corollary 3.2).

Lemma 3.6. *Given a connected graph G in which v is a cut-vertex (so $G - v = \cup_{i \in I} H_i$, where each H_i is connected). Let $k_i = m(\mu; H_i)$, where $\sum_{i \in I} k_i > 0$. Then v*

is the Parter vertex in G if and only if it is the Parter vertex in $H_j + v$ for some $j \in I$.

Observe next that in bipartite graphs there are no neutral vertices for $\mu = 0$ (due to symmetry of the spectrum with respect to the origin). In view this we have:

Lemma 3.7. *Consider a U-partition of a graph $G \in \mathscr{S}'$, with $H_0 = C_4$ and $|U| \geq 3$. Then*

$$\eta(G - U) \geq |G| - 2a - |U|. \tag{6}$$

Moreover, $\eta(G - U)$ and $n - 2a - |U|$ have the same parity.

Proof. Assume first that each vertex of U is adjacent to just two vertices of H_0. But then $\lambda_1(H_0 + U) > 3$ (see Lemma 3.10) below). So there exists a vertex $u \in U$ adjacent to just one vertex of H_0. Let $H_u = H_0 + u$ and $U' = U \setminus \{u\}$. Then u is the Parter vertex in H_u (a computational observation[4]), and also in $G' = G - U'$ (by Lemma 3.6 – observe the component of G' containing u). Therefore $\eta(G - U) = \eta(G') + 1$. On the other hand, $|\eta(G) - \eta(G'| \leq |U| - 1$, and (6) follows. The "parity claim" follows since the nullity in bipartite graphs changes just by 1 each time a vertex of G is deleted from it.

Lemma 3.8. *If $H \subset G$ then*

$$\eta(H) \geq |H| - 2a - 2. \tag{7}$$

If equality holds and if $|H| \geq 2a + 2$, then each vertex in $V(G) \setminus V(H)$ is adjacent to a vertex in $V(H)$. In particular, if $v \in V(H_i) \subset V(G) \setminus V(H)$ then v is adjacent to a vertex in $U \cap V(H)$.

Proof. Clearly, $\text{rank}(A(G)) \geq \text{rank}(A(H))$. Therefore, $|G| - \eta(G) \geq |H| - \eta(H)$. So the first claim follows since $\eta(G) = |G| - 2a - 2$. For the second claim observe first that the vertices in $V(G) \setminus V(H)$ belong to the star cell of G (for $\mu = 0$) if $\eta(G) \geq |G| - |H|$, i.e. if $|H| \geq 2a + 2$ (see, for example, [**5**], Chapter 7). If so each star complement of H is also a star complement of G. The rest immediately follows from the domination property of vertices from star complements ([**5**], Theorem 7.3.1). □

The next result is taken from [**9**] (see Theorem 4.3). Since $G \in \mathscr{S}'$ is a reflexive graph we can put some further constraints on it. Recall, any graph H_i ($i \leq s$) is a Smith graph (so with index equal to 2). Let $\mathbf{x}_i$ be the eigenvector of H_i corresponding to its index (see Fig. 2.2 for the entries of $\mathbf{x}_i$'s). If $u \in U$ let

$$\sigma_i(u) = \sum_{v \in \Gamma_i(u;G)} \mathbf{x}_i(v),$$

where $\Gamma_i(u; G) = \Gamma(u; G) \cap V(H_i)$. Note $\Gamma_i(u; G) \neq \emptyset$ (see Lemma 3.1(3^o)). Next let

$$\sigma_{i/j}(u) = \sigma_i(u)/\sigma_j(u).$$

[4] For this aim, here and later, we will use newGRAPH [**11**] for similar computations.

Lemma 3.9. *Under the assumptions above, for any i and j ($0 \leq i < j \leq s$) we have:*

$$\textit{if} \quad \sigma_{i/j}(u') \neq \sigma_{i/j}(u'') \quad \textit{for some } u', u'' \in U \quad \textit{then} \quad \lambda_2(G) > 2.$$

Equivalently, if $\lambda_2(G) = 2$ then $\sigma_{i/j}(x)$, as a function in x on U, is a constant.[5]

3.4. Some forbidden and/or constraint configurations

If $a \geq 6$, as will be seen in Section 4, almost all instances from Table 2.1 can be rejected without examining too much the structure of tentative graphs $G \in \mathscr{S}'$. On the other hand, for $a \leq 5$ and $H_0 = C_4$ it turns that we have to examine H_i's ($i \leq s$). Then, since $|U| \geq a - 1$ (by Lemma 3.2(3^o)), it turns that cases with $|U| = 4$ (or even those with $|U| = 3$) deserve special attention. In addition, we then assume that $a \geq 4$ (otherwise, if $a = 3$, the situation is rather simple). In what follows we will need some results involving κ and h (see Subsection 3.2).

Lemma 3.10. *Let $\hat{U} = \{x \in U : \sigma_0(x) = 2\}$. Then $|\hat{U}| \leq 2$, and if $|\hat{U}| = 2$ we have:*

1^o *if two vertices in $\hat{U}$ belong to the same colour class then $\kappa + h \geq 6$ and $q \geq 6$;*

2^o *if two vertices in $\hat{U}$ belong to opposite colour classes then $\kappa + h \geq 7$ and $q \geq 5$.*

Proof. Let $H = H_0 + \hat{U}$. If $|\hat{U}| \geq 3$ then $\lambda_1(G) \geq \lambda_1(H) > 3$ and the first claim follows. Next, 1^o follows since $H = K_{2,4}$ (then $\kappa \geq 6$ since $q \geq 6$). Finally, 2^o follows since $H = K_{3,3} - e$ (note $H \neq K_{3,3}$ – otherwise $\lambda_1(G) > 3$); next, $\kappa \geq 5$ since $q = 5$, while $h \geq h(H) = 2$.

Lemma 3.11. *If $a \in \{4, 5\}$ and $|U| = 4$ we have:*

- *if $\kappa + h \geq 7$ then $\hat{E} > \hat{F}$ for all instances* (**5** − **23**) *from Table* 2.1;
- *if $\kappa + h = 6$ then $\hat{E} > \hat{F}$ for instances above but* (**8**) *and* (**19**).

Proof. From Subsection 3.2 we have that $\hat{E} \geq (f(0,1) + 2q|U|) + 2(\kappa + h)$. Here $f(0,1)$ is computed by minMAX-algorithm (see Subsection 3.2); to compute q we use Lemma 2.2 (see also Tables 4.8 – 4.12). Next we easily check that $(f(0,1) + 2q|U|) + 2(\kappa + h) > \hat{F}$ under the imposed conditions. □

Since G is reflexive, the same holds for $H_{i/j} \subset G$ (since reflexivity is a hereditary property). So a necessary condition for G to be reflexive is that the function $\sigma_{i/j}(x)$ is a constant on U (see Lemma 3.9). This fact we first consider in the next lemma

Lemma 3.12. *Let $H_0 = C_4$, $H_1 = C_{2k}$ ($k \geq 2$) and $|U| = 4$. If $a \geq 4$ then $\sigma_{0/1}(x) = 1$. In particular, if $k = 2$ then $\sigma_0(x) = 1$ (and also $\sigma_1(x) = 1$).*

Sketch proof. Assume first that $\sigma_0(x) = 1$. Then $\sigma_1(x) = y$ (by Lemma 3.9). If $y = 1$ we are done. Otherwise, if $y \geq 2$ then $\epsilon_v \geq 4$. So $\epsilon \geq \epsilon_c + \epsilon_v \geq 5$. On the

[5] Note, edges joining vertices in U do not have any influence in validity of the Lemma.

other hand (by Lemma 3.3(1^o)) $\epsilon \leq s^* + \epsilon = ||G|| - |G| - s(|U| - 1) \leq 5$ if $a \in \{4, 5\}$ (see Table 2.1; note also that $|U| > 4$ if $a \geq 6$ (by Lemma 3.2(3^o)). Equality arises only if $a = 4$, $|G| = 17$, $s = 1$ and $s^* = 0$, a contradiction (since $|G|$ cannot be odd).

Secondly, assume that $\sigma_0(x) \neq 1$. Then $\sigma_0(u) = 2$ for some $u \in U$. So $\sigma_{0/1}(u) \geq 1$ (since $\deg(u) \leq 4$). On the other hand (by Lemma 3.10), there exists $u' \in U$ for which $\sigma_0(u') = 1$, and therefore $\sigma_{0/1}(u') \leq 1$. So $\sigma_{0/1}(x) = 1$ (by Lemma 3.9), whence $\sigma_1(u) = 2$. Finally, let $k = 2$ and $\sigma_0(u) = 2$ for some $u \in U$. If so $\sigma_1(u) = 2$, and consequently $\lambda_1(G) > \lambda_1(H_{0/1}) \geq 3$, a contradiction.

Lemma 3.13. *Let* $H_0 = C_4$, $|U| = 4$ *and* $a \in \{4, 5\}$. *Then* $H_i \neq C_4$ $(i \geq 1)$.

Sketch proof. Suppose to the contrary, that $H_1 = C_4$. Then, by Lemma 3.12, $H_{0/1}$ (ignoring the edges in $\langle U \rangle$) should have at least one of the patterns from Fig. 3.2, but $\mathbf{p_9} - \mathbf{p_{11}}$ (then, by Lemma 3.4(3^o), $\mathrm{diam}(G) > 4$). To reject the remaining patterns but $\mathbf{p_2}$, we will use Lemma 3.11. Note here that $\nu_3 \geq 4$ for all instances in question (i.e. $\mathbf{5} - \mathbf{23}$) but (**8**) and (**19**) (for them $nu_3 = 2$; so they cannot contain patterns which make ν_3 greater than 4).

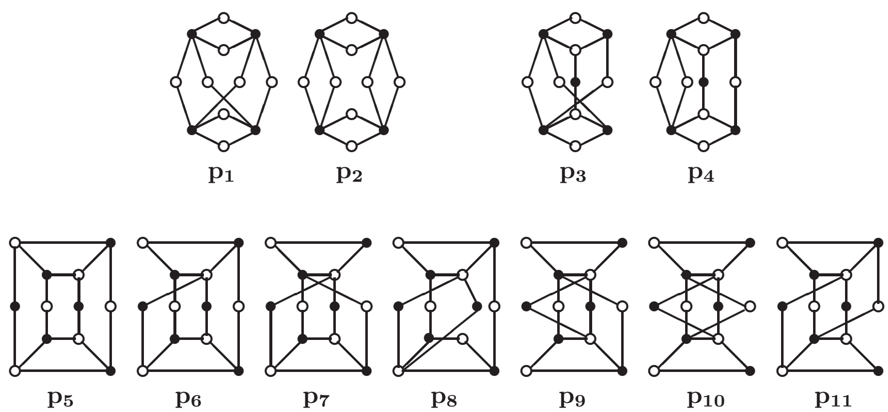


Fig. 3.2. The (reduced) patterns of $H_{0/1}$

(i) patterns $\mathbf{p_1}$, $\mathbf{p_3}$ and $\mathbf{p_7}$: Now $h(H_{0/1}) \geq 7$ (a computational observation) and we are done.

(ii) pattern $\mathbf{p_2}$: Let $H_v = H_{0/1} + v$, where $v \notin V(H_{0/1})$ is a jumper. Then H_v is feasible only if $\deg(v; H_v) = 2$ and $K_{2,3} \not\subset H_v$ (otherwise $\lambda_1(H_v) > 3$ or $\lambda_2(H_v) > 2$). Let v' be another vertex of G such as v (it exists since G is 2-connected). But then $\lambda_1(H_v + v') > 3$, a contradiction.

(iii) pattern $\mathbf{p_4}$: Now $h(H_{0/1}) \geq 3$ (since $\min(U) = 1$) and $\nu_3 \geq 4$ (see Fig. 3.2). So it suffices to prove that $\kappa + h \geq 6$. Let u be a (unique) black vertex in U, and let $Y = V(G) \setminus V(H_{0/1})$. There exists in Y a black vertex, say v, being a jumper (otherwise, u is a cut-vertex in G). If jump$(v) \geq 2$ or $\epsilon_u \geq 2$, then $\kappa + h \geq 6$ and we are done. Otherwise, we have:

If $a = 4$, let $H_v = H_{0/1} + v$, and observe that $\eta(H_v) \geq 3$ (by Lemma 3.8). On the other hand, $\eta(H_v) = 3$ (a computational observation). So it turns that each vertex in Y is a jumper (by Lemma 3.8). If $||\langle Y \rangle|| \geq 2$ then four additional hexagons arise. So $h \geq 7$ and we are done. Otherwise, $s^* \geq 4$, and then $s^* + \epsilon_c \geq 5$. On the other hand, $s^* + \epsilon = ||G|| - |G| - s(|U| - 1) \leq 4$ (by Lemma 3.3(1^0)).

If $a = 5$, then $s = 2$ (so $\kappa \geq 2$), and H_2 is a Smith graph. In addition, each black vertex of H_2 is a jumper (by Lemma 3.4(3^o)), and also at least one white. So an additional hexagon arises, and $\kappa + h \geq 6$, as required.

(iv) patterns $\mathbf{p_5}$, $\mathbf{p_6}$ and $\mathbf{p_8}$: Now $h(H_{0/1}) \geq 4$ (since $\min(U) = 2$), and $\nu_3 \geq 4$ for all patterns (see Fig 3.2). So it suffices to prove that $\kappa + h \geq 6$. Let P be a shortest path in G joining two distinct vertices of U, not passing through H_0, nor H_1 (so $H_P = H_{0/1} + V(P)$ is an induced subgraph of G). Let $\ell(P)$ be the length of P. If $\ell(H_P) \leq 3$ then two additional hexagons arise. So $h \geq 6$, as required.

For $\mathbf{p_5}$, let $\ell(P) = 4$. Then (by Lemma 3.8) $\eta(H_P) \geq 5$ if $a = 4$, or $\eta(H_P) \geq 3$ if $a = 5$. If $a = 4$ then $\eta(H_P) = 3$ (a computational observation). So $a \neq 4$. If $a = 5$, then all interior vertices of P are jumpers (see Lemma 3.8), again a contradiction. If $l(P) \geq 5$ then $\eta(H_P) \leq 3$ (a computational observation), while $\eta(H_P) \geq 4$ (by Lemma 3.8).

For $\mathbf{p_6}$, $h(H_{0/1}) \geq 6$ and we are done.

For $\mathbf{p_8}$, let $\ell(P) = 4$. Then either $\kappa \geq 2$, or otherwise $\eta(H_P) = 5$ (a computational observation) but only if $a = 4$. In the former case $\kappa + h \geq 6$ and we are done. In the latter case $\eta(H_P) \geq 5$ (by Lemma 3.8), and thus the central vertex of P which should be a non-jumper by the choice of P is a jumper (by Lemma 3.8). Finally, if $\ell(P) \geq 5$ then we encounter on P two adjacent non-jumpers, a contradiction (by Lemma 3.4(4^o)).

Lemma 3.14. *Let $H_0 = C_4$ and $a \geq 4$. Then $H_i \neq S_5$ $(i \geq 1)$.*

Proof. Suppose to the contrary, that $H_1 = S_5$. Then $\min(U) = 0$, for otherwise $\Delta(G) > 4$ (so all vertices in U are white). Clearly $|U| \leq 4$, and $|U| \geq 3$ with equality if $a = 4$ and $s = 2$ (see Lemma 3.2(3^o)).

Assume that $\sigma_0(x) \neq 1$. Then for some $u \in U$ $\sigma_0(u) = 2$. So $\sigma_1(u) \leq 2$ (note $\deg(u) \leq 4$), and $\sigma_{0/1}(u) \geq 1$. By Lemma 3.10, there exists $u' \in U$ such that $\sigma_0(u') = 1$, and therefore $\sigma_{0/1}(u') \leq 1$. So $\sigma_{0/1}(x) = 1$ (by Lemma 3.9), and $\sigma_1(u) = 2$. But then we have: if $|U| = 3$ then $a = 4$ and $s = 2$, and therefore $\deg(u) > 4$; if $|U| = 4$, then some vertex in U cannot not have a (black) neighbour in H_0. So $\sigma_0(x) = 1$, and we have:

If $|U| = 3$ then $\sigma_1(u) \geq 2$ for some $u \in U$ (otherwise $\delta(G) < 2$). But then $\sigma_1(x) = y$, where $y \geq 2$. Next $y < 3$ (otherwise, $\deg(u) > 4$, since $s = 2$). If $y = 2$, then each vertex in U is of degree 4 (since $s = 2$). Let H be a subgraph of G obtained by extending $H_{0/1}$ by three cross-edges between U and H_2. But then

$\lambda_1(G) > \lambda_1(H) > 3$.

If $|U| = 4$ then $\sigma_{0/1}(x) = 1$ (otherwise, $\sigma_1(x) = y$, where $y \geq 2$, and then $\lambda_1(H_{0/1}) > 3$). So $\sigma_1(x) = 1$, and this gives rise to the unique graph $H_{0/1}$. Let $H_v = H_{0/1} + v$, where $v \notin V(H_{0/1})$ is a jumper, and also a black vertex. Note first that $\eta(H_v) = 2$ independently of $\deg(v; H_v)$ (a computational observation). If $a = 4$ then $\eta(H_v) \geq 4$ (by Lemma 3.8), and we are done. If $a = 5$ then $\eta(H_v) = 2$, and (by the same lemma) each vertex in $Y = V(G) \setminus V(H_v)$ is adjacent either to v or to a vertex in U (depending on its colour class). Observe next that $|Y| \geq 6$ (see Table 2.1). If all vertices in Y are black then each of them is a 2-jumper, and then $\deg(u) > 4$ for some $u \in U$. So each white vertex in Y is adjacent to v, and also to each black vertex in Y (by putting it in the role of v). So $\langle Y \cup \{v\}\rangle$ is a bi-complete graph on at least 7 vertices, a thus $\lambda_1(G) > 3$. □

For the sake of simplifications, in what follows we say that H_0 is a *minimizer* if, whenever $H_0, H_0' \subset G$ and $H_0 = H_0'$, then $|U| \leq |U'|$. Next we prove:

Lemma 3.15. *Let $H_0 = C_4$, $|U| = 4$ and $a \in \{4, 5\}$. If $H_0 \subset G$ is a minimizer then $H_i \neq W_6$ ($i \geq 1$).*

Sketch proof. Suppose to the contrary, that $H_1 = W_6$. Then we have:

(i) $\sigma_0(x) = 1$. Then $\min(U) \neq 0$ (otherwise $\delta(G) < 2$). If $\min(U) = 1$ let u be a (unique) black vertex in U. Then $\sigma_1(u) = 2$ (other possibilities can be easily rejected), and therefore $\sigma_1(x) = 2$. But then $\kappa + h \geq 7$ (a computational observation obtained over all $H_{0/1}$ patterns). So $\min(U) \neq 1$ (see Lemma 3.11).

If $\min(U) = 2$ then $\sigma_1(x) = y$, where $y \leq 2$. If $y = 1$ then all vertices in H_0, and pendant vertices in H_1, are jumpers, while both interior vertices in H_1 are non-jumpers (see Lemma 3.4(4^o)). So $H_{0/1}$ is determined up to edges in $\langle U\rangle$. Let $a = 4$. If $\epsilon_u \geq 1$ then $\eta(H_{0/1}) = 2$ (a computational observation). On the other hand, $\eta(H_{0/1}) \geq 4$ (by Lemma 3.8). If $\epsilon_u = 0$, let $H_v = H_{0/1} + v$, where $v \notin V(H_{0/1})$ is a jumper. Now $\eta(H_v) = 3$ (a computational observation). On the other hand, $\eta(H_v) \geq 5$ (by Lemma 3.8). So $a \neq 4$, and let $a = 5$. If $\epsilon_u \geq 1$ then $\eta(H_{0/1}) = |H_{0/1}| - 2a - 2$ (a computational observation), and therefore all vertices in $Y = V(G) \setminus V(H_{0/1})$ are jumpers. So $h \geq ||H_2|| + 2\epsilon_u \geq 7$, as required (note $||H_2|| \geq 5$, by Lemmas 3.13 and 3.14). If $\epsilon_u = 0$ let v and v' be two adjacent vertices in H_2. Then, as before, we have $\eta(H_{0/1} + v + v') = 2$ (so it is too small, since $\eta(H_{0/1} + v + v') \geq 4$, by Lemma 3.8). If $y = 2$ there are several patterns for $H_{0/1}$, and then $\lambda_1(H_{0/1}) \geq 3$ (if both interior vertices in H_1 are non-jumpers), or otherwise, $\kappa + h \geq 7$ (a computational observation obtained over all $H_{0/1}$ patterns).

(ii) $\sigma_0(x) \neq 1$. So $\sigma_0(u) = 2$ for some $u \in U$. By Lemma 3.10, there exists $u' \in U$ such that $\sigma_0(u') = 1$. But then, by Lemma 3.9, $\sigma_1(u) = 2\sigma_1(u')$. So $\sigma_1(u) = 2$, and $\sigma_{0/1}(x) = 1$. Observe next that $\sigma_0(x) = 1$ for all $x \in U$ but u, since otherwise, by Lemmas 3.10 and 3.11, we are done for all instances in question (with (**8**) and (**19**) included by considering q, if necessary). In addition, we are done if u is adjacent to an interior vertex in W_6 (then $\delta(G) < 2$). If u is adjacent to two pendant vertices in H_1 then a quadrangle, say Q^*, arises with $l(Q^*) \geq 4$, and we easily obtain that $\kappa \geq 3$, $h \geq 3$ and $q \geq 5$, as required.

Lemma 3.16. *Let $H_0 = C_4$, $|U| = 4$, $a = 4$ and $n \in \{18, 19\}$. Then $H_1 \neq C_k$ ($k = 6, 8, 10$).*

Sketch proof. Suppose to the contrary, that $H_1 = C_k$ for some $k \in \{6, 8, 10\}$. Then $s = 1$ (otherwise $|H_2| \leq 5$, an obvious contradiction). Observe next that $\eta(G-U) \geq 6$ (by Lemma 3.7). Since $s^* + \epsilon = 22 - n$ (by Lemma 3.3(1^o)), it follows that $s^* \leq 3$ (since $\epsilon_c = 1$). But then $\eta(H_1) \neq 0$ (otherwise $\eta(G-U) \leq 5$; see Remark 2.5). Therefore, let $H_1 = C_8$.

(i) $n = 18$: Then $s^* = 2$, and $H_1^* = H_2^* = P_1$ (otherwise $\eta(G-U) = 4$ is too small, i.e. less than 6). Therefore $\epsilon_u + \epsilon_v = 1$. By Lemma 3.9 $\sigma_{0/1}(x) = 1$, whence $\sigma_0(x) = 1$ (otherwise $\epsilon_v \geq 2$). So $\sigma_1(x) = 1$. Let v and v' be the vertices corresponding to H_1^* and H_2^*. If $\min(U) = 0$ then $\epsilon_u = 0$, and thus $\deg(v) = 3$. Also, by Lemma 3.4(4^o), all jumpers in H_1 are simple jumpers. Consequently, $h \geq h(v) \geq 6$ (since $h(H_0+U)+v) \geq 4$, $h((H_1+U)+v) \geq 2$), and also $\nu_3 > 2$. But then we are done (by Lemma 3.11). If $\min(U) = 1$ then $\operatorname{diam}(G) > 4$ (by Lemma 3.4(4^o)). If $\min(U) = 2$ then $\epsilon_v = 0$, and $\deg(v) = \deg(v') = 2$. In addition, all vertices in H_0 must be simple jumpers (by Lemma 3.4(4^o)). But then $h \geq 6$ (since $h((H_0+U)+v), h((H_1+U)+v') \geq 3$). So $h \geq 6$ and $\nu_3 > 2$, and we are done as before.

(ii) $n = 19$: Then $\eta(G-U) < 7$ unless $s^* = 3$ and $H_i^* = P_1$ for each i. But then $s^* + \epsilon_c \geq 4$, a contradiction.

Lemma 3.17. *Let $H_0 = C_4$, $|U| = 4$, $a = 4$ and $n \in \{18, 19\}$. If $H_0 \subset G$ is a minimizer then $H_1 \neq W_k$ ($k = 7, 8, \ldots, 11$).*

Sketch proof. Suppose to the contrary, that $H_1 = W_k$ for some $k \in \{7, 8, 9, 10, 11\}$. Then, as in the proof of Lemma 3.16, $s = 1$, $\eta(G-U) \geq n - 12 \geq 6$ and $s^* + \epsilon = 22 - n \leq 4$. Also, $s^* \geq 1$ (otherwise $\eta(G-U) \leq 5$). Therefore $k \leq 9$ if $n = 18$, and $k \leq 10$ if $n = 19$. Since $k \geq 7$ at least one interior vertex of W_k is a jumper (see Lemma 3.4(1^o)). Depending on the function $\sigma_0(x)$, we next have:

(i) $\sigma_0(x) = 1$. By Lemma 3.9, $\sigma_1(x) = y$, where $y \geq 2$ (since $\sigma_1(u) \geq 2$). In addition, $y < 3$ (otherwise $s^* + \epsilon_v \geq 5$). So $y = 2$ and $\sigma_{0/1}(x) = \frac{1}{2}$. Consequently, each vertex in U is adjacent either to two pendant vertices of W_k (so is of *type* (a)), or to at least one of interior vertices of W_k (so is of *type* (b)). Clearly, at least two vertices in U are of type (a), and at least one of type (b). Let $H = (H_0 \cup H_1) + \{u_1, u_2\}$, where $u_1, u_2 \in U$.

Let k be odd. Assume first that $k = 7$, and that u_1 is of type (a) and gives rise to a hexagon, while u_2 of type (b). Then $\eta(H) = 3$ (a computational observation over all patterns). But then equality holds in (7). So each vertex not in H is adjacent to u_1 or u_2 (by Lemma 3.8), and therefore $\Delta(G) > 4$. So each vertex of type (a) (in U) gives rise to a quadrangle in $H_1 + U$, and for each of them, say Q, $l(Q) \geq 4$ (since H_0 is a minimizer). But then $\lambda_2(H_1 + U) > 2$ (by Lemma 2.4(1^o)).

Secondly, if $k = 9$, assume that u_1 and u_2 are of different types. Then $\eta(H) = 3$ (a computational observation over all patterns), a contradiction ($\eta(H) \geq 5$, by Lemma 3.8).

Let k be even. Then, we can choose u_1 and u_2 to be of type (a), and in opposite colours. Then two (separated) quadrangles arise in H, and consequently we easily obtain (by Lemma 2.4(1^o)) that $\lambda_2(H) > 2$.

(ii) $\sigma_0(x) \neq 1$. Let $\hat{U} = \{x \in U : \sigma_0(x) = 2\}$. Since $|\hat{U}| \leq 2$ (by Lemma 3.10), there exists $u' \in U \setminus U'$ with $\sigma_0(u') = 1$. By Lemma 3.9, $\sigma_1(u) = 2\sigma_1(u')$. If $\sigma_1(u) \geq 4$ then $s^* + \epsilon_v \geq 5$ or $\deg(u) > 4$. So $\sigma_1(u) = 2$, and $\sigma_{0/1}(x) = 1$. If $|\hat{U}| = 2$ then $s^* + \epsilon_v \geq 5$. Otherwise, if $|\hat{U}| = 1$ then $\delta(G) < 2$, a final contradiction.

Lemma 3.18. *Let $H_0 = C_4$, $|U| = 4$, $a = 4$ and $n \in \{18, 19\}$. Then $H_1 \neq E_k$ ($k = 7, 8, 9$).*

Sketch proof. Suppose to the contrary, that $H_1 = E_k$ for some $k \in \{7, 8, 9\}$. Then $s = 1$, $\eta(G - U) \geq n - 12 \geq 6$ and $s^* + \epsilon = 22 - n$ (see the proof of Lemma 3.16). Observe first that $H_1 \neq E_9$ (then $\eta(G - U) \leq 5$, by Remark 2.5). In addition, by the same arguments, if $H_1 = E_7$ or E_8 then each H_i^* is an isolated vertex, i.e. equal to P_1.

Let $H_1 = E_7$. For $n = 18$, $s^* = 3$. So $\epsilon = 1$. Consequently, if $\sigma_0(x) \neq 1$, there exists in U a unique element, say u, such that $\sigma_0(u) = 1$, while $\sigma_0(u') = 1$ if $u' \neq u$. Then $\sigma_1(u) = 2$, while $\sigma_1(u') = 1$. So $diam(G) > 4$ (see Lemma 3.4(4^o); note also that $\min(U) = 1$). Otherwise, if $\sigma_0(x) = 1$, then $\sigma_1(x) = y$ for some $y \geq 1$; if $y = 1$ then $diam(G) > 4$ (see Lemma 3.4(4^o); now $\min(U) = 0$). If $y \geq 2$ then $\epsilon_v \geq 2$, a contradiction. For $n = 19$, $s^* = 4$, and therefore $s^* + \epsilon \geq 4$, a contradiction.

Let $H_1 = E_8$. For $n = 18$, $s^* = 2$ and $\epsilon = 2$.

Observe first that $\sigma_{0/1}(x) = \dfrac{1}{k}$, where $k \in \{1, 2, 3, 4\}$ (since $\epsilon_v \leq 2$). Let $\sigma_i = \sum_{u \in U} \sigma_i(u)$. So $\sigma_1 = k\sigma_0$. Next $\sigma_0 \in \{4, 5\}$ (otherwise, $\mathrm{diam}(G) \geq \mathrm{diam}(H_1 + U) > 4$). If $\sigma_0 = 4$ then $\sigma_1 \in \{8, 12\}$ (other values are too small or too big). So $k \in \{2, 3\}$. Next, for each $u \in U$, $\sigma_1(u) \leq k$. If $k = 2$ then all vertices in H_0 must be jumpers (by Lemma 3.4(4^o); then, as well, $\min(U) = 2$). But now, we easily obtain (see Subsection 3.2(**H**)) that $\kappa + h \geq 7$, a contradiction (by Lemma 3.11). If $k = 3$ then $\mathrm{diam}(H_{0,1}) > 4$, a contradiction. If $\sigma_0 = 5$ then $\sigma_1 \in \{5, 10\}$ (other values are too big). So $k \in \{1, 2\}$. If $k = 1$ then $\min(U) = 0$, and consequently $\Delta > 4$. If $k = 2$ then, we easily obtain $\kappa + h \geq 7$, or otherwise $\mathrm{diam}(G) > 4$. So $\eta(G - U) \leq 5$ (by Remark 2.5), a contradiction. For $n = 19$, $s^* = 3$ and $\epsilon = 0$. So $\sigma_0(x) = 1$. On the other hand, $\sigma_1(x) \neq y$, for any $y \geq 1$ (see Fig. 2.2), and we are done.

4. THE PROOF OF THEOREM 1.1

In this section we consider 43 instances from Table 2.1 (addressed by their identifiers). To prove the main theorem, we have to discard all instances but one (namely, **(7)**). For this purpose, instances are grouped according to some common features. Some data relevant in proofs are displayed in separate tables. Besides some ad hoc strategies (like in Proposition 4.1), the proofs are most frequently based on the following two dominating strategies:

(a) to estimate $\hat{E} = p + 2(e + h)$ by $\mathcal{E}$ in order to show that $\hat{E} \geq \mathcal{E} > \hat{F}$, where $\hat{F} = 114 + 4a + \nu_3$;

(b) to deduce contradictions in the structure of graphs from $\mathscr{S}'$ (see Fig. 3.1) based on tools developed in Section 3.

We first give some more details concerning strategy (a). Recall, quantities p, e and h were estimated in Subsection 3.2 (provided $H_0 = C_4$). Here we add only a few arguments for estimating parameters α, β (or α^*, β^*), and thus p. Recall, $p \geq \max\{f(\alpha, \beta), f^*(\alpha^*, \beta^*)\}$, where α, β denote the number of red, blue edges (RB-edges), while α^*, β^* denote the number of yellow, green edges (YG-edges), respectively.

If a is big, say $a \geq 7$ (or even $a \geq 6$) we can estimate α and β as follows:

(i) If $a \geq 7$, consider the subgraph D $(= D(4); see Subsection 3.1)$ of G, induced by vertices of degree 4 (so it contains all red edges). Usually, it is a forest; if not, then $\alpha \geq 4$ (or even 6 if $q = 0$). In the former case, let P $(= P_k)$ be a "small component" of D, say with $k \leq 3$. Denote by p_k the number of components in D equal to P_k. Let v, and v' if $k > 1$, be the vertices of P with smallest degree in D (not G). Let $S(v) = S_5$ be the star having v as its center. Consider a U-partition in G with $H_0 = S(v)$. Then $|U| \geq a - 1$ (by Lemma 3.2(3^o) – with equality only if $s = 2$). Next we have:

If $k = 1$ then $P = P_1$ (so v is an isolated vertex in D). It is incident in G to at least $a - 5$ $(= (a - 1) - 4)$ blue edges. So we have found $(a - 5)p_1$ blue edges. Observe here (and forth) that each blue edge is counted only once (i.e. within its end-vertex of degree 4).

If $k = 2$ then $P = P_2$ (so vv' is an isolated edge in D). If $|U| = a - 1$ then v' is incident to 3 blue edges (since $\deg(u) \geq 3$ if $u \in U$). If $|U| \geq a$ then v is incident to at least one blue edge (since $|U| \geq 7$). By exchanging the roles of v and v', we encounter now at least 2 blue edges. So we have found $2p_2$ blue edges.

If $k = 3$ then P $(= P_3)$ (let w be the central vertex of P). If $|U| = a - 1$ then w is incident to just two blue edges (since $\deg(u) \geq 3$ if $u \in U$). If $|U| \geq a$ then v is incident to at least one blue edge (since $|U| \geq 7$). By exchanging the roles of v and v', we now encounter at least 2 blue edges. So we have found $2p_3$ blue edges.

Summarizing the above conclusions we obtain: if $a \geq 7$ then

$$\beta \geq (a - 5)p_1 + 2(p_2 + p_3). \tag{8}$$

(ii) If $a \geq 6$ then $\beta \geq (a - 5)(\nu_4 - 2\alpha)$ (since $p_1 \geq \nu_4 - 2\alpha$). Since $\beta \leq 3\nu_3$ we obtain: if $a \geq 6$ then

$$\alpha \geq \frac{(a-5)\nu_4 - 3\nu_3}{2(a-5)} \quad \text{and} \quad \beta \geq (a-5)(\nu_4 - 2\alpha). \tag{9}$$

(iii) In addition, if $a \geq 6$, to estimate α and β, one can count RB-edges incident to quadrangle(s). Then, since $|U| \geq a-1 \geq 5$ for each quadrangle, at least $a-5$ vertices on each quadrangle are of degree 4. So we can find on each quadrangle at least two RB-edges (since $a \geq 6$). Some further RB-edges can arise if $|U| = a-1$ (then $\deg(u) \geq 3$ if $u \in U$), but more can arise for sure if $|U| \geq a$.

On the other hand, to estimate α^* and β^* (if $a \leq 6$) we put focus on vertices of degree 2 in $G-U$. Any such vertex, say v, is either a non-jumper, or 1-jumper, or 2-jumper. If v is a non-jumper, then it is an interior vertex of Smith, or reduced Smith graphs, and so is incident to one or two YG-edges, unless both neighbours are of degree 4 in G. But then, since $H_i \neq S_5$ (by Lemma 3.14), they are both jumpers, and possibly 2-jumpers. If v is a simple jumper, then it is a pendant vertex in some of acyclic H_i's. So it has only one neighbour in them. Then either one YG-edge arises, or two hexagons. Finally, if v is a 2-jumper then either one or two YG-edges arise, or none (if both neighbours are of degree 4 in G, but then two RB-edges arise). More details will be given later, when more details on the structure of G is known.

It is also worth mentioning that "RB-variant" is preferable if $f(0,1) > f(0,0)$; otherwise (if $f(0,1) = f(0,0)$) then "YG-variant" can be superior.

We now give a short overview of dominating tools for strategy (b), which turns to be very helpful if $a \in \{4,5\}$.

- $|U| \geq a+1-s$, with equality if and only if $s=2$ (Lemma 3.2);
- $s^* + \epsilon = m - n - s(|U|-1)$ (Lemma 3.3);
- $\sum_{u \in U} \deg(u) = (s+1)|U| + (s^* + \epsilon) - 2\epsilon_c - \epsilon_v$ (Lemma 3.3);
- $\eta(G-U) \geq n - 2a - |U|$ (Lemma 3.7);
- $\eta(G-U) = \sum_{i=0}^{t} \eta(H_i)$ (see also Remark 2.5);
- $\sigma_{i/j}(x)$ $(= \sigma_i(x)/\sigma_j(x))$ is a constant on U (Lemma 3.9);

(I) – Instances of Table 4.1. Given a graph $G \in \mathscr{S}'$, let G^* be the smallest (multi)-graph homeomorphic to G (so G^* has no vertices of degree 2).

identifier	a	(ν_4, ν_3, ν_2)	q
2	3	$(3,2,12)$	9
3	3	$(2,2,14)$	11
4	3	$(1,2,16)$	13

Table 4.1.

Proposition 4.1. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table* 4.1.

Sketch proof. Observe first that $|G^*| = \nu_3 + \nu_4 \le 5$, and that G^* has no loops (otherwise G is not 2-connected). So all possible graphs G^* can be easily constructed (just by hand). To obtain any tentative graph G starting from G^*, we need to insert in total ν_2 vertices of degree 2 into edges of G^*. Since G is simple and bipartite of girth 4 (note $q > 0$), we first insert vertices that "destroy" cycles of lengths less than 4. But further on it turns that each obtained graph has a smaller number of quadrangles than required (see Table 4.1).

(II) – Instances of Table 4.2. Recall, $\ell_{\min} \ge a - 1$. Here we resolve some very simple cases.

identifier	a	(ν_4, ν_3, ν_2)	q	$\mathcal{E}$	$\hat{F}$
23	5	$(1, 10, 12)$	9	145	144
24	6	$(1, 18, 4)$	4	161	156
28	6	$(4, 10, 10)$	4	149	148
31	6	$(5, 6, 14)$	5	147	144
33	7	$(4, 14, 8)$	2	157	156
36	7	$(5, 10, 12)$	3	157	152
39	7	$(6, 6, 16)$	4	157	148

Table 4.2.

Proposition 4.2. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table 4.2.*

Proof. Let $\mathcal{E} = f(0,1) + 2q(a-1)$. Then $\mathcal{E} > \hat{F}$ for all instances of Table 4.2, and we are done.

(III) – Instances of Table 4.3. Let $\mathcal{E} = f(\alpha, \beta)$ (note $q = 0$ for all instances).

identifier	a	(ν_4, ν_3, ν_2)	q	$\hat{F}$
35	7	$(8, 6, 12)$	0	148
41	8	$(4, 18, 6)$	0	164
43	8	$(7, 10, 12)$	0	156

Table 4.3.

Proposition 4.3. *There are no graphs $G \in \mathscr{S}'$ with parameters as in Table* 4.3.

Sketch proof. First, we are done if $\alpha \ge 6$ (then, for all instances, $\mathcal{E} = f(6,0) > \hat{F}$). So assume that D is a forest (since $q = 0$). Then, for a fixed α, we show (using (8)) that β is enough large to ensure that $\mathcal{E} > \hat{F}$.

(35) If $\alpha = 5$ then $\beta \ge 2$; if $\alpha = 4$ then $\beta \ge 5$; if $\alpha = 3$ then $\beta \ge 8$; if $\alpha = 2$ then $\beta \ge 11$; if $\alpha = 1$ then $\beta \ge 13$; if $\alpha = 0$ then $\beta \ge 15$. Therefore, $\mathcal{E} \ge \min\{f(\alpha, \beta)\} = 149 > \hat{F} = 148$.

(41) If $\alpha \ge 3$ then $\mathcal{E} = 165$; if $\alpha = 2$ then $\beta \ge 4$; if $\alpha = 1$ then $\beta \ge 8$; if $\alpha = 0$ then $\beta \ge 9$. Therefore, $\mathcal{E} \ge \min\{f(\alpha, \beta)\} = 165 > \hat{F} = 164$.

(43) If $\alpha \ge 5$ then $\mathcal{E} = 159$; if $\alpha = 4$ then $\beta \ge 1$; if $\alpha = 3$ then $\beta \ge 4$; if $\alpha = 2$ then $\beta \ge 7$; if $\alpha = 1$ then $\beta \ge 10$; if $\alpha = 0$ then $\beta \ge 13$. Therefore, $\mathcal{E} \ge \min\{f(\alpha, \beta)\} = 157 > \hat{F} = 156$. □

In what follows $q > 0$. We assume that $Q_1, Q_2, \ldots, Q_q$ are the quadrangles of a tentative graph G, and that $\ell(Q_1) \le \ell(Q_2) \le \cdots \le \ell(Q_q)$. Moreover, since $H_0 = C_4$ if not told otherwise, then $H_0 = Q_1$, or just Q for short.

(IV) – Instances of Table 4.4. Let $\mathcal{E} = f(\alpha, \beta) + 2q\ell_{\min}$, where $\ell_{\min} = \ell(Q)$.

identifier	a	(ν_4, ν_3, ν_2)	q	$f(0,1)$	$\hat{F}$
25	6	$(3, 14, 6)$	3	121	152
29	6	$(6, 6, 12)$	3	109	144
32	6	$(7, 2, 16)$	4	97	140
34	7	$(6, 10, 10)$	1	136	152
37	7	$(7, 6, 14)$	2	121	148
38	7	$(9, 2, 16)$	1	124	144
40	7	$(8, 2, 18)$	3	109	144
42	8	$(5, 14, 10)$	1	145	160

Table 4.4.

Proposition 4.4. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table 4.4.*

Proof Recall $\ell_{\min} \geq a - 1$ (with equality if $s = 2$). So we have:

(25) Let $\ell_{\min} = 5$. Then $s = 2$, and $\deg(u) \geq 3$ if $u \in U$. So if v is a vertex of Q of degree 4 it is incident either to at least one red edge (then $\alpha \geq 1$), or to at least three blue edges (then $\beta \geq 3$). So $\mathcal{E} = \min\{f(1,0), f(0,3)\} + 2q\ell_{\min} = 153 > \hat{F} = 152$. If $\ell_{\min} \geq 6$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} = 157 > \hat{F} = 152$.

(29) Let $\ell_{\min} = 5$. If $\alpha \geq 3$, then $\mathcal{E} \geq f(3,0) + 2q\ell_{\min} = 147 > \hat{F} = 144$. Next, using we obtain: if $\alpha = 2$ then $\beta \geq 2$; if $\alpha = 1$ then $\beta \geq 4$; if $\alpha = 0$ then $\beta \geq 6$ (while $\beta \geq 7$ is required). So, in the latter case, let $\beta = 6$. Then each vertex of D is incident to just one blue edge (note one blue edge is guaranteed by reasoning as in Proposition 4.3, taking that $H_0' = S_5$). On the other hand, each vertex of Q of degree 4 is incident to at least two blue edges, a contradiction (note, each vertex in U is of degree at least 3). So $\mathcal{E} = \min\{f(\alpha, \beta)\} + 2q\ell_{\min} = 145 > \hat{F} = 144$. If $\ell_{\min} \geq 6$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 145 > \hat{F} = 144$.

(32) Let $\ell_{\min} = 5$. Then $\alpha \geq 1$ (by (9)). If $\alpha \geq 2$ then $\mathcal{E} \geq f(2,0) + 2q\ell_{\min} = 143 > \hat{F} = 140$; if $\alpha = 1$ then $\beta \geq 3$ (by (9)), and we are done (since $f(1,3) = 103$). If $\ell_{\min} \geq 6$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 145 > \hat{F} = 140$.

(34) Let $\ell_{\min} = 6$. Then $s = 2$, and $\deg(u) \geq 3$ if $u \in U$. Counting RB-edges around Q we obtain $\alpha + \beta \geq 7$. So either $\alpha \geq 3$ (then $f(3,0) \geq 141$), or (by (9)) we have: if $\alpha = 2$ then $\beta \geq 5$; if $\alpha = 1$ then $\beta \geq 6$. In addition, if $\alpha = 0$ then $\beta \geq 9$ (since each vertex of degree 4 now has two neighbours of degree 3). Consequently, $\mathcal{E} = \min\{f(\alpha, \beta)\} + 2q\ell_{\min} = 153 > \hat{F} = 152$. If $\ell_{\min} \geq 7$, then $\alpha + \beta \geq 4$ (with $\alpha \geq 2$). So $\mathcal{E} = \min\{f(\alpha, \beta)\} + 2q\ell_{\min} \geq 154 > \hat{F} = 152$.

(37) Let $\ell_{\min} = 6$. Then, as above, either $\alpha \geq 2$, or $\alpha = 1$ and $\beta \geq 2$, or $\beta \geq 5$. But then $\mathcal{E} = \min\{f(\alpha, \beta)\} + 2q\ell_{\min} = 149 > \hat{F} = 148$. If $\ell_{\min} \geq 7$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 149 > \hat{F} = 148$.

(38) Now $\ell_{\min} \geq 6$. First, if $\alpha \geq 5$, then $\mathcal{E} = f(\alpha, 0) + 2q\ell_{\min} \geq 147 > \hat{F} = 144$. From (9) we obtain that $\alpha \geq 3$, and next we have: if $\alpha = 4$ then $\beta \geq 1$; if $\alpha = 3$ then $\beta \geq 3$. So $\mathcal{E} = \min\{f(\alpha, \beta)\} + 2q\ell_{\min} \geq 145 > \hat{F} = 144$.

(40) Now $\ell_{\min} \geq 6$. Then $\alpha \geq 3$ (by (9)), and $\mathcal{E} = f(3,0)+2q\ell_{\min} \geq 151 > \hat{F} = 144$.

(42) Now $\ell_{\min} \geq 7$. Now all vertices in Q, but possibly one, are of degree 4. So either all edges of Q are red (then $\alpha \geq 4$), or two are red while other two are blue (so $\alpha, \beta \geq 2$). Then $\mathcal{E} = \min\{f(\alpha,\beta)\} + 2q\ell_{\min} \geq 166 > \hat{F} = 160$.

(V) – Instances of Table 4.5. We now consider instances with more structural considerations involved.

identifier	a	n	m	(ν_4, ν_3, ν_2)	q	$f(0,1)$	$\hat{F}$
1	3	17	21	$(1,6,10)$	10	49	132
5	4	17	25	$(1,14,2)$	5	97	144
30	6	24	33	$(8,2,14)$	2	112	140

Table 4.5.

Proposition 4.5. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table* 4.5.

Sketch proof. Recall $\ell_{\min} \geq a-1$ (with equality if $s=2$). So we have:

(1) Let $\ell_{\min} = 2$. Then $s = 2$ and $\deg(u) \geq 3$ if $u \in U$. So $s^* = 0$ (otherwise $\nu_4 > 1$) and consequently $\eta(G-U) \leq 8$. On the other hand $\eta(G-U) \geq 9$.

Let $\ell_{\min} = 3$. Then $s \geq 1$. If $s = 1$ then $s^*+\epsilon = 2$. So $s^* \leq 2$, and consequently $\eta(G-U) \leq 8$, while $\eta(G-U) \geq 8$. So $\eta(G-U) = 8$, and then $H_1 = S_5$, $H_1^* = S_4$ and $H_2^* = P_1$. But then $\epsilon_v \geq 1$ (otherwise, due to H_1, $\delta(G) < 2$). So $s^*+\epsilon \geq s^*+\epsilon_v \geq 3$, a contradiction. If $s = 2$ then $s^*+\epsilon = 0$, while $\epsilon_v > 0$ (as above).

Let $\ell_{\min} \geq 4$. If $K_{2,3} \not\subset G$, there are at most 6 (or 3, or 1) quadrangles passing through each vertex of degree 4 (resp. 3 and 2). But then $q \leq (6\nu_4+3\nu_3+\nu_2)/4 < 10$. Otherwise, if $K_{2,3} \subset G$, then $\mathcal{E} = f(0,1)+2(q\ell_{\min}+\kappa) \geq 135 > \hat{F} = 132$ (since $\kappa \geq 3$).

(5) Let $\ell_{\min} = 3$. Then $s = 2$, and therefore $\deg(u) \geq 3$ if $u \in U$. Next $s^* = \epsilon_u = 0$, $\epsilon_v \leq 1$ (otherwise $\nu_4 > 1$) and $\epsilon_c \leq 2$. So $s^*+\epsilon \leq 3$, a contradiction (since $s^*+\epsilon = 4$).

Let $\ell_{\min} = 4$. Then $s \geq 1$. Next $|H_1| \geq 6$ (by Lemmas 3.12 and 3.13), and therefore $s = 1$ (otherwise $|H_2| < 4$). So $s^*+\epsilon = 5$, whence $\epsilon_v \leq 5$. But then $\sum_{u \in U} \deg(u) = 18 - 2\epsilon_c - \epsilon_v \leq 13$ (note $\nu_4 = 1$). So $2\epsilon_c + \epsilon_v \geq 5$. Since $\epsilon_c \leq 1$ we obtain: if $\epsilon_c = 0$ then $\epsilon_v = 5$; if $\epsilon_c = 1$ then $\epsilon_v \geq 3$. In the former case $s^* = 0$, while in the latter case $s^* = 1$ (note $|R^*|$ is odd). But in both cases $\deg(u) = 4$ for some $u \in U$. So $\min(U) = 2$ (otherwise there exists a vertex of degree 4 in H_0). If $s^* = 0$ then $H_1 = W_9$, or E_9, and if $s^* = 1$ then $H_1 = C_6$, or C_8. Then $\eta(G-U) \geq 5$. So $H_1 \neq E_9, C_6$. If $H_1 = W_9$ then $\nu_4 > 1$ (one interior vertex in W_9 is of degree 4 in G). Finally, if $H_1 = C_8$, then either $\nu_4 > 1$, of all vertices in C_8 but one are simple jumpers, and therefore (observing also H_1^* $(= P_1)$) $h \geq 8$, a contradiction (by Lemma 3.11).

Let $\ell_{\min} \geq 5$. Then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 147 > \hat{F} = 144$.

(30) Let $\ell_{\min} = 5$. Then then $s = 2$ and $\deg(u) \geq 3$ if $u \in U$. In addition, $s^* + \epsilon = 1$ (so $\epsilon_v \leq 1$). Since $\nu_3 = 2$ at least three vertices in U are of degree 4. Therefore, if $s^* = 0$ then $\epsilon_v \geq 3$. Otherwise, if $s^* = 1$ then $\epsilon_v \geq 1$.

If $\ell_{\min} \geq 6$ then $\alpha \geq 1$ and $\beta \geq 6$ (see (9)). So $\mathcal{E} = f(\alpha, \beta) + 2q\ell_{\min} \geq 143 > \hat{F} = 140$.

(VI) – Instances of Table 4.6. The remaining instances with $a \geq 6$.

identifier	a	n	m	(ν_4, ν_3, ν_2)	q	$f(0,1)$	$\hat{F}$
26	6	23	33	$(5, 10, 8)$	2	124	148
27	6	23	33	$(7, 6, 10)$	1	128	144

Table 4.6.

Proposition 4.6. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table 4.6.*

Sketch proof. Since $\ell_{\min} \geq a - 1$ (with equality if $s = 2$) we have:

(26) Let $\ell_{\min} = 5$. Now, if $\alpha = 0$, then $\beta \geq \nu_4 = 5$ (since each vertex of degree 4 is incident to at least one blue edge). So $\mathcal{E} = \min\{f(1,0), f(0,5)\} + 2q\ell_{\min} = 145$, and we are done if, say $h \geq 2$. To see this, observe first that $s = 2$, and $\min(U) > 0$ (since $\ell_{\min} > 4$). So all vertices in H_1 and H_2, belonging to one of the colour classes, are jumpers (by Lemma 3.4(3^o)). If so we can find in H_i an edge e_i ($i = 1, 2$) so that $h(e_i) \geq 1$ – note, at least one vertex in H_i in opposite colour is a jumper (since $\min(U) > 0$). So $h \geq 2$.

Let $\ell_{\min} \geq 6$. Then, $\beta \geq 5$ if $\alpha = 0$, and $\mathcal{E} \geq \min\{f(1,0), f(0,5)\} + 2q\ell \geq 149 > \hat{F} = 148$.

(27) Let $\ell_{\min} = 5$. Now, if $\alpha = 0$, then $\beta \geq \nu_4 + 2 = 9$ (now, in contrast to situation from above, the additional blue edges are are found on the (unique) quadrangle). So $\mathcal{E} = \min\{f(1,0), f(0,9)\} + 2q\ell_{\min} = 139$, and we are now done, if say $h \geq 3$. Reasoning as above, we again have that $h \geq 2$. To prove that $h \geq 3$, observe first that $H_1 \neq C_4$ (since $q = 1$), and also that $H_1 \neq S_5$ (by Lemma 3.14). So $H_1 \in \{C_6, W_6, W_7, E_7\}$. But then $h(H_{0/1}) \geq 2$, as required. Indeed, if there exists a 2-jumper in H_1 then latter follows at once (since $q = 1$); otherwise, we can find in H_1 an additional edge in the role of e_1.

Let $\ell_{\min} = 6$. Then $s \geq 1$. As above, $s \neq 2$. So let $s = 1$ (then $s^* + \epsilon = 5$). As before, $\beta \geq 9$ if $\alpha = 0$, whence $\mathcal{E} = \min\{f(1,0), f(0,9)\} + 2q\ell_{\min} = 141$. So we are done if, say $h \geq 2$. The latter is true if there are 2-jumpers in R^*. Otherwise, due to simple jumpers, $h \geq 2$ if $|H_1| \leq 7$ (then $h \geq ||H_1|| - \Delta(H_1)$; note the edges incident to a non-jumper, if any, do not give rise to hexagons). So $|H_1| \geq 8$. Also $|H_1| \neq 13$. Otherwise, only $\epsilon_v \neq 0$ (note, $\epsilon_u = 0$ since $q = 1$), whence $\epsilon_v = 5$. But then $h \geq 2$ (since $h \geq ||H_1|| - 2\Delta(H_1)$ – note, there are 11 ($= |U| + \epsilon_v$) cross edges between U and H_1). So $s^* \geq 1$, and $|H_1^*| \leq 5$ (since $|R^*| = 13$). If $H_1^* = P_i$ ($i \leq 5$), then either $h \geq 2$, or $h \geq 1$ and $\alpha^* \geq 1$ (then $f^*(1,0) \geq 131$), or $\alpha^* \geq 2$ (then $f^*(2,0) \geq 134$), and we are done. If $H_1^* = Y_4$ (or Y_5) then $h \geq 2$ (then $s^* = 1$, and therefore all vertices either in H_1, or in H_1^* are jumpers).

Let $\ell_{\min} \geq 7$. Then either $\alpha \geq 3$, or $\alpha = 2$ and $\beta \geq 6$ (note, if $\ell_{\min} = 7$ then one vertex in H_0 of degree 4 is incident to at least two red edges, while other vertices of degree 4 are incident to a blue edge). So $\mathcal{E} = \min\{f(\alpha,\beta)\} + 2ql - min \geq 146 > \hat{F} = 144$.

(VII) – Instances of Table 4.7. Cases with $a = 5$ and $n = 20$.

identifier	a	n	m	(ν_4, ν_3, ν_2)	q	$f(0,1)$	$\hat{F}$
14	5	20	29	$(2, 14, 4)$	4	109	148
15	5	20	29	$(4, 10, 6)$	3	112	144

Table 4.7.

Proposition 4.7. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table 4.7.*

Sketch proof. Now $\ell_{\min} \geq a - 1 = 4$ (with equality if $l = 4$). Next we have:

Let $\ell_{\min} = 4$. Then $s = 2$, and $|H_1| + |H_2| \leq 12$. So $H_1 = C_6$ (by Lemmas 3.13 – 3.15), and also $H_2 = C_6$. But then $\eta(G - U) \leq 2$, a contradiction (since $\eta(G - U) \geq 6$). So $l \neq 4$ and we have:

(14) Let $\ell_{\min} \geq 5$. Then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 149 > \hat{F} = 148$.

(15) Let $\ell_{\min} = 5$. Then $\mathcal{E} = f(0,1) + 2q\ell_{\min} = 142$. So we are done if one of the following holds: $h \geq 2$, or $\alpha^* \geq 1$ (then $f^*(1,0) = 115$), or $\beta^* \geq 3$ (then $f^*(0,3) = 115$), or $h = 1$ and $\beta^* \geq 1$ (then $f^*(0,1) = 113$). So $H_1 \neq C_4$ (then $h \geq 4$), and $H_1 \neq S_5$ (by Lemma 3.14). If $H_1 = C_{2k}$ ($k \geq 3$) then each non-jumper in H_1 is adjacent to jumpers in H_1 (otherwise $\alpha^* \geq 1$). Next, at least one of these jumpers is adjacent to an additional non-jumper (otherwise $h \geq 2$). Proceeding in this way, we obtain that all vertices in H_1 belonging to one colour class are non-jumpers, while the others are jumpers, a contradiction (since $\ell_{\min} > 4$). Observe now that there are at most two 2-jumpers in R^*, neither of them being a 3-jumper (otherwise, $\kappa \geq 3$ or $h \geq 2$ – see Subsection 3.2($\mathbf{H}(b)$)). Observe also that $\sigma_0(x) = 1$ (otherwise, $\kappa \geq 3$). Let $H_1 = W_k$ ($k \geq 6$). Then both vertices of degree 3 (in W_k) are non-jumpers (otherwise $h \geq 2$). If $k = 6$ let $\sigma_1(x) = y$. Clearly, $y \leq 2$. So we have: if $y = 1$ then $\beta^* \geq 3$; if $y = 2$ then $\kappa \geq 3$. If $k = 7$ then $h \geq 2$. If $k \geq 8$ then $\beta \geq 2$ (two green edges are incident to vertices of degree 3 in W_k because number of 2 jumpers in W_k is at most 2). Next, since there are no more green edges in W_k, all other vertices in the interior path must be simple jumpers, whence either $\beta^* \geq 3$, or $h \geq 1$ in addition to $\beta^* \geq 1$). So let $H_1 = E_k$. If $k = 7$ then $h \geq 1$ and $\beta^* \geq 1$. If $k \geq 8$ consider H_i^*'s. It easily follows that $s^* = 1$ if $k = 8$ (then $H_1^* = P_3$), and $s^* = 2$ if $k = 9$ (then $H_1^* = H_2^* = P_1$). So there are no 2-jumpers in H_1, and we easily obtain that either $\alpha^* \geq 1$, or $h \geq 1$ and $\beta^* \geq 1$.

Let $\ell_{\min} \geq 6$. Then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 148 > \hat{F} = 144$.

(VIII) – Instances of Table 4.8. Cases with $a = 5$ and $n = 21$.

identifier	a	n	m	(ν_4, ν_3, ν_2)	q	$f(0,1)$	$\hat{F}$
16	5	21	29	$(1, 14, 6)$	6	97	148
17	5	21	29	$(3, 10, 8)$	5	97	144
18	5	21	29	$(5, 6, 10)$	4	97	140
19	5	21	29	$(7, 2, 12)$	3	100	136

Table 4.8.

Proposition 4.8. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table 4.8.*

Sketch proof. Now $\ell_{\min} \geq a - 1 = 4$ (with equality if $s = 2$). Let $\ell_{\min} = 4$. Then $|H_1| \leq 6$, whence $H_1 = C_6$ (see Lemmas 3.13 – 3.15). But then $\eta(G - U) \leq 5$ is too small (since $\eta(G_U) \geq 7$). Next we have:

(16) Let $\ell_{\min} \geq 5$. Then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 157 > \hat{F} = 148$.

(17) Let $\ell_{\min} \geq 5$. Then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 147 > \hat{F} = 144$.

(18) Let $\ell_{\min} \geq 5$. If $\ell_{\min} = 5$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} + 2(\kappa + h) \geq 137$, and we are done if $\kappa + h \geq 2$. So there are at most two quadrangles sharing just one common edge, and then $\alpha + \beta \geq 6$ (since each quadrangle contains at least two RB-edges). But then $\mathcal{E} = \min\{f(\alpha, \beta)\} + 2q\ell_{\min} \geq 142 > \hat{F} = 140$. If $\ell_{\min} \geq 6$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 142 > \hat{F} = 140$.

(19) Let $\ell_{\min} \geq 5$. If $\ell_{\min} = 5$ then $K_{2,3} \not\subset G$. Otherwise, $\kappa \geq 3$ and $\alpha \geq 1$ (since $\nu_3 = 2$). But then $\mathcal{E} = f(1,0) + 2(ql + \kappa) + h = 137 > \hat{F} = 136$, and we are done. So we have two patterns for $H_0 + U$, and in both of them one vertex in H_0 (say white one) is of degree 2. If so all vertices in R^* of in opposite colour (so black ones) are jumpers (by Lemma 3.4(3^o)). Let $H_1 = C_{2k}$ ($k \geq 2$). If $k = 2$ then $h \geq 4$ and we are also done. If $k \geq 3$ we are also done. Indeed, from the above, $\sigma_0(x) \equiv 1$. Also, $\sigma_{0/1}(x) \equiv 1$, and therefore $\sigma_1(x) \equiv 1$. So just five cross edges join U with H_1, and then either $\nu_3 > 2$, or some black vertices are not jumpers (note, at most one vertex in R^* is of degree 3, since one vertex of degree 3 is already in H_0). Let $H_1 \neq W_k$ (with $k \geq 6$). Now at least one vertex of degree 3 in W_k must be a jumper, so $h \geq 2$. The other must be a non-jumper but its neighbours which are pendant vertices in W_k must be jumpers, but not 2-jumpers (otherwise, either $\nu_3 > 2$, or $h \geq 4$). So $\beta^* \geq 2$, and then $\mathcal{E} = f^*(0,2) + 2(q\ell_{\min} + h) = 137 > \hat{F} = 136$. If $H_1 = E_k$, then $s = 1$ (note, if $s = 2$ then $n > 21$). So $s^* + \epsilon = 4$. Let $\sigma_1(x) \equiv y$. Let $k = 7$. Then: $y \neq 1$ (since $\sigma_1(u) > 1$ for some $u \in U$); $y \neq 2$ (then $\nu_3 > 2$); $y \neq 3$ (then $|\{u : \sigma_1(u) = 3\}| = 1$); $y \ngeq 4$ (then $\epsilon_v \geq 5$). Let $k = 8$. Then: $y > 2$ (otherwise, by Lemma 3.4(4^o), two non-jumpers in H_1 are in opposite colours, a contradiction); $y \neq 3$ (then either $\nu_3 > 2$, or two non-jumpers in H_1 are in opposite colours); $y \ngeq 4$ (then $s^* + \epsilon_v \geq 5$). Let $k = 9$: Then $s^* = 3$ (otherwise, $\eta(G - U) < 6$, a contradiction (since $\eta(G - U) \geq 6$)). So $\epsilon_v \leq 1$. Next we have: $y > 3$ (otherwise, two non-jumpers in H_1 are in opposite colours); $y \ngeq 4$ (otherwise, $\epsilon_v \geq 2$).

Let $\ell_{\min} \geq 6$. Then $\alpha \geq 1$, and therefore $\mathcal{E} = f(1,0) + 2q\ell_{\min} \geq 137 > \hat{F} = 136$.

(IX) – Instances of Table 4.9. Cases with $a = 5$ and $n = 22$.

identifier	a	n	m	(ν_4, ν_3, ν_2)	q	$f(0,1)$	$\hat{F}$
20	5	22	29	$(2, 10, 10)$	7	85	144
21	5	22	29	$(4, 6, 12)$	6	85	140
22	5	22	29	$(6, 2, 14)$	5	85	136

Table 4.9.

Proposition 4.9. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table* 4.9.

Sketch proof. Now $\ell_{\min} \geq a - 1 = 4$ (with equality only if $s = 2$). If $\ell_{\min} = 4$ then $s = 2$, and $s^* + \epsilon = 1$. Since $|H_1| + |H_2| \leq 14$, $|H_1| \leq 7$. On the other hand, if $|H_1| \leq 6$ then $H_1 = C_6$ (by Lemmas 3.13 – 3.15. But then $\eta(G - U) \leq 7$, a contradiction (since $\eta(G - U) \geq 8$). So $|H_1| = 7$. If $H_1 = E_7$ then $\eta(G - U) \leq 6$ (again too small). So let $H_1 = H_2 = W_7$. But then $\epsilon_v \geq 2$ (by Lemma 3.4(1^o)). So $\ell_{\min} \neq 4$, and we have:
(20) If $\ell_{\min} \geq 5$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 155 > \hat{F} = 144$.
(21) If $\ell_{\min} \geq 5$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 144 > \hat{F} = 140$.
(22) If $\ell_{\min} \geq 5$ then $\beta \geq 2$ (it easily follows), and $\mathcal{E} = f(0,2) + 2q\ell_{\min} \geq 137 > \hat{F} = 136$.

(X) – Instances of Table 4.10. Cases with $a = 4$ and $n = 20$.

identifier	a	n	m	(ν_4, ν_3, ν_2)	q	$f(0,1)$	$\hat{F}$
12	4	20	25	$(2, 6, 12)$	9	61	136
13	4	20	25	$(4, 2, 14)$	8	65	132

Table 4.10.

Proposition 4.10. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table* 4.10.

Sketch proof. Now $\ell_{\min} \geq a - 1 = 3$ (with equality only if $s = 2$). If $\ell_{\min} = 3$, then $s = 2$, and $s^* + \epsilon = 1$. If $s^* = 0$ then $\eta(G - U) \leq 8$, a contradiction (since $\eta(G - U) \geq 9$). If $s^* = 1$ then $|H_1| + |H_2| \leq 12$, and therefore $|H_1| \leq 6$. So $H_1 = W_6$ (note, H_1 is a tree since $\epsilon_c = 0$; also $H_1 \neq S_5$ by Lemma 3.14). But then $\delta(G) < 2$ (since $\epsilon_v = 0$). So $\ell_{\min} \neq 3$.

If $\ell_{\min} = 4$ then $s \geq 1$, and $\eta(G - U) \geq 8$. If $s = 1$ then $s^* + \epsilon = 2$. If $s^* = 2$ then $\epsilon_v = 0$, and H_1^* and H_2^* are paths. So $\eta(G - U) \leq 7$ (too small). The same holds if $s^* \leq 1$. If $s = 2$ then $s^* + \epsilon < 0$. So $\ell_{\min} \neq 4$, and we have:
(12) If $\ell_{\min} \geq 5$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 151 > \hat{F} = 136$.
(13) If $\ell_{\min} \geq 5$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 145 > \hat{F} = 132$.

(XI) – Instances of Table 4.11. Cases with $a = 4$ and $n = 19$.

identifier	a	n	m	(ν_4, ν_3, ν_2)	q	$f(0,1)$	$\hat{F}$
9	4	19	25	$(1, 10, 8)$	8	73	140
10	4	19	25	$(3, 6, 10)$	7	73	136
11	4	19	25	$(5, 2, 12)$	6	73	132

Table 4.11.

Proposition 4.11. *There are no graphs $G \in \mathscr{S}'$ with parameters given in Table* 4.11.

Sketch proof. Now $\ell_{\min} \geq a-1 = 3$ (with equality only if $s = 2$). If $\ell_{\min} = 3$ then $s = 2$, and $\deg(u) \geq 3$ for $u \in U$. Next $s^* + \epsilon = 2$ and $\eta(G-U) \geq 8$ (by Lemmas 3.3(1^o) and 3.7). So $s^* \leq 1$ (otherwise $\Delta(G) > 4$). If $s^* = 0$ then $\eta(G-U) \leq 8$, with equality only if $(H_1, H_2) = (S_5, W_7)$. But then $\min(U) = 0$ and $\epsilon_v = 2$. So $\nu_4 = 4$, a contradiction (for all instances). If $s^* = 1$ then $|H_1| + |H_2| \leq 11$. So $|H_1| \leq 5$, and $H_1 = C_4$ or S_5. But then either $\epsilon_c + \epsilon_v = 1$, and H_2 must be a tree with 3 pendant vertices (otherwise $\epsilon \geq 2$). So $H_2 = E_7$, while $H_1 = C_4$. But then $\eta(G-U) \leq 6$. So $\ell_{\min} \neq 3$.

If $\ell_{\min} = 4$ then $s \geq 1$. But then H_1 cannot be any of possible Smith graphs up to 11 vertices (by Lemmas 3.13–3.18). So $\ell_{\min} \neq 4$, and we have:

(9) If $\ell_{\min} \geq 5$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 153 > \hat{F} = 140$.

(10) If $\ell_{\min} \geq 5$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 143 > \hat{F} = 136$.

(11) If $\ell_{\min} \geq 5$ then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 133 > \hat{F} = 132$.

(XII) – Instances of Table 4.12. Cases with $a = 4$ and $n = 18$. Now we encounter graphs that satisfy the most of our constraints so that the integrality of the spectrum has to be checked. To avoid a rather involved case study, we will occasionally replace it by some (local) computer search which is manageable even by hand with newGRAPH.

identifier	a	n	m	(ν_4, ν_3, ν_2)	q	$f(0,1)$	$\hat{F}$
6	4	18	25	$(2, 10, 6)$	6	85	140
7	4	18	25	$(4, 6, 8)$	5	85	136
8	4	18	25	$(6, 2, 10)$	4	88	132

Table 4.12.

Proposition 4.12. *There is only one graph $G \in \mathscr{S}'$ with parameters given in Table* 4.12 (*see the third graph of Fig.* 1.1 *– it arises from instance* (**7**)).

Sketch proof. Now $\ell_{\min} \geq a-1 = 3$ (with equality only if $s = 2$). If $\ell_{\min} = 3$ then $s = 2$, and $s^* \leq 1$ (otherwise $\Delta(G) > 4$). Since $|H_1| + |H_2| \leq 11$, it follows that $H_1 = C_4$ (recall $H_1 \neq S_5$). If $s^* = 0$ then $H_2 = W_7$. Note, $H_2 \neq E_7$ (otherwise, $\eta(G-U) \leq 5$, while $\eta(G-U) \geq 7$). If $(H_1, H_2) = (C_4, W_7)$ then $G - U = C_4 \cup C_4 \cup W_7$. So, to obtain G, we need to add three vertices to $G-U$ (i.e. those which comprise U). This can be easily done (say by using newGRAPH). It turns that none of the resulting graphs G, except just one arising from (**7**), is integral. Otherwise, if $s^* = 1$ then $|H_2| = 4$, or 6. So $(H_1, H_2, H_1^*) = (C_4, C_4, P_3)$, or (C_4, W_6, P_1). Now, once again (since $|U| = 3$), we can easily construct all tentative graphs G, and verify that none of them is integral.

If $\ell_{\min} = 4$ then $s \geq 1$. But then H_1 cannot be any of possible Smith graphs up to 10 vertices (by Lemmas 3.13–3.18). So $\ell_{\min} \neq 4$, and we have:

(6) Let $\ell_{\min} \geq 5$. Then $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 145 > \hat{F} = 140$.

(7) Let $\ell_{\min} \geq 5$. Then either $\alpha \geq 1$, or $\beta \geq 3$ (note, if $\alpha = 0$ then each quadrangle contains just two blue edges). So $\mathcal{E} = \min\{f(1,0), f(0,3)\} + 2q\ell_{\min} \geq 137 > \hat{F} = 136$.

(8) Let $\ell_{\min} \geq 5$. Now $\mathcal{E} = f(0,1) + 2q\ell_{\min} \geq 128$. So we are done if $\kappa + h \geq 3$. This happens if $K_{2,3} \subset G$ (then $\kappa \geq 3$), or if $A_7, B_8 \subset G$ (then $h \geq 3$). Otherwise, observe that each quadrangle has at least two RB-edges, and just one vertex of degree 3 (note $\nu_3 = 2$). So, in view of the forbidden configurations, there exist two pairs of quadrangles with one common vertex of degree 3, and also one common edge. If so, $\alpha + \beta \geq 6$ and $h \geq 2$, and then $\mathcal{E} = \min\{f(1,0), f(0,5)\} + 2q\ell_{\min} + 2h \geq 133 > \hat{F} = 132$.

Concluding remarks. Collecting the results of Propositions 4.1 – 4.12 we arrive at the proof of Theorem 1.1. At this place some further facts deserve to be mentioned. Namely, in this paper we have not only reproduced a part of the results of computer search carried on in [**7**], but also have shown why these types of problem are indeed too hard. Next, we hope that the methods (or, according to referee "the tricks and lemmas") developed in this paper can be used for studying the other classes of integral graphs.

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